PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 7, July 2010, Pages 2513–2519
S 0002-9939(10)10301-3
Article electronically published on March 4, 2010

# UNIFORM ASYMPTOTIC EXPANSIONS OF THE TRICOMI-CARLITZ POLYNOMIALS

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(Communicated by Walter Van Assche)

*Dedicated to the Lord*

Abstract. The Tricomi-Carlitz polynomials satisfy the second-order linear difference equation

$$(n+1)f_{n+1}^{(\alpha)}(x) - (n+\alpha)\,x\,f_n^{(\alpha)}(x) + f_{n-1}^{(\alpha)}(x) = 0, \qquad n \geq 1,$$

with initial values $f_0^{(\alpha)}(x) = 1$ and $f_1^{(\alpha)}(x) = \alpha x$, where $x$ is a real variable and $\alpha$ is a positive parameter. An asymptotic expansion is derived for these polynomials by using the turning-point theory for three-term recurrence relations developed by Wang and Wong [Numer. Math. **91**(2002) and **94**(2003)]. The result holds uniformly in regions containing the critical values $x = \pm 2/\sqrt{\nu}$, where $\nu = n + 2\alpha - 1/2$.

## 1. Introduction

The Tricomi polynomials are defined by

$$t_n^{(\alpha)}(x) = \sum_{k=0}^{n} (-1)^k \binom{x-\alpha}{k} \frac{x^{n-k}}{(n-k)!}, \qquad n = 0, 1, 2, \ldots, \tag{1.1}$$

which satisfy the recurrence relation

$$(n+1)t_{n+1}^{(\alpha)}(x) - (n+\alpha)t_n^{(\alpha)}(x) + x\,t_{n-1}^{(\alpha)}(x) = 0, \qquad n \geq 1, \tag{1.2}$$

with initial values $t_0^{(\alpha)}(x) = 1$, $t_1^{(\alpha)}(x) = \alpha$. Tricomi [7] observed that $\{t_n^{(\alpha)}(x)\}$ is not a system of orthogonal polynomials, since the recurrence relation (1.2) fails to have the required form [6, p.135]. However, Carlitz [1] discovered that if one sets

$$f_n^{(\alpha)}(x) = x^n t_n^{(\alpha)}(x^{-2}), \tag{1.3}$$

then $f_n^{(\alpha)}(x)$ satisfies

$$(n+1)f_{n+1}^{(\alpha)}(x) - (n+\alpha)\,x\,f_n^{(\alpha)}(x) + f_{n-1}^{(\alpha)}(x) = 0, \qquad n \geq 1, \tag{1.4}$$

Received by the editors June 23, 2009, and, in revised form, November 6, 2009.
2010 *Mathematics Subject Classification.* Primary 41A60, 39A10; Secondary 33C45.
*Key words and phrases.* Tricomi-Carlitz polynomials, uniform asymptotic expansion, difference equation.

with initial values $f_0^{(\alpha)}(x) = 1$, $f_1^{(\alpha)}(x) = \alpha x$. Furthermore, he gave the orthogonal relation

$$\int_{-\infty}^{\infty} f_m^{(\alpha)}(x)\, f_n^{(\alpha)}(x)\, d\psi^{(\alpha)}(x) = \frac{2\, e^{\alpha}}{(n+\alpha)n!}\, \delta_{mn}, \tag{1.5}$$

where $\psi^{(\alpha)}(x)$ is the step function whose jumps are

(1.6)

$$d\psi^{(\alpha)}(x) = \frac{(k+\alpha)^{k-1}e^{-k}}{k!} \quad \text{at} \quad x = x_k = \pm(k+\alpha)^{-1/2}, \qquad k = 0, 1, 2, \ldots.$$

The generating function

$$\exp\left\{\frac{w}{x} + \frac{1-\alpha x^2}{x^2}\log(1-wx)\right\} = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)\, w^n \tag{1.7}$$

can be derived from the recurrence relation (1.4) when $x \neq 0$. The series in (1.7) is convergent for $|wx| < 1$.

Asymptotic behavior of $f_n^{(\alpha)}(x)$ has been investigated by Goh and Wimp [2, 3]. In their first paper, they used an elementary approach to show that

$$\frac{f_n^{(\alpha)}(y/\sqrt{\alpha})}{\alpha^{-n/2} y^n n^{-\alpha(1-y^2)/y^2-1}} \to \frac{e^{\alpha/y^2}}{\Gamma(-\alpha(1-y^2)/y^2)}, \qquad \text{as} \quad n \to \infty, \tag{1.8}$$

uniformly for all $y$ in a compact set $K \subseteq \mathbb{C}\backslash[-1,1]$. (There is a minor error in the statement of their result; the factor $\alpha^{-n/2}$ in the denominator on the left-hand side of (1.8) is missing in their equation (23) in [2].) The validity of (1.8) can be verified by a direct application of Darboux's method [10, p. 116]. Goh and Wimp also observed that all zeros of $f_n^{(\alpha)}(y/\sqrt{\alpha})$ lie in the interval $[-1,1]$. In their second paper, they used a saddle-point method to study the asymptotics of $f_n^{(\alpha)}(z/\sqrt{n})$ for $z$ in neighborhoods of $z = \pm i/2$. Note that the scales in their two papers are different; in [2] the scale is $y/\sqrt{\alpha}$, whereas in [3] the scale is $z/\sqrt{n}$. The behavior of $f_n^{(\alpha)}(x)$ as $\alpha \to \infty$ has been studied by López and Temme [4]. Their result is expressed in terms of Hermite polynomials.

The purpose of this paper is to present an asymptotic expansion for $f_n^{(\alpha)}(t/\sqrt{\nu})$, which holds uniformly for $t$ in $[0,\infty)$, where $\nu = n + 2\alpha - 1/2$. In view of the reflection formula $f_n^{(\alpha)}(-x) = (-1)^n f_n^{(\alpha)}(x)$, a corresponding result can be given for $t$ in $(-\infty, 0]$. Note that our result is not covered in the two papers of Goh and Wimp; in fact, it complements those in [2] and [3]. We also point out that the Tricomi-Carlitz polynomials do not satisfy a second-order differential equation; hence the powerful tools developed for differential equations (see, e.g., [5]) are not applicable. Our approach is to use a turning-point theory recently introduced by Wang and Wong [8, 9] for three-term recurrence relations.

## 2. Difference equation

Returning to (1.4), we write

$$f_{n+1}^{(\alpha)}(x) - \frac{n+\alpha}{n+1}\, x\, f_n^{(\alpha)}(x) + \frac{1}{n+1} f_{n-1}^{(\alpha)}(x) = 0 \tag{2.1}$$

and introduce the sequence $\{K_n\}$ defined by

$$(n+1)K_{n+1} = K_{n-1} \tag{2.2}$$

with $K_0 = 1$ and $K_1 = \sqrt{2/\pi}$. Induction shows that

$$K_n = \frac{1}{2^{n/2}\Gamma\left(\frac{1}{2}n+1\right)}. \tag{2.3}$$

With the notation

$$F_n^{(\alpha)}(x) := \frac{f_n^{(\alpha)}(x)}{K_n}, \tag{2.4}$$

equation (2.1) can be put in the canonical form considered in [9],

$$F_{n+1}^{(\alpha)}(x) - (A_n x + B_n)F_n^{(\alpha)}(x) + F_{n-1}^{(\alpha)}(x) = 0 \tag{2.5}$$

with

$$A_n = \frac{n+\alpha}{n+1}\frac{K_n}{K_{n+1}} = \frac{n+\alpha}{\sqrt{2}}\frac{\Gamma\left(\frac{1}{2}(n+1)\right)}{\Gamma\left(\frac{1}{2}n+1\right)} \tag{2.6}$$

and $B_n = 0$. To find an asymptotic expansion for $A_n$, we recall the well-known result [10, p. 47]

$$\frac{\Gamma\left(\frac{1}{2}n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}n+1\right)} \sim \sqrt{\frac{2}{n}}\left[1 - \frac{1}{4n} + \frac{1}{32n^2} + \cdots\right].$$

Thus,

$$A_n \sim n^{1/2}\left[1 + (\alpha - \frac{1}{4})\frac{1}{n} + (\frac{1}{32} - \frac{\alpha}{4})\frac{1}{n^2} + \cdots\right]. \tag{2.7}$$

In terms of the notation

$$A_n \sim n^{-\theta}\sum_{s=0}^{\infty}\frac{\alpha_s}{n^s} \qquad \text{and} \qquad B_n \sim \sum_{s=0}^{\infty}\frac{\beta_s}{n^s}, \tag{2.8}$$

used in [9], we have

$$\theta = -\frac{1}{2}, \quad \alpha_0 = 1, \quad \alpha_1 = \alpha - \frac{1}{4}, \quad \alpha_2 = (\frac{1}{32} - \frac{\alpha}{4}), \cdots \tag{2.9}$$

and $\beta_0 = \beta_1 = \beta_2 = \cdots = 0$. If these expansions are recast in the form

$$A_n \sim \nu^{-\theta}\sum_{s=0}^{\infty}\frac{\alpha'_s}{\nu^s} \qquad \text{and} \qquad B_n \sim \sum_{s=0}^{\infty}\frac{\beta'_s}{\nu^s}, \tag{2.10}$$

where $\nu = n + \tau_0$ and $\tau_0$ is some fixed real number to be determined, it is easily found that

$$\alpha'_0 = 1, \quad \alpha'_1 = (\alpha - \frac{1}{4}) - \frac{\tau_0}{2}, \cdots \tag{2.11}$$

and $\beta'_0 = \beta'_1 = \beta'_2 = \cdots = 0$. To apply the result in [9], we first choose $\tau_0$ so that $\alpha'_1 = 0$. From (2.11), it is obvious that the choice is

$$\tau_0 = 2\alpha - \frac{1}{2}. \tag{2.12}$$

According to equation (2.4) in [9], the characteristic equation is

$$\lambda^2 - t\lambda + 1 = 0, \tag{2.13}$$

where $t$ is the rescaled variable $x = \nu^{-\frac{1}{2}}t$. The two roots of this equation are

$$\lambda_\pm = \frac{1}{2}(t \pm \sqrt{t^2 - 4}). \tag{2.14}$$

The points $t_\pm = \pm 2$ where the two roots coincide are called the *turning points* of equation (2.5). In view of the symmetry relation $F_n^{(\alpha)}(-x) = (-1)^n F_n^{(\alpha)}(x)$, we may restrict ourselves just to the case $0 < t < \infty$.

We now define the function $\zeta(t)$ introduced in [9, (4.10)]. With $t_+ = 2$, $\theta = -\frac{1}{2}$, $\alpha_0' = 1$ and $\beta_0' = 0$, this function is given by

$$\frac{2}{3}[\zeta(t)]^{3/2} := \log\frac{t+\sqrt{t^2-4}}{2} - \frac{1}{t^2}\int_2^t \frac{s^2}{\sqrt{s^2-4}}\,ds, \qquad t \geq 2,$$

and

$$\frac{2}{3}[-\zeta(t)]^{3/2} := t^{-2}\int_t^2 \frac{s^2}{\sqrt{4-s^2}}\,ds - \cos^{-1}\frac{t}{2}, \qquad 0 < t < 2.$$

By direct calculation, one obtains

(2.15)
$$\frac{2}{3}[\zeta(t)]^{3/2} = \log\frac{t+\sqrt{t^2-4}}{2} - \frac{1}{t^2}\left[\frac{1}{2}t\sqrt{t^2-4} + 2\log|t+\sqrt{t^2-4}| - 2\ln 2\right],$$

for $t \geq 2$, and

$$\frac{2}{3}[-\zeta(t)]^{3/2} = \frac{1}{t^2}\left[-2\sin^{-1}\frac{t}{2} + \pi + \frac{t}{2}\sqrt{4-t^2}\right] - \cos^{-1}\frac{t}{2}, \tag{2.16}$$

for $0 < t < 2$. We also define the functions $H_0(\zeta)$ and $\Phi(\zeta)$ introduced in [9, (4.19) and (4.28)]. In the present situation, these functions are given by

$$H_0(\zeta) = -\sqrt{\frac{t^2-4}{4\zeta}} \qquad \text{and} \qquad \Phi(\zeta) = 0, \tag{2.17}$$

where $\zeta$ is the function defined in (2.15) and (2.16). Note that in our special case, $\alpha_1' = \beta_1' = 0$; hence, according to the definition of $\Phi(\zeta)$ given in (4.28) of [9], the second equation in (2.17) holds for $0 < t < \infty$, instead of $t \geq \delta$, $0 < \delta < 2$.

With this preliminary work done, we can now apply the main result in [9] to conclude that there are constants $C_1(x)$ and $C_2(x)$ such that the polynomials $F_n^{(\alpha)}(x)$ in (2.4) can be expressed as

$$F_n^{(\alpha)}(x) = C_1(x)P_n(x) + C_2(x)Q_n(x), \tag{2.18}$$

where, with $x = \nu^{-\frac{1}{2}}t$, $P_n(x)$ and $Q_n(x)$ have the asymptotic expansions

$$P_n(\nu^{-\frac{1}{2}}t) \sim \left(\frac{4\zeta}{t^2-4}\right)^{\frac{1}{4}}\left[\mathrm{Ai}\left(\nu^{\frac{2}{3}}\zeta\right)\sum_{s=0}^{\infty}\frac{\widetilde{A}_s(\zeta)}{\nu^{s-\frac{1}{6}}} + \mathrm{Ai}'\left(\nu^{\frac{2}{3}}\zeta\right)\sum_{s=0}^{\infty}\frac{\widetilde{B}_s(\zeta)}{\nu^{s+\frac{1}{6}}}\right] \tag{2.19}$$

and

$$Q_n(\nu^{-\frac{1}{2}}t) \sim \left(\frac{4\zeta}{t^2-4}\right)^{\frac{1}{4}}\left[\mathrm{Bi}\left(\nu^{\frac{2}{3}}\zeta\right)\sum_{s=0}^{\infty}\frac{\widetilde{A}_s(\zeta)}{\nu^{s-\frac{1}{6}}} + \mathrm{Bi}'\left(\nu^{\frac{2}{3}}\zeta\right)\sum_{s=0}^{\infty}\frac{\widetilde{B}_s(\zeta)}{\nu^{s+\frac{1}{6}}}\right]. \tag{2.20}$$

In (2.19) and (2.20), $\mathrm{Ai}(\cdot)$ and $\mathrm{Bi}(\cdot)$ are the Airy functions, the leading coefficients are given by

$$\widetilde{A}_0(\zeta) = 1 \qquad \text{and} \qquad \widetilde{B}_0(\zeta) = 0, \tag{2.21}$$

and the expansions hold uniformly for $0 \leq t < \infty$.

## 3. Determination of $C_1(x)$ and $C_2(x)$

First we examine the behavior of $f_n^{(\alpha)}(x)$ as $n \to \infty$. By using the classical steepest decent method, one can show that to leading order, we have

$$\begin{aligned} f_n^{(\alpha)}(\frac{t}{\sqrt{\nu}}) \sim & \frac{\nu^{-\frac{n}{2}} e^{\nu\left(\frac{\lambda_+}{t}+(\frac{2}{t^2}-1)\log\lambda_+\right)}}{\sqrt{2\pi\nu}(t^2-4)^{\frac{1}{4}}} \cos(\pi\alpha - \frac{\nu\pi}{t^2}) \\ & + \frac{\nu^{-\frac{n}{2}} e^{\nu\left(\frac{\lambda_-}{t}+(\frac{2}{t^2}-1)\log\lambda_-\right)}}{\sqrt{2\pi\nu}(t^2-4)^{\frac{1}{4}}} 2\sin(\pi\alpha - \frac{\nu\pi}{t^2}), \end{aligned} \tag{3.1}$$

when $t > 2$;

$$\begin{aligned} f_n^{(\alpha)}(\frac{t}{\sqrt{\nu}}) \sim & \frac{\sqrt{2}\nu^{-\frac{n}{2}} e^{\frac{\nu}{2}}}{\sqrt{\pi\nu}(4-t^2)^{\frac{1}{4}}} \sin\left[\nu\left(\frac{1}{2t}\sqrt{4-t^2} + \frac{\pi}{t^2} - \frac{2}{t^2}\sin^{-1}\frac{t}{2} - \cos^{-1}\frac{t}{2}\right)\right. \\ & \left. + \pi\alpha - \frac{\nu\pi}{t^2} + \frac{\pi}{4}\right], \end{aligned} \tag{3.2}$$

when $0 < t < 2$ and

$$f_n^{(\alpha)}(\frac{2}{\sqrt{\nu}}) \sim \nu^{-\frac{n}{2}} e^{\frac{\nu}{2}} 3^{\frac{1}{3}} \nu^{-\frac{1}{3}} \left(\frac{1}{3\Gamma(\frac{2}{3})}\cos(\pi\alpha - \frac{\nu\pi}{4}) + \frac{\sqrt{3}}{3\Gamma(\frac{2}{3})}\sin(\pi\alpha - \frac{\nu\pi}{4})\right), \tag{3.3}$$

when $t = 2$. Next we recall the well-known asymptotic formulas

$$\mathrm{Ai}(\eta) \sim \frac{\eta^{-\frac{1}{4}}}{2\sqrt{\pi}} \exp(-\frac{2}{3}\eta^{\frac{3}{2}}) \tag{3.4}$$

and

$$\mathrm{Bi}(\eta) \sim \frac{\eta^{-\frac{1}{4}}}{\sqrt{\pi}} \exp(\frac{2}{3}\eta^{\frac{3}{2}}) \tag{3.5}$$

as $\eta \to +\infty$; see [5, p. 392]. Furthermore, since $\nu = n + 2\alpha - 1/2$, it is readily seen from (3.4) and (3.5) that

$$\frac{P_n(\nu^{-\frac{1}{2}}t)}{2^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)} \sim \frac{\nu^{-\frac{n}{2}}\nu^{-\frac{1}{2}}}{\sqrt{2}\,\pi(t^2-4)^{\frac{1}{4}}} e^{\nu\left(\frac{\lambda_+}{t}+(\frac{2}{t^2}-1)\log\lambda_+\right)},$$

$$\frac{Q_n(\nu^{-\frac{1}{2}}t)}{2^{\frac{n}{2}}\Gamma(\frac{n}{2}+1)} \sim \frac{\sqrt{2}\nu^{-\frac{n}{2}}\nu^{-\frac{1}{2}}}{\pi(t^2-4)^{\frac{1}{4}}} e^{\nu\left(\frac{\lambda_-}{t}+(\frac{2}{t^2}-1)\log\lambda_-\right)}.$$

Comparing the two sides of (2.18), one concludes that

$$C_1(x) = \sqrt{\pi}\cos(\pi\alpha - \frac{\pi}{x^2}), \qquad C_2(x) = \sqrt{\pi}\sin(\pi\alpha - \frac{\pi}{x^2}). \tag{3.6}$$

Note that in obtaining (3.6), any one of the three formulas (3.1), (3.2) or (3.3) could have been used. In summary, we have from (2.4), (2.18), (2.19) and (2.20)

$$\begin{aligned} f_n^{(\alpha)}(\frac{t}{\sqrt{\nu}}) = & \frac{\sqrt{\pi}\cos(\pi\alpha - \frac{\nu\pi}{t^2})}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)} \left(\frac{4\zeta}{t^2-4}\right)^{\frac{1}{4}} \nu^{\frac{1}{6}} \left[\mathrm{Ai}\left(\nu^{\frac{2}{3}}\zeta\right) + O(\frac{1}{\nu})\right] \\ & + \frac{\sqrt{\pi}\sin(\pi\alpha - \frac{\nu\pi}{t^2})}{2^{\frac{1}{2}n}\Gamma(\frac{1}{2}n+1)} \left(\frac{4\zeta}{t^2-4}\right)^{\frac{1}{4}} \nu^{\frac{1}{6}} \left[\mathrm{Bi}\left(\nu^{\frac{2}{3}}\zeta\right) + O(\frac{1}{\nu})\right], \end{aligned} \tag{3.7}$$

where $x = \nu^{-\frac{1}{2}} t$ and $\nu = n + 2\alpha - 1/2$. This result holds uniformly for $0 \le t < \infty$. As a check on the validity of (3.7), we note that

$$f_{2n}^{(\alpha)}(0) = \frac{(-1)^n}{2^n n!}, \quad f_{2n+1}^{(\alpha)}(0) = 0, \quad n = 0, 1, 2, \ldots. \tag{3.8}$$

From (3.7), we obtain

$$f_n^{(\alpha)}(0) \sim \frac{e^{n/2}}{\sqrt{\pi} n^{(n+1)/2}} \sin(-\frac{n\pi}{2} + \frac{\pi}{2}). \tag{3.9}$$

In view of Stirling's formula, (3.9) agrees with (3.8).

## 4. Numerical results

The expansion (3.7) is particularly useful near $t = 2$, where the two characteristic roots in (2.14) coincide. As an illustration, we take $\alpha = 1.9$ and $n = 100$. Table 1 provides exact and approximate values of $2^{n/2}\Gamma(n/2 + 1) f_n^{(\alpha)}(t/\sqrt{\nu})$. The last column of the table shows the percentage error of the approximations.

Table 1. Numerical Results

| t | Exact values | Approximate values | Error |
|---|---|---|---|
| 1.0 | $-7.46841313\times10^{-1}$ | $-7.37169075\times10^{-1}$ | 0.0130 |
| 1.2 | $6.30560302\times10^{-1}$ | $6.34644034\times10^{-1}$ | 0.0065 |
| 1.4 | $8.46491439\times10^{-1}$ | $8.31094179\times10^{-1}$ | 0.0182 |
| 1.6 | -1.30620957 | -1.29091632 | 0.0117 |
| 1.8 | 1.51877275 | 1.49426911 | 0.0161 |
| 2.0 | 1.90931070 | 1.89864389 | 0.0056 |
| 2.2 | $3.26791053\times10^{2}$ | $3.16621347\times10^{2}$ | 0.0311 |
| 2.4 | $-9.21372780\times10^{4}$ | $-8.83780892\times10^{4}$ | 0.0408 |
| 2.6 | $3.94171949\times10^{9}$ | $3.74229636\times10^{9}$ | 0.0506 |
| 2.8 | $1.54972056\times10^{13}$ | $1.45539824\times10^{13}$ | 0.0609 |
| 3.0 | $7.21884326\times10^{16}$ | $6.70106781\times10^{16}$ | 0.0717 |

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