ZEROS OF THE EISENSTEIN SERIES $E_2$

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Abstract. In this paper we investigate the zeros of the Eisenstein series $E_2$. In particular, we prove that $E_2$ has infinitely many SL$_2$(Z)-inequivalent zeros in the upper half-plane $\mathcal{H}$, yet none in the standard fundamental $\mathcal{F}$. Furthermore, we go on to investigate other fundamental regions in the upper half-plane $\mathcal{H}$ for which there do or do not exist zeros of $E_2$. We establish infinitely many such regions containing a zero of $E_2$ and infinitely many which do not.

1. Introduction

Let $\mathcal{H} = \{\tau \in \mathbb{C}, \text{Im}(\tau) > 0\}$ be the upper half-plane. The Eisenstein series are defined for every even integer $k \geq 2$ and $\tau \in \mathcal{H}$ by

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$$= 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \frac{n^{k-1} q^n}{1 - q^n}, \quad q = e^{2\pi i \tau}.$$

Here $B_k$ is the $k$-th Bernoulli number and $\sigma_k(n) = \sum_{d|n} d^k$.

These series play an important role in the theory of modular forms and quasi-modular forms. They have been the topic of extensive investigation for a long time from various points of view. For instance, from the analytic point of view, the study of the zeros of $E_k(z)$, $k \geq 4$, has been carried out by several authors. In 1963, K. Wohlfahrt proved in [6] that the zeros of $E_k$, $4 \leq k \leq 26$, are simple and lie in the arc of the unit circle $\{z = e^{i\theta} : \pi/2 \leq \theta \leq 3\pi/2\}$ in the fundamental domain $\mathfrak{F} = \{\tau \in \mathfrak{H}, |\tau| \geq 1 \text{ and } |\text{Re}(\tau)| \leq 1/2\}$ of the modular group SL$_2$(Z). He also conjectured that this holds for all $k \geq 4$. In 1970, F.K.C. Rankin and H.P.F. Swinnerton-Dyer [5] proved Wohlfahrt’s conjecture. In 1982, R.A. Rankin [4] generalized their result to a certain class of Poincaré series. However, nothing has been proven for the Eisenstein series $E_2$, which is important in many fields. In fact, even whether it has a finite or an infinite number of zeros has not been known.

In this paper, we prove that there are infinitely many non-equivalent zeros of $E_2$ in $\mathfrak{F}$. In fact, since $E_2$ is not exactly a modular form but rather a quasi-modular form, two zeros $\tau_0$ and $\tau_1$ of $E_2$ are SL$_2$(Z)-equivalent, that is $\tau_1 = \gamma \cdot \tau_0$ for $\gamma \in \text{SL}_2(\mathbb{Z})$ if and only if $\tau_1 = \tau_0 + n$ for an integer $n$. Thus, we restrict our investigation to the half-strip $\mathfrak{S} = \{\tau \in \mathfrak{H}, -\frac{1}{2} < \text{Re}(\tau) \leq \frac{1}{2}\}$, in which we prove

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that there are infinitely many zeros for $E_2$. Moreover, these zeros present a strange distribution in $\mathfrak{H}$. More precisely, the fundamental domain $\mathfrak{F}$ and infinitely many of its conjugates within $\mathfrak{H}$ contain no zero of $E_2$, while there are infinitely many conjugates of $\mathfrak{F}$ which contain zeros of $E_2$.

2. Eisenstein series: Some properties

The most familiar Eisenstein series are

\begin{align}
E_2(\tau) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n, \\
E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n, \\
E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n.
\end{align}

The series $E_4$ and $E_6$ are, respectively, modular forms of weight 4 and 6. However, the Eisenstein series $E_2$ is not a modular form. In fact, it transforms under the action of the modular group as follows (see [3]).

**Proposition 2.1.** For $\alpha = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$, we have

\begin{equation}
E_2(\alpha \cdot \tau) = (c\tau + d)^2 E_2(\tau) + \frac{6c}{\pi^4} (c\tau + d),
\end{equation}

where

\[ \alpha \cdot \tau = \frac{a\tau + b}{c\tau + d}. \]

This proposition can be proved using the fact that $E_2$ is the logarithmic derivative of the modular discriminant $\Delta = \frac{1}{1728} (E_4^3 - E_6^2)$ of weight 12, the derivation being $\frac{1}{2\pi i} \frac{d}{d\tau}$.

These three functions were especially studied by Ramanujan [2], who proved that they satisfy the following differential equations:

\begin{align}
\frac{1}{2\pi i} \frac{dE_2}{d\tau} &= \frac{1}{12} (E_2^2 - E_4), \\
\frac{1}{2\pi i} \frac{dE_4}{d\tau} &= \frac{1}{3} (E_2 E_4 - E_6), \\
\frac{1}{2\pi i} \frac{dE_6}{d\tau} &= \frac{1}{2} (E_2 E_6 - E_4^2).
\end{align}

Thus the graded ring $\mathbb{C}[E_2, E_4, E_6]$ is closed under the differential operator $\frac{d}{d\tau}$. It is known that the space of all modular forms is exactly the graded ring $\mathbb{C}[E_4, E_6]$. We shall at this stage give some special values of $E_2$ at $i$ and at the cubic root of unity $\rho = \frac{-1 + i\sqrt{3}}{2}$:

\begin{align}
E_2(i) &= \frac{3}{\pi}, \\
E_2(\rho) &= \frac{2\sqrt{3}}{\pi}.
\end{align}

This follows from the transformation formula for $E_2$ together with the appropriate transformations that fix $i$ and $\rho$. 

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Zeros of the Eisenstein series $E_2$

In this section we prove that the series $E_2$ has infinitely many zeros, a fact that has not been known before. Set $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $S_n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ for positive integers $n$.

**Proposition 3.1.** The Eisenstein series $E_2$ has a zero $\tau_0$ on the imaginary axis and a zero $\tau_1$ on the axis $\text{Re}(z) = \frac{1}{2}$.

**Proof.** It is clear that for $\tau = iy$, the series $E_2(\tau)$ is real and increasing on $(0, \infty)$. Meanwhile, $\lim_{y \to 0} E_2(iy) = -\infty$ and $\lim_{y \to \infty} E_2(iy) = 1$. It follows that $E_2$ has a unique zero, say $\tau_0$, on the purely imaginary axis. Similarly, $E_2(\tau)$ is real for $\tau = \frac{1}{2} + iy$, $y > 0$. Furthermore, we have

$$\lim_{y \to \infty} E_2(\frac{1}{2} + iy) = -\infty.$$ 

Indeed, for $\alpha = S^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ we have

$$E_2 \left( \frac{1}{2} + iy \right) = -\frac{1}{y^2} \left( \frac{1}{4} E_2 \left( -\frac{1}{2} + \frac{i}{4y} \right) - \frac{6y}{\pi} \right).$$

This gives the desired limit since $E_2 \left( -\frac{1}{2} + \frac{i}{4y} \right)$ tends to 1 as $y$ tends to 0. Combining this with the fact that $E_2(\rho) = E_2(\rho + 1) = \frac{2\sqrt{3}}{\pi}$ yields the existence of a zero $\tau_1$ of real part $1/2$ and whose imaginary part is less than $\sqrt{3}/2$. Here again we used the transformation formula in Proposition 2.1 with $\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

As for the location of these two zeros, and taking into account the special value of $E_2$ at $i$ and $\rho$ given respectively by (9) and (10), we have

**Proposition 3.2.** The zeros $\tau_0$ and $\tau_1$ are contained respectively in the fundamental domains $S_F$, $S_2F$.

It is worth mentioning that numerical values of these two zeros appear in [1], where they are studied as equilibrium points of Green’s functions.

Unlike the case of modular forms, the set of zeros of $E_2$ is not invariant under every conjugation by elements of $\text{SL}_2(\mathbb{Z})$. In fact we have

**Proposition 3.3.** Two zeros of $E_2$ are equivalent if and only if one is a translate of the other by an integer.

**Proof.** Suppose that $z_1$, $z_2$ are any two zeros of $E_2$ in the half-plane $\mathcal{H}$ that are equivalent modulo $\text{SL}_2(\mathbb{Z})$. Say, $z_1 = \alpha \cdot z_2$, $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then, by the transformation formula for $E_2$ in Proposition 2.1 we have

$$E_2(z_1) = 0 = E_2(\alpha \cdot z_2) = (cz_2 + d)^2 E_2(z_2) + \frac{6c}{\pi i} (cz_2 + d)$$

which is possible only when $c = 0$, and in this case we have $a = d = \pm 1$; that is, $\alpha$ is a translation. The converse follows from the invariance of $E_2$ under translation. □
As a consequence we have

**Corollary 3.4.** No two distinct zeros of $E_2$ in the half-strip $\mathcal{S}$ are equivalent modulo the modular group $SL_2(\mathbb{Z})$.

We now state the main results of this section.

**Theorem 3.5.** The Eisenstein series $E_2$ has infinitely many zeros in the half-strip $\mathcal{S} = \{ \tau \in \mathbb{H}, -\frac{1}{2} < Re(\tau) \leq \frac{1}{2} \}$.

**Proof.** Let $\tau_0$ be the unique zero of $E_2$ on the imaginary axis. Let $\alpha = \left( \begin{array}{ll} t & u \\ v & w \end{array} \right) \in SL_2(\mathbb{Z})$, where $tv \neq 0$. Then, by Equation (5), we have

$$E_2(\tau_0) = 0 = E_2(\alpha^{-1} \cdot \tau_0) = (v(\alpha \cdot \tau_0) + t)^2 E_2(\alpha \cdot \tau_0) - \frac{6v}{\pi i} (v(\alpha \cdot \tau_0) + t).$$

It follows that

$$(-v(\alpha \cdot \tau_0) + t)E_2(\alpha \cdot \tau_0) = \frac{6v}{\pi i},$$

which is equivalent to saying that

$$\frac{E_2(\alpha \cdot \tau_0)}{(\alpha \cdot \tau_0) E_2(\alpha \cdot \tau_0) + \frac{6v}{\pi i}} = \frac{v}{t}.$$ 

This means that the map $f(z)$ defined by

$$f(z) = \frac{E_2(z)}{(z E_2(z) + \frac{6v}{\pi i})}$$

carries $\alpha \cdot \tau_0$ onto $r_0 = v/t$, and thus it maps any open neighborhood $D_0$ of $\alpha \cdot \tau_0$, which we choose in the interior of the fundamental domain $\alpha S\mathcal{F}$ and on which it is holomorphic, onto an open neighborhood $U_0$ of $r_0$. Let $r_1 = a_1/b_1$ be a reduced fraction in $\mathbb{Q} \cap U_0 \setminus \{r_0\}$. Then there exists $z_1 \in D_0 \setminus \{\alpha \cdot \tau_0\}$ such that $f(z_1) = a_1/b_1$. Therefore,

$$(-a_1 z_1 + b_1) E_2(z_1) = \frac{6a_1}{\pi i}. \quad (11)$$

Choose $c_1, d_1 \in \mathbb{Z}$ such that $b_1 d_1 - a_1 c_1 = 1$. Then

$$\gamma_1 := \left( \begin{array}{ll} d_1 & -c_1 \\ -a_1 & b_1 \end{array} \right) \in SL_2(\mathbb{Z}).$$

If we set $\tau_1 = \gamma_1 \cdot z_1$, then, using (6) and (11), we have $E_2(\tau_1) = 0$. Moreover, $\tau_1$ is not equivalent to $\tau_0$ modulo $SL_2(\mathbb{Z})$; otherwise we would have, according to Proposition 3.3, that $\tau_0 := T^n \gamma_1 \cdot z_1$ for some $n \in \mathbb{Z}$ with $T = \left( \begin{array}{ll} 1 & 1 \\ 0 & 1 \end{array} \right)$. Since $z_1 \in \alpha S\mathcal{F}$, write $z_1 = \alpha \cdot z'_1$ for some $z'_1 \in S\mathcal{F}$. We have $\tau_0 = T^n \gamma_1 \alpha \cdot z'_1$ with $\tau_0$ and $z'_1$ being in the fundamental domain $S\mathcal{F}$. Therefore, $T^n \gamma_1 \alpha = 1$, and hence $\tau_0 = z'_1$ and $\alpha \cdot \tau_0 = z_1$, a contradiction since we have chosen $z_1 \in D_0 \setminus \{\alpha \cdot \tau_0\}$. Thus $\tau_1$ is a zero of $E_2$ that is not equivalent to $\tau_0$.

It remains to show that two distinct rational numbers lead to two distinct zeros of $E_2$. Let $r_2 = a_2/b_2$ be a rational number in $U_0 \setminus \{r_0, r_1\}$. In the same way we construct a zero of $E_2$, $\tau_2 = \gamma_2 \cdot z_2$, that is not equivalent to $\tau_0$ modulo $SL_2(\mathbb{Z})$, with $z_2 \in \alpha S\mathcal{F}$. Then $\tau_2$ is not equivalent to $\tau_1$ modulo $SL_2(\mathbb{Z})$. Indeed if $\tau_1 = T^m \cdot \tau_2$ for some $m \in \mathbb{Z}$, then $\gamma_1 \alpha \cdot z'_1 = T^m \gamma_2 \alpha \cdot z'_2$ with $z'_1$ and $z'_2$ being in the same fundamental domain $S\mathcal{F}$. It follows that $\gamma_1 \alpha = T^m \gamma_2 \alpha$, and consequently $r_1 = r_2$. 


This contradicts our choice of \( r_2 \). Hence, \( \tau_2 \) is another zero of \( E_2 \) that is not equivalent to either \( \tau_0 \) or \( \tau_1 \). Finally, since the open set \( U_0 \) contains infinitely many rational numbers, we deduce that \( E_2 \) has infinitely many zeros in the half-strip \( \mathcal{S} \). \( \square \)

Since \( E_2 \) is the logarithmic derivative of the discriminant \( \Delta \), from the above theorem we deduce

**Corollary 3.6.** The discriminant \( \Delta \) has infinitely many critical points.

We now look at the multiplicity of the zeros of \( E_2 \).

**Theorem 3.7.** The zeros of the Eisenstein series \( E_2 \) are all simple.

**Proof.** Let \( z_0 \) be a zero of \( E_2 \). By (12), we have

\[
\frac{1}{2\pi i} \frac{dE_2(z_0)}{d\tau} = \frac{1}{12} \left( E_2(z_0)^2 - E_4(z_0) \right) = \frac{-1}{12} E_4(z_0).
\]

Therefore, to prove that this zero is simple, it suffices to show that \( E_4(z_0) \neq 0 \).

It is known that \( E_4 \) has all its zeros at \( \rho = \frac{1+\sqrt{3}}{2} \) and its conjugates modulo \( SL_2(\mathbb{Z}) \) (see for instance [3]). Thus, it is enough to show that \( E_2(\alpha \cdot \rho) \neq 0 \) for all \( \alpha \in SL_2(\mathbb{Z}) \). Using (3) and (11), we have for \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \):

\[
E_2(\alpha \cdot \rho) = (cp + d)^2 \frac{2\sqrt{3}}{\pi} + \frac{6c}{\pi i} (cp + d) = \frac{2\sqrt{3}}{\pi} (c^2 - cd + d^2),
\]

which does not vanish unless \( c = d = 0 \), which is not the case since \( ad - bc = 1 \). This shows that \( E_2 \) does not vanish on the orbit of \( \rho \) and that consequently \( E_4 \) and \( E_2 \) have no common zeros. \( \square \)

4. DISTRIBUTION OF THE ZEROS OF \( E_2 \)

In this section, we will show that there are infinitely many fundamental regions within the half-strip \( \mathcal{S} \) that contain zeros of \( E_2 \), and we will also show that there are infinitely many such regions that do not contain any zero of \( E_2 \).

**Theorem 4.1.** There is a positive integer \( c_0 \) such that for all integers \( c \geq c_0 \), there is a fundamental domain with a vertex at \( 1/c \) containing a zero of \( E_2 \).

**Proof.** Let \( \tau_0 \) again denote the unique zero of \( E_2 \) on the imaginary axis, and let \( \alpha = \left( \begin{array}{cc} 1 & w \\ 0 & 1 \end{array} \right) \in SL_2(\mathbb{Z}) \), so that \( tv \neq 0 \). As in the proof of Theorem 3.3, the map

\[
f(z) = \frac{E_2(z)}{(zE_2(z) + \frac{6}{\pi})}
\]

maps any neighborhood of \( \alpha \cdot \tau_0 \) onto a neighborhood of \( v/t \). In particular, \( f \) maps a neighborhood \( D_0 \) of \( S_1 \tau_0 \), chosen to be in the interior of \( S_1 \mathcal{S} \), onto a neighborhood \( U_0 \) of 1 (recall that \( S_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)). There exists a positive integer \( c_0 \) such that for all \( c \geq c_0 \), \( 1 + 1/c \in U_0 \). For each \( c \geq c_0 \), let \( z_c \in D_0 \) be such that \( f(z_c) = 1 + 1/c \).

Therefore, if \( \gamma_c = \begin{pmatrix} -1 & -1 \\ c & 1 \end{pmatrix} \), then, as in the proof of Theorem 3.3, \( \gamma_c^{-1} \cdot z_c \) is a zero \( E_2 \) belonging to \( \gamma_c^{-1} S_1 \mathcal{S} \). If we set \( S_c = \gamma_c^{-1} S_1 S = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \) for \( c \geq c_0 \), then we have constructed a zero of \( E_2 \) in the fundamental domain \( S_c \mathcal{S} \) which has a vertex at the cusp \( 1/c \). \( \square \)
Remark 4.1.

- Thanks to Proposition 3.3, the above theorem can be extended to include the cusps 0 and 1/2.
- An immediate consequence of this theorem is again the infiniteness of the number of zeros of the Eisenstein series $E_2$. Furthermore, it follows from Corollary 3.4 that all these zeros are inequivalent modulo $SL_2(\mathbb{Z})$, as all these fundamental domains are contained in the half-strip $S$.

We now focus on the fundamental domains that contain no zeros of $E_2$.

**Proposition 4.2.** The Eisenstein series $E_2$ has no zeros in the fundamental domain $\mathfrak{F}$ of $SL_2(\mathbb{Z})$.

**Proof.** Let $\tau_0 = iy_0$ be the unique zero of $E_2$ on the imaginary axis. Using the transformation formula for $E_2$, we have

$$0 < E_2(-1/\tau_0) = \frac{6}{\pi} y_0 < 1.$$ 

This follows from the fact that $\text{Im}(\tau_0) < 1$ (since $\tau_0 \in S \mathfrak{F}$) and thus $\text{Im}(-1/\tau_0) > \text{Im}(\tau_0)$, and the fact that $E_2$ is strictly increasing on the imaginary axis with the value 0 at $\tau_0$ and the value 1 at $i\infty$. Therefore

$$y_0 < \frac{\pi}{6}.$$ 

If $\tau = x + iy \in \mathfrak{F}$ is a zero of $E_2$, then $\text{Im}(\tau) > \sqrt{3}/2 > \pi/6 > y_0$ and therefore

$$\frac{1}{24} |1 - E_2(\tau)| = \sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi in\tau} \leq \sum_{n=1}^{\infty} \sigma_1(n)e^{-2\pi ny} < \sum_{n=1}^{\infty} \sigma_1(n)e^{-2\pi ny_0}.$$ 

The latter sum is simply $1/24(1 - E_2(\tau_0)) = 1/24$. Therefore

$$\frac{1}{24} |1 - E_2(\tau)| < \frac{1}{24}.$$ 

Hence $E_2(\tau)$ cannot be 0 if $\tau \in \mathfrak{F}$. \quad \square

In the above proof we have used the inequality $\sqrt{3}/2 > \pi/6$, which is obvious numerically but is a consequence of a simpler inequality such as $\pi < 4$. In what follows we will rely on another inequality which is also numerically obvious:

$$e^{-\pi\sqrt{3}} < \frac{1}{200}.$$ 

It simply says that 0.00433 < 0.005.

We will now investigate more fundamental domains that do not contain any zeros of $E_2$. For a fixed integer $c \geq 2$ we again set $S_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$ and $S_{b,d}(c) = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $b, d \in \mathbb{Z}$, and $\delta_b = \begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix} \in SL_2(\mathbb{Z})$, $b \in \mathbb{Z}$. The fundamental domain $S_{b,d}(c) \mathfrak{F}$ has a vertex at the cusp $1/c$, as does $S_c \mathfrak{F}$. Also $\delta_b \mathfrak{F}$ has a vertex at the cusp 0, as does $S_b \mathfrak{F}$.

Let us examine more closely the fundamental domain $S_c \mathfrak{F}$. Its vertices are

$$\frac{1}{c}, \quad S_c \cdot \rho = c - \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad S_c \cdot (\rho + 1) = c + \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$ 

It is clear that $\text{Im}(S_c \cdot \rho) > \text{Im} S_c \cdot (\rho + 1)$ and $\text{Re} S_c \cdot \rho > 1/c > \text{Re} S_c \cdot (\rho + 1)$. Thus
we have the following situation for the fundamental region $S_c$ (see Figure 1).

The edge joining $1/c$ and $S_c \cdot \rho$ is an arc of the circle $C_1(c)$ centered at $c_1(c) = (c - 1)/c(c - 2)$ and having radius $r_1(c) = 1/c(c - 2)$, while the edge joining $1/c$ and $S_c \cdot (\rho + 1)$ is an arc of the circle $C_2(c)$ centered at $c_2(c) = (c + 1)/c(c + 2)$ with radius $r_2(c) = 1/c(c + 2)$. In particular, any other fundamental domain having the cusp $1/c$ as a vertex is either within the circle $C_1(c)$ or within the circle $C_2(c)$.

The case $c = 2$ needs to be clarified, as the radius $r_1(2)$ is infinite and in this case the arc joining $1/2$ and $S_2 \cdot \rho$ is the vertical segment $[1/2, 1/2 + i\sqrt{3}/6]$ (see Figure 2). Moreover, as we are restricting the study to the half-strip $\mathfrak{S}$, we only consider those fundamental domains with vertex at the cusp $1/2$ that lie under the arc of the circle $C_2(2)$. It has center at $c_2(2) = 3/10$ and radius $r_2(2) = 1/10$.

**Lemma 4.3.** If we set

$$M = \frac{1}{24} \left(1 - E_2 \left(\frac{i\sqrt{3}}{2}\right)\right),$$

then we have

$$24^2 \left(M^2 + \frac{M}{\pi}\right) < 1. \quad (14)$$

**Proof.** Set $q = \exp(-\pi\sqrt{3})$. We have

$$0 < M = \sum_{n \geq 1} \sigma_1(n) q^n = \sum_{n \geq 1} \frac{nq^n}{1 - q^n} \leq \frac{1}{1 - q} \sum_{n \geq 1} nq^n = \frac{q}{(1 - q)^2}.$$ 

Hence, using \cite{13}, we have

$$M \leq \frac{40000}{7880599}.$$ 

Therefore,

$$24^2 \left(M^2 + \frac{M}{\pi}\right) < 24^2 \left(M^2 + \frac{M}{3}\right) \leq \frac{61444600320000}{62103840598801} < 1.$$ 

□
In the following, we will prove that the only fundamental domains having a vertex at the cusp $1/c$ that might contain a zero of $E_2$ are the $\beta_c \mathcal{F}$, and the only fundamental domain having a vertex at the cusp $0$ that might contain a zero is $S \mathcal{F}$.

**Theorem 4.4.** If $b \neq 0$, then $E_2$ has no zeros in $S_{b,d}(c) \mathcal{F}$ or in $\delta_b \mathcal{F}$.

**Proof.** Suppose first that $c \geq 3$, and suppose there is a zero $z_0$ of $E_2$ in the fundamental domain $S_{b,d}(c) \mathcal{F}$ where $S_{b,d}(c) = \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$. If $b \neq 0$, then, according to the discussion preceding the above lemma, the fundamental domain $S_{b,d}(c) \mathcal{F}$ is either within the circle $C_1(c)$ or $C_2(c)$. We will show that in fact $z_0$ is outside the circles $C_1(c)$ and $C_2(c)$, which is a contradiction.

We have 

$$E_2(S_{b,d}(c)^{-1} \cdot z_0) = \frac{-6c}{\pi i} (-cz_0 + 1),$$

so that

$$\sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inS_{b,d}(c)^{-1} \cdot z_0} = \frac{1}{24} + \frac{c}{4\pi i} (-cz_0 + 1) = -\frac{c^2}{4\pi i} \left( z_0 - \left( \frac{1}{c} + \frac{\pi i}{6c^2} \right) \right).$$

(15)
Since \( S_{b,d}(c)^{-1} \cdot z_0 \in \mathfrak{F} \), we have
\[
\text{Im} \left( S_{b,d}(c)^{-1} \cdot z_0 \right) \geq \frac{\sqrt{3}}{2}
\]
Hence
\[
\left| \sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi niS_{b,d}^{-1}z_0} \right| \leq \sum_{n=1}^{\infty} \sigma_1(n)e^{-n\pi\sqrt{3}} = M.
\]
Therefore
\[
\left| z_0 - \left( \frac{1}{c} + \frac{\pi i}{6c^2} \right) \right| \leq M \frac{4\pi}{c^2};
\]
that is, \( z_0 \) belongs to the disk \( D_0(c) \) of center \( c_0(c) = \frac{1}{c} + \frac{\pi i}{6c^2} \) and radius \( r_0(c) = \frac{M\pi}{288} \). We will now show that the disk \( D_0(c) \) lies outside the circles \( C_1(c) \) and \( C_2(c) \) by showing respectively that \( |c_0(c) - c_1(c)| > r_1(c) + r_0(c) \) and that \( |c_0(c) - c_2(c)| > r_2(c) + r_0(c) \). Because the cusp \( 1/c \) and \( c_0(c) \) are on the same vertical axis, we have
\[
|c_1(c) - c_0(c)|^2 = r_1(c)^2 + \left( \frac{\pi}{6c^2} \right)^2.
\]
Thus in order to prove that \( |c_0(c) - c_1(c)| > r_0(c) + r_1(c) \) we only need to prove that
\[
\frac{r_0(c)^2 + 2r_0(c)r_1(c)}{\left( \frac{\pi}{6c^2} \right)^2} < \frac{4\pi}{c^2}.
\]
In other words,
\[
2\pi M^2 + \frac{M c}{c - 2} < \frac{\pi}{288}.
\]
In the meantime, for \( c \geq 4 \), we have \( c/(c - 2) = 1 + 2/(c - 2) \leq 2 \). Thus it is enough to prove that \( 2\pi M^2 + 2M < \pi/288 \), which is a consequence of Lemma 4.3.

Similarly, we prove that \( |c_2 - c_0| > r_2 + r_0 \). Indeed, as above, it is enough to show that
\[
2\pi M^2 + \frac{M c}{c + 2} < \frac{\pi}{288},
\]
which is a consequence of Lemma 4.3 since \( c/(c + 2) < 1 \). Notice that \( |c_2 - c_0| > r_2 + r_0 \) is also valid for the cases \( c = 2 \) and \( c = 3 \). This proves the theorem for \( c \geq 4 \) and also for \( c = 2 \) since the circle \( C_1(c) \) is the vertical line \( \text{Re} \; z = 1/2 \), and thus we only need to estimate the distance \( |c_2 - c_0| \).

The case \( c = 3 \) involves different estimates since we cannot apply Lemma 4.3 for the above choice of \( M \). As we noticed above \( z_0 \) is outside the circle \( C_2(3) \), and we only need to show that it is outside \( C_1(3) \). On the other hand, the fundamental domain \( S_{-1,-2}(3)\mathfrak{F} \) is adjacent (on the right) to \( S_3\mathfrak{F} \) (see Figure 3), and the disc \( D_0(3) \) is outside the circle \( C_3 \) which joins the vertices \( 1/3 \) and \( S_{-1,-2}(3)\cdot\rho \). Indeed, this circle is centered at \( 8/21 \) and has radius \( 1/21 \). Moreover
\[
|c_0(3) - 8/21| = \frac{\sqrt{324 + 49\pi^2}}{378} \approx 0.07518,
\]
and
\[
r_0(3) + \frac{1}{21} = \frac{4\pi M}{9} + \frac{1}{21} < \frac{4\pi}{9 \cdot 200} + \frac{1}{21} \approx 0.0546.
\]
It follows that the only possible values of \((b, d)\) for which \( S_{b,d}\mathfrak{F} \) might contain a zero are \((b, d) = (-1, -2)\) leading to \( S_{-1,-2}(3)\mathfrak{F} \) and \((b, d) = (0, 1)\) leading to \( S(3)\mathfrak{F} \). We now show that \( z_0 \notin S_{-1,-2}(3)\mathfrak{F} \) by exhibiting a smaller disc \( D(3) \) containing \( z_0 \).
and lying outside the circle \( C_1(3) \) as the disc \( D_0(3) \) does not necessarily meet this condition. The transformation \( S_{-1,-2} \) maps \( D_0(3) \) onto a disc \( D'_0(3) \) centered at

\[
c_0'(3) = S_{-1,-2}(3)^{-1} \cdot c_0(3) = \frac{6i}{\pi} + \frac{2}{3}
\]

and with radius \( r'_0(3) \) that can easily be shown to satisfy \( r'_0(3) < 0.26 \). Therefore, we obtain a more precise lower bound to \( \text{Im} \ S_{-1,-2}(3) \cdot z_0 \) as compared to (16):

\[
\text{Im} \ S_{-1,-2}(3) \cdot z_0 > \frac{6}{\pi} - 0.26.
\]

We now replace \( M \) in Lemma 4.3 by

\[
M' = \frac{1}{24} (1 - E_2 (i(6/\pi - 0.26)))
\]

and obtain

\[
2\pi M'^2 + 3M' < \pi/288.
\]

Hence, as in the general case, we conclude that

\[
\left| z_0 - \left( \frac{1}{3} + \frac{i\pi}{54} \right) \right| \leq M' \frac{4\pi}{9},
\]

and therefore, the disc \( D(3) = D(1/3 + i\pi/54, 4\pi M'/9) \) is outside the circle \( C_1(3) \). It follows that there is no zero of \( E_2 \) in \( S_{-1,-2}(3) \) and thus in any \( S_{b,a}(3) \) for \( b \neq 0 \).
Finally, for the case of the cusp at 0, if \( z_0 \) is a zero of \( E_2 \) in \( \delta_b \mathfrak{F} \), then \( z_0 \) is contained inside the circle centered at \( \frac{e^{2 \pi i}}{b} \) and having radius \( 4M\pi \) which is clearly contained in \( S \mathfrak{F} \). Therefore \( b = 0 \), since, otherwise, \( \delta_b \mathfrak{F} \) and \( S \mathfrak{F} \) are disjoint. □

References


