PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 6, June 2010, Pages 2209–2215
S 0002-9939(10)10263-9
Article electronically published on February 9, 2010

A HIGHER-ORDER GENUS INVARIANT AND KNOT FLOER HOMOLOGY

PETER D. HORN

(Communicated by Daniel Ruberman)

ABSTRACT. It is known that knot Floer homology detects the genus and Alexander polynomial of a knot. We investigate whether knot Floer homology of K detects more structure of minimal genus Seifert surfaces for K. We define an invariant of algebraically slice, genus one knots and provide examples to show that knot Floer homology does not detect this invariant. Finally, we remark that certain metabelian L^2-signatures bound this invariant from below.

1. INTRODUCTION

Knot Floer homology was defined by Peter Ozsváth and Zoltán Szabó [OS04b] and by Jacob Rasmussen [Ras03]. Knot Floer homology is a powerful knot invariant, and it detects such information as the Alexander polynomial [OS04b], [Ras03] and knot genus [OS04a, Theorem 1.2]. Either of these invariants can be computed from a minimal genus Seifert surface. We investigate whether knot Floer homology contains more information about any minimal genus Seifert surface.

In [Hor09], the author defined a geometric invariant for knots in S^3 called the **first-order genus**. Roughly, the first-order genus of K is obtained by adding the individual genera (in $S^3 - K$) of the curves in a symplectic basis on a minimal genus Seifert surface for K, and taking the minimum over all minimal genus Seifert surfaces and all symplectic bases. The first-order genus of a knot is difficult to compute, as there are many symplectic bases for a given Seifert surface. While difficult to compute in general, the first-order genus is a notion of higher-order genus defined for all knots.

In this paper, we define a similar invariant, though it is only defined for algebraically slice, genus one knots. We take a minimum over Seifert surfaces, but what we record is the genus (in $S^3 - K$) of a basis curve which inherits the zero framing from the surface. This invariant is called the **differential genus**. We provide many examples and show that the differential genus is not detected by knot Floer homology.

Theorem 1.1. *There are examples of knots that cannot be distinguished by knot Floer homology, but have different differential genera.*

Received by the editors April 23, 2009, and, in revised form, July 23, 2009.
2010 *Mathematics Subject Classification.* Primary 57M25.
The author was partially supported by National Science Foundation grant DMS-0706929, the Lodieska Stockbridge Vaughn Fellowhip at Rice University, and the NSF Mathematical Sciences Postdoctoral Research Fellowship DMS-0902786.

Cochran, Harvey and Leidy [CHL08] defined the first-order L^2-signatures of a knot. By their definition, each algebraically slice, genus one knot has (at most) three first-order L^2-signatures. We will discuss the relationship between our higher-order genus invariant and these first-order L^2-signatures.

2. Motivation and definition

Let Σ_g be a compact, oriented surface with one boundary component. If $f : \Sigma_g \hookrightarrow S^3$ is an embedding with $K = f(\partial\Sigma_g)$, then some invariants of K can be computed using this embedded surface $f(\Sigma_g)$. Such a surface is called a **Seifert surface** for K. For example, any Seifert surface can be used to compute the knot's Alexander polynomial. This polynomial is encoded in the knot Floer homology $\widehat{HFK}(K)$. The smallest genus of such embedded surfaces with boundary K is called the genus of K, $g(K)$, and this invariant is also detected by $\widehat{HFK}(K)$. We naïvely ask whether $\widehat{HFK}(K)$ contains any more information about the embedded surfaces with boundary K. For example, we are interested in whether $\widehat{HFK}(K)$ contains information about the knottedness of certain simple closed curves on Seifert surfaces $f(\Sigma_g)$ for K. In this paper we will restrict our attention to genus one, algebraically slice knots.

Our motivating example is the positively-clasped, untwisted Whitehead double of a knot K, denoted $D(K)$ and depicted in Figure 1. There is an obvious genus one Seifert surface for $D(K)$. In [Hor09], the author defined a knot invariant, $g_1(K)$, that measures the knottedness of the bands in Seifert surfaces for a knot. For many knots K, this invariant applied to $D(K)$ 'detects' the genus of K; i.e. $g_1(D(K)) = 1 + g(K)$. One may ask if $\widehat{HFK}(D(K))$ detects $g_1(D(K)) \approx g(K)$, and by Hedden's formula [Hed07, Theorem 1.2], the answer is 'yes' in the sense that $\widehat{HFK}(D(K), 1)$ has as a direct summand $\bigoplus_{j=-g(K)}^{g(K)} G_j(K)$, where the $G_j(K)$ are certain groups depending on K. Due to computational difficulties, it is unknown whether $\widehat{HFK}(K)$ detects $g_1(K)$ in general. We aim to define an invariant that is more computable than g_1 and which measures, more or less, the same thing.

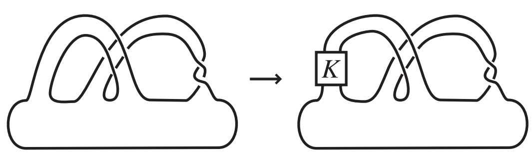

Figure 1. $D(K)$: the positively-clasped, untwisted Whitehead double of K

Definition 2.1. Let K be an algebraically slice knot in S^3 of genus one. Let Σ be any genus one Seifert surface for K. Then Σ has a **metabolizer** $\mathfrak{m}$, a rank one submodule of $H_1(\Sigma;\mathbb{Z})$ on which the Seifert form vanishes. One can show that Σ has exactly two metabolizers $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let $[\alpha_1]$ and $[\alpha_2] \in H_1(\Sigma;\mathbb{Z})$ be generators of $\mathfrak{m}_1$ and $\mathfrak{m}_2$, respectively. By the classification of essential closed curves on a punctured torus [Min99], each $[\alpha_i]$ determines a unique oriented knot in Σ; denote this knot by α_i. The knot α_i is called a **derivative of K with respect to the metabolizer $\mathfrak{m}_i$**.

To sum up, each genus one Seifert surface Σ for an algebraically slice knot K has exactly two (up to orientation) derivatives α_1 and α_2. We denote this set of derivatives as $\partial(K,\Sigma) = \{\alpha_1, \alpha_2\}$.

Let $\mathcal{G}(K)$ denote the set of isotopy classes (in $S^3 - K$) of genus one Seifert surfaces for K, and if α is a null-homologous knot in $S^3 - K$, let $g^K(\alpha)$ denote the genus of α in $S^3 - K$. We define the **differential genus** of K to be

$$\operatorname{dg}(K) = \min_{\Sigma \in \mathcal{G}(K)} \left\{ \max_{\partial(K,\Sigma)=\{\alpha_1,\alpha_2\}} \left\{ g^K(\alpha_1), g^K(\alpha_2) \right\} \right\}.$$

Remark 2.2. The differential genus measures the knottedness of self-linking zero curves on (genus one) Seifert surfaces for K. One may define metabolizers and derivatives of algebraically slice knots of higher genus (see [CHL08]), but in the higher genus setting, a metabolizer may have infinitely many distinct derivatives. One may try to generalize the definition of differential genus to higher genus algebraically slice knots; this may be taken up in a future paper.

3. Examples

Example 3.1. Let K be a knot that is non-trivial and not a cable. By [Whi73], the untwisted Whitehead double of K, $D(K)$, has a unique minimal genus Seifert surface. Each of the untwisted curves on this Seifert surface has the same knot type as K. One can further argue that $\operatorname{dg}(D(K)) = g(K)$.

Example 3.2. Let $R = 9_{46}$ as depicted in Figure 2. A symplectic basis of curves α and β have been drawn for the implied Seifert surface Σ. One can check that α and β have self-linking zero, and so the two derivatives for Σ are α and β. Each of α and β is unknotted; however, $g^R(\alpha) = g^R(\beta) = 1$. The knot Floer homology of R is

$$\begin{matrix} 0 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 0 \end{matrix}$$

where the 5 appears in bigrading $(0,0)$. The genus of R is one, and since $\operatorname{rank} \widehat{HFK}(R,1) = 2 < 4$, we may apply Theorem 2.3 of [Juh08] to conclude that Σ is the unique genus one Seifert surface for R up to isotopy. We conclude that $\operatorname{dg}(R) = 1$.

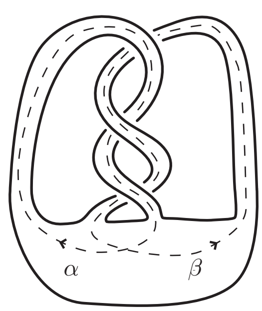

FIGURE 2. The 9_{46} knot

Example 3.3. Now consider the knot K_n in Figure 3. Observe that $K_0 = 9_{46}$. A symplectic basis of curves x and y have been drawn for the implied Seifert surface F. One can check the Seifert form of F to be

$$\begin{pmatrix} 3n & -2 \\ -1 & 0 \end{pmatrix}.$$

If $n \in \mathbb{N}$, the two curves of self-linking zero are $\alpha_n = x + ny$ and $\beta_n = y$. As in the calculation for 9_{46}, one can check that $g^{K_n}(\beta_n) = 1$. The other curve α_n is more complicated; see Figure 4. The knot α_n can be represented by the braid on $n+1$ strands depicted in Figure 5.

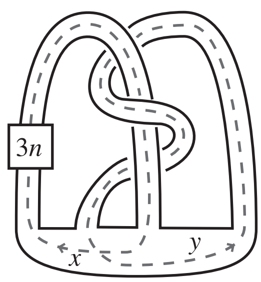

FIGURE 3. A diagram for K_n, and a basis for a Seifert surface

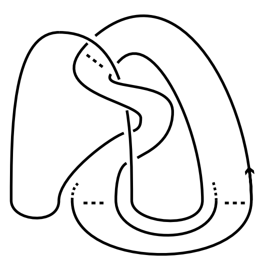

FIGURE 4. A knot diagram of α_n, a curve of self-linking zero

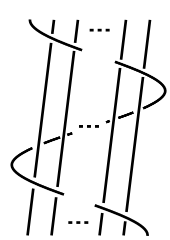

FIGURE 5. A braid representative of α_n using $n+1$ strands

By [Cro89, Corollary 4.1], the Seifert surface constructed by applying Seifert's algorithm to the braid diagram in Figure 5 has minimal genus. In particular,

$g(\alpha_n) = n$, and hence $g^{K_n}(\alpha_n) \geq n$. We will argue that $\mathrm{dg}(K_n) \geq n$. For a given n, one may easily construct a grid diagram for K_n. For several small values of n, we used Marc Culler's Gridlink [Cul] to compute the knot Floer homology of K_n. We found that $\widehat{HFK}(K_n) \cong \widehat{HFK}(9_{46})$ for the values $n = -1, \frac{-2}{3}, \frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$. A recent result of M. Hedden [Hed08] implies that $\widehat{HFK}(K_n) \cong \widehat{HFK}(9_{46})$ for all n. By [Juh08], F is the unique genus one Seifert surface for K_n. By previous arguments, we conclude that $\mathrm{dg}(K_n) = g^{K_n}(\alpha_n) \geq g(\alpha_n) = n$.

Proof of Theorem 1.1. One can show by calculating the Alexander polynomials of the derivatives of $K_{1/3}$ (which have the knot type of the unknot and 10_{132}) that $\mathrm{dg}\left(K_{1/3}\right) \geq 2$. The knot $K_{1/3}$ is called 11_{n139} in the knot tables [CL09]. Thus, $K_{1/3}$ is an explicit example of a knot with the same knot Floer homology as 9_{46} and distinct differential genus. $\square$

Theorem 3.4. *There exists an infinite family of knots K_n such that*

- $\widehat{HFK}(K_n) \cong \widehat{HFK}(K_m)$ *for all m and n, and*
- $\mathrm{dg}(K_n) \neq \mathrm{dg}(K_m)$ *for $m \neq n$.*

Proof. The family is constructed by taking a subsequence of the knots K_n from Example 3.3. $\square$

4. First-order L^2-signatures and the differential genus

Metabelian signatures of knots have been defined by Casson-Gordon, Letsche, Cochran-Orr-Teichner, Friedl, and Cochran-Harvey-Leidy [CG78], [CG86], [Let00], [COT03], [Fri04], [CHL08]. We are interested in those of Cochran, Harvey, and Leidy because each genus one, algebraically slice knot has two "first-order L^2-signatures." We now recall some of the background needed to define these signatures.

Suppose K is an oriented knot in S^3; M_K denotes the closed, oriented 3-manifold obtained by zero-surgery on K, and $G = \pi_1(M_K)$. Let $G^{(1)}$ denote the commutator subgroup of G and let $G^{(2)}$ denote the commutator subgroup of $G^{(1)}$. The **classical rational Alexander module** of K is

$$\mathcal{A}_0(K) := \frac{G^{(1)}}{G^{(2)}} \bigotimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Q}\left[t, t^{-1}\right].$$

Here $G^{(1)}/G^{(2)}$ is identified with the classical Alexander module $H_1\left(M_K; \mathbb{Z}\left[t, t^{-1}\right]\right)$. The **Blanchfield pairing** of K:

$$\mathcal{B}\ell_0^K : \mathcal{A}_0(K) \times \mathcal{A}_0(K) \to \mathbb{Q}(t)/\mathbb{Q}\left[t, t^{-1}\right],$$

is defined by

$$\mathcal{B}\ell_0^K(x, y) = \sum_{n \in \mathbb{Z}} \frac{(d \cdot y t^n) t^n}{\Delta_K(t)},$$

where $\Delta_K(t)$ is the Alexander polynomial of K and d is a 2-chain with $\partial d = \Delta_K(t) \cdot x$. We say a submodule $P \subset \mathcal{A}_0(K)$ is **Lagrangian** (respectively **isotropic**) if $P = P^{\perp}$ (respectively $P \subset P^{\perp}$) with respect to the Blanchfield pairing. To a submodule $P \subset \mathcal{A}_0(K)$, we can associate a metabelian quotient $\phi_P : G \to G/\widetilde{P}$ by setting $\widetilde{P} = \ker\left(G^{(1)} \to G^{(1)}/G^{(2)} \to \mathcal{A}_0(K) \to \mathcal{A}_0(K)/P\right)$. To this quotient we can associate a real number, called the Cheeger-Gromov von Neumann ρ-invariant, $\rho\left(M_K, \phi_P\right)$ [CG85].

Definition 4.1. The **first-order L^2-signatures of a knot** K are the real numbers $\rho(M_K, \phi_P)$, where P is a Lagrangian submodule of $\mathcal{A}_0(K)$ with respect to $\mathcal{B}\ell_0^K$.

Remark 4.2. These are a subset of the metabelian L^2-signatures of Cochran, Harvey, and Leidy [CHL08, Definition 4.1], who allow for P to be isotropic.

Assume K is a genus one, algebraically slice knot with a Seifert surface Σ. The reader will recall that $H_1(\Sigma;\mathbb{Z})$ generates $\mathcal{A}_0(K)$ as a $\mathbb{Q}\left[t,t^{-1}\right]$-module (one must pick a lift of Σ to the infinite cyclic cover). If $\Delta_K(t)=1$, then $\mathcal{A}_0(K)=0$ has no Lagrangian submodules. On the other hand, if $\Delta_K(t)\neq 1$, then $\Delta_K(t)=f(t)f(t^{-1})$ for some linear polynomial $f(t)$. $\mathcal{A}_0(K)$ must be isomorphic to $\frac{\mathbb{Q}\left[t,t^{-1}\right]}{\langle f(t)f(t^{-1})\rangle}$. Thus, any proper submodule P must be

$$\frac{\mathbb{Q}\left[t,t^{-1}\right]}{\langle f(t)\rangle} \qquad \text{or} \qquad \frac{\mathbb{Q}\left[t,t^{-1}\right]}{\langle f(t^{-1})\rangle}.$$

Since the Blanchfield pairing is primitive, $\mathcal{A}_0(K)$ will have precisely two Lagrangians. By Definition 2.1, K will have precisely two Lagrangians and hence two first-order L^2-signatures.

Definition 4.3. Suppose $P\subset\mathcal{A}_0(K)$ is a Lagrangian. The metabolizer $\mathfrak{m}$ **represents** P if the image of $\mathfrak{m}$ under the map

$$i_*\circ(\mathrm{id}\otimes 1): H_1(\Sigma;\mathbb{Z})\hookrightarrow H_1(\Sigma;\mathbb{Z})\otimes\mathbb{Q}\twoheadrightarrow\mathcal{A}_0(K)$$

spans P as a $\mathbb{Q}$-vector space. (To define i_*, it is necessary to choose a lift of Σ to the infinite cyclic cover, but this definition is independent of the choice.)

Proposition 4.4 (Lemma 5.5 of [CHL08])**.** *Let K be an algebraically slice knot and let P be a Lagrangian of $\mathcal{A}_0(K)$. If Σ is any Seifert surface for K, then some metabolizer of $H_1(\Sigma)$ represents P.*

Proposition 4.5 (Corollary 5.8 of [CHL08])**.** *Let K be a genus one, algebraically slice knot. Suppose P is a Lagrangian for K, Σ is a genus one Seifert surface for K, $\mathfrak{m}$ is the metabolizer of Σ representing P, and J is the derivative with respect to $\mathfrak{m}$. Then the first-order L^2-signature of K with respect to P is equal to $\rho_0(J)=\int_{S^1}\sigma_\omega(J)\,d\omega$, the integral of the Levine-Tristram signature function.*

Determining $\mathrm{dg}(K)$ involves computing the genus of two curves from each genus one Seifert surface, of which there may be many. Examples of knots that have an arbitrary number of non-isotopic Seifert surfaces are known [Suz91, p. 47]. Yet we have the following remarkable fact: if just one of the first-order L^2-signatures is large, then the differential genus must be large.

Proposition 4.6. *Let K be a genus one, algebraically slice knot with non-trivial Alexander polynomial. Let ρ_1 and ρ_2 denote the first-order L^2-signatures of K with respect to the two Lagrangians P_1 and P_2. Then $2\,\mathrm{dg}(K)\geq\max\{|\rho_1|,|\rho_2|\}$.*

Proof. Let Σ be the Seifert surface where the minimum is attained. For either derivative $J_i\subset\Sigma$, where J_i represents the Lagrangian P_i, we have

$$2\,\mathrm{dg}(K)\geq 2\,g^K(J_i)\geq 2\,g(J_i)\geq\left|\int_{S^1}\sigma_\omega(J_i)\,d\omega\right|=|\rho_i|. \qquad \square$$

References

[CG78] Andrew J. Casson and Cameron McA. Gordon, *On slice knots in dimension three*, Algebraic and Geometric Topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, CA, 1976), Proceedings of Symposia in Pure Mathematics, vol. XXXII, American Mathematical Society, Providence, RI, 1978, pp. 39–53. MR520521 (81g:57003)

[CG85] Jeff Cheeger and Mikhael Gromov, *Bounds on the von Neumann dimension of L^2-cohomology and the Gauss-Bonnet theorem for open manifolds*, Journal of Differential Geometry **21** (1985), 1–34. MR806699 (87d:58136)

[CG86] Andrew J. Casson and Cameron McA. Gordon, *Cobordism of classical knots*, À la recherche de la topologie perdue, Progress in Mathematics, vol. 62, Birkhäuser Boston, Boston, MA, 1986, pp. 181–199. MR900252

[CHL08] Tim Cochran, Shelly Harvey, and Constance Leidy, *Derivatives of knots and second-order signatures*, `http://arXiv.org/pdf/0808.1432`, 2008. To appear in Algebra & Topology

[CL09] Jae Choon Cha and Charles Livingston, *Knotinfo: Table of knot invariants*, `http://www.indiana.edu/~knotinfo`, February 2010.

[COT03] Tim Cochran, Kent Orr, and Peter Teichner, *Knot concordance, Whitney towers and L^2-signatures*, Annals of Mathematics **157** (2003), 433–519. MR1973052 (2004i:57003)

[Cro89] Peter Cromwell, *Homogeneous links*, Journal of the London Mathematical Society **39** (1989), no. 2, 535–552. MR1002465 (90f:57001)

[Cul] Marc Culler, *Gridlink 2.0*, software, available at `http://www.math.uic.edu/~culler/gridlink`.

[Fri04] Stefan Friedl, *Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants*, Algebraic & Geometric Topology **4** (2004), 893–934. MR2100685 (2005j:57016)

[Hed07] Matthew Hedden, *Knot Floer homology of Whitehead doubles*, Geometry and Topology **11** (2007), 2277–2338. MR2372849 (2008m:57030)

[Hed08] ———, *in preparation*, 2008.

[Hor09] Peter D. Horn, *The first-order genus of a knot*, Mathematical Proceedings of the Cambridge Philosophical Society **146** (2009), no. 1, 135–149. MR2461873 (2009i:57012)

[Juh08] András Juhász, *Knot Floer homology and Seifert surfaces*, Algebraic and Geometric Topology **8** (2008), 603–608. MR2443240

[Let00] Carl F. Letsche, *An obstruction to slicing knots using the eta invariant*, Mathematical Proceedings of the Cambridge Philosophical Society **128** (2000), no. 2, 301–319. MR1735303 (2001b:57017)

[Min99] Yair Minsky, *The classification of punctured-torus groups*, Annals of Mathematics **149** (1999), 559–626. MR1689341 (2000f:30028)

[OS04a] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and genus bounds*, Geometry and Topology **8** (2004), 311–334. MR2023281 (2004m:57024)

[OS04b] ———, *Holomorphic disks and knot invariants*, Advances in Mathematics **186** (2004), no. 1, 58–116. MR2065507 (2005e:57044)

[Ras03] Jacob Rasmussen, *Floer homology and knot complements*, Ph.D. thesis, Harvard University, 2003.

[Suz91] Shinichi Suzuki, *Introduction to knot theory*, Saien-su-sha (in Japanese), 1991.

[Whi73] Wilbur Whitten, *Isotopy types of knot spanning surfaces*, Topology **12** (1973), 373–380. MR0372842 (51:9046)

Department of Mathematics, Rice University–MS 136, P.O. Box 1892, Houston, Texas 7725-1892

Current address: Department of Mathematics, Columbia University, 2990 Broadway, New York, New York 10025

E-mail address: `pdhorn@math.columbia.edu`