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ALGEBRAIC CYCLES OF A FIXED DEGREE

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ABSTRACT. In this paper, the homotopy groups of Chow variety $C_{p,d}(\mathbb{P}^n)$ of effective p-cycles of degree d are proved to be stable in the sense that p or n increases. We also obtain a negative answer to a question by Lawson and Michelsohn on homotopy groups for the space of degree two cycles.

1. Introduction

Let $\mathbb{P}^n$ be the complex projective space of dimension n and let $C_{p,d}(\mathbb{P}^n)$ be the space of effective algebraic p-cycles of degree d on $\mathbb{P}^n$. A fact proved by Chow and Van der Waerden is that $C_{p,d}(\mathbb{P}^n)$ carries the structure of a closed complex algebraic variety [CW]. Hence it carries the structure of a compact Hausdorff space.

Consider the spaces

$$\mathcal{D}(d) := \lim_{p,q\to\infty} C_{p,d}(\mathbb{P}^{p+q})$$

of cycles of a fixed degree (with arbitrary dimension and codimension), as introduced in [LM], where the limit for p is given by suspension $\Sigma : C_{p,d}(\mathbb{P}^n) \to C_{p+1,d}(\mathbb{P}^{n+1})$ and the limit for q is induced by the inclusion $\mathbb{P}^{p+q} \subset \mathbb{P}^{p+q+1}$; i.e., a p-cycle in $C_{p,d}(\mathbb{P}^{p+q})$ is viewed as a p-cycle in $C_{p,d}(\mathbb{P}^{p+q+1})$.

Then there is a filtration (cf. [LM, §7], [L1])

$$BU = \mathcal{D}(1) \subset \mathcal{D}(2) \subset \cdots \subset \mathcal{D}(\infty) = K(even, \mathbb{Z}),$$

where $BU = \lim_{q\to\infty} BU_q$ and $K(even,\mathbb{Z}) = \prod_{i=1}^{\infty} K(2i,\mathbb{Z})$ (the weak product of Eilenberg-MacLane spaces).

The inclusion map $\mathcal{D}(d) \subset \mathcal{D}(\infty)$ induces maps on homology and homotopy groups. It was proved in [LM] that $\mathcal{D}(1) \subset \mathcal{D}(\infty)$ induces an injective map on homotopy groups. Moreover, as abstract groups $\pi_*(\mathcal{D}(1)) \cong \pi_*(\mathcal{D}(\infty))$.

The following natural question was proposed in [LM]:

Question 1.1. Is $\pi_*(\mathcal{D}(d)) \to \pi_*(\mathcal{D}(\infty))$ injective for $d \geq 1$?

An affirmative answer to Question 1.1 implies that $\pi_*(\mathcal{D}(d)) \cong \pi_*(\mathcal{D}(\infty))$ as abstract groups.

The first main result in this paper is the following negative answer to Question 1.1 for $d = 2$.

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Theorem 1.2. *There is an integer $k > 0$ such that the induced map $\pi_k(\mathcal{D}(2)) \to \pi_k(\mathcal{D}(\infty))$ from the inclusion $\mathcal{D}(2) \subset \mathcal{D}(\infty)$ is not injective. Moreover, there is an integer $k > 0$ such that, as abstract groups, $\pi_k(\mathcal{D}(2)) \ncong \pi_k(\mathcal{D}(\infty))$.*

The proof of this theorem is based on Theorem 1.3 below and direct calculations under the assumption of a positive answer to Question 1.1.

The second main result is the following:

Theorem 1.3. $\pi_k(\mathcal{D}(d)) \cong \pi_k(C_{p,d}(\mathbb{P}^n))$ *for* $k \le min\{2p+1, 2(n-p)\}$.

The proof of Theorem 1.3 is given in section 3.

2. Homology groups of the space of algebraic cycles with degree two

Note that $C_{p,1}(\mathbb{P}^n)$ is the Grassmannian $G(p+1, n+1)$ of linear p-spaces in $\mathbb{P}^n$. Furthermore,

$$C_{p,2}(\mathbb{P}^n) = \mathrm{SP}^2(G(p+1, n+1)) \cup Q_{p,n}, \tag{2.1}$$

where $\mathrm{SP}^i(X)$ denotes the i-th symmetric product X and $Q_{p,n}$ consists of effective irreducible p-cycles of degree 2 in $\mathbb{P}^n$. Every degree 2 variety of dimension p in $\mathbb{P}^n$ is contained as a hypersurface in a linear space of dimension $p+1$ (cf. [GH, pp. 173-4]). Hence $Q_{p,n}$ is a fiber bundle over the Grassmannian $G(p+2, n+1)$ with fiber S, the space of all smooth quadrics in $\mathbb{P}^{p+1}$. Note that S is isomorphic to $\mathbb{P}^{\binom{p+3}{2}-1} - \mathrm{SP}^2(\mathbb{P}^{p+1})$, i.e., the complement of non-irreducible quadrics (which is a pair of p-planes) in the space of all quadric hypersurfaces in $\mathbb{P}^{p+1}$.

To prove Theorem 1.2, we assume that the answer to Question 1.1 is affirmative for $d = 2$. Then $\pi_{2k}(\mathcal{D}(2))$ is a subgroup of $\mathbb{Z}$ and so $\pi_{2k}(\mathcal{D}(2)) \cong 0$ or $\mathbb{Z}$. Note that the map $\pi_{2k}(\mathcal{D}(1)) \to \pi_{2k}(\mathcal{D}(\infty)) \cong \mathbb{Z}$ is multiplication by $(k-1)!$ (cf. [LM], Theorem 4.4) and it factors through $\pi_{2k}(\mathcal{D}(2))$. So $\pi_{2k}(\mathcal{D}(2))$ is nontrivial and $\pi_{2k}(\mathcal{D}(2)) \cong \mathbb{Z}$ for all k if Question 1.1 has a positive answer. Similarly, $\pi_{2k-1}(\mathcal{D}(2)) = 0$ by assuming a positive answer to Question 1.1.

By Theorem 1.3, we have

$$\pi_k(C_{p,2}(\mathbb{P}^n)) \cong \pi_k(\mathcal{D}(2)) = \begin{cases} \mathbb{Z}, & k \le 2p+1 \text{ and even}, \\ 0, & k \le 2p+1 \text{ and odd}. \end{cases}$$

In the following computation, we take $p = 4, d = 2$ as our example.

Lemma 2.1. *Let $X \to B$ be a fibration between CW complexes with fiber F. Suppose that B is simply connected, $H_{2i}(F, \mathbb{Q})$ is finite dimensional, and $H_{2i-1}(B, \mathbb{Q})$ and $H_{2i-1}(F, \mathbb{Q})$ vanish. Then $H_k(X, \mathbb{Q}) \cong \bigoplus_{i+j=k} H_i(B, \mathbb{Q}) \otimes H_j(F, \mathbb{Q})$; that is, the Leray spectral sequence degenerates at E^2.*

Proof. By Leray's Theorem for singular homology, we get the E^2 term

$$E^2_{p,q} = H_p(B, H_q(F, \mathbb{Q})) \cong H_p(B, \mathbb{Q}) \otimes H_q(F, \mathbb{Q}), \quad d^2 : E^2_{p,q} \to E^2_{p-2,q+1}$$

since B is simply connected.

From the assumption, all odd dimensional homology groups of B and F vanish, so at least one of $E^2_{p,q}$ and $E^2_{p-3,q+2}$ vanishes. This implies that d^2 is a zero map. Hence we get $E^3_{p,q} = E^2_{p,q}$ and $d^3 : E^2_{p,q} \to E^2_{p-3,q+2}$. By the same reason, $d^3 = d^4 = \cdots = 0$. Therefore, the Leray spectral sequence degenerates at E^2, i.e.,

$$\bigoplus_{p+q=k} H_p(B, \mathbb{Q}) \otimes H_q(F, \mathbb{Q}) \cong \bigoplus_{p+q=k} E^2_{p,q} = \bigoplus_{p+q=k} E^\infty = H_k(X, \mathbb{Q}). \qquad \square$$

Proposition 2.2. *Let X be a connected CW complex such that*

$$\pi_k(X) = \begin{cases} \mathbb{Z}, & 0 < k \leq 9 \text{ and even;} \\ 0, & k \leq 9 \text{ and odd.} \end{cases}$$

Then the first 10 Betti numbers $\beta_i(X)$ of X are

$$\beta_i(X) = \begin{cases} 1,1,2,3,5, & \text{for } i = 0,2,4,6,8, \\ 0, & \text{for } i = 1,3,5,7,9. \end{cases} \tag{2.2}$$

Proof. Let $\cdots \to Y_n \to Y_{n-1} \to \cdots \to Y_1 = K(\pi_1(X), 1)$ be the Postnikov approximation of X, where $Y_n \to Y_{n-1}$ is a fibration with $K(\pi_n(X), n)$ as fibers (cf. [W], Chapter IX). For a fixed n, we have isomorphisms of homotopy groups $\pi_q(X) \cong \pi_q(Y_n)$ and homology groups $H_q(X, \mathbb{Q}) \cong H_q(Y_n, \mathbb{Q})$ for $1 \leq q \leq n$. Therefore, the first 10 Betti numbers of X coincide with those of Y_9.

Note that Y_2 is homotopy equivalent (denoted by $\simeq$) to $K(\mathbb{Z}, 2)$ since Y_1 is contractible. Since $Y_3 \to Y_2$ is a fibration with $K(\pi_3(X), 3) \simeq *$ as fibers, we get $Y_3 \simeq Y_2$. Note that $Y_4 \to Y_3$ is a fibration with $K(\pi_4(X), 4) = K(\mathbb{Z}, 4)$ as fibers, we obtain $H_k(X, \mathbb{Q}) \cong \bigoplus_{i+j=k} H_i(Y_3, \mathbb{Q}) \otimes H_j(K(\mathbb{Z}, 4), \mathbb{Q})$ by Lemma 2.1. Using Lemma 2.1 for several times, we get (modulo $H_*(-, \mathbb{Q})$ for $* > 9$)

$$\begin{aligned} H_*(X, \mathbb{Q}) &\cong H_*(Y_9, \mathbb{Q}) \\ &\cong H_*(Y_8, \mathbb{Q}) \ (\text{since } Y_9 \simeq Y_8) \\ &\cong H_*(Y_7, \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}) \ (\text{since } K(\mathbb{Z}, 8) \to Y_8 \to Y_7 \text{ is a fibration}) \\ &\cong H_*(Y_6, \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}) \\ &\cong H_*(Y_4, \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 6), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}) \\ &\cong H_*(Y_2, \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 4), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 6), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}) \\ &\cong H_*(K(\mathbb{Z}, 2), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 4), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 6), \mathbb{Q}) \otimes H_*(K(\mathbb{Z}, 8), \mathbb{Q}). \end{aligned}$$

Therefore, the first 10 Betti numbers $\beta_i(X)$ of X are given as follows:

$$\beta_i(X) = \begin{cases} 1,1,2,3,5, & \text{for } i = 0,2,4,6,8; \\ 0, & \text{for } i = 1,3,5,7,9. \end{cases}$$ □

The combination of Theorem 1.3 and Proposition 2.2 implies the following result:

Corollary 2.3. *If the answer to Question* 1.1 *is affirmative for $d = 2$, then the first 10 Betti numbers of $C_{4,2}(\mathbb{P}^n)$ $(n \geq 9)$ are given by*

$$\beta_i(C_{4,2}(\mathbb{P}^n)) = \begin{cases} 1,1,2,3,5, & \text{for } i = 0,2,4,6,8; \\ 0, & \text{for } i = 1,3,5,7,9. \end{cases}$$

The proof of Proposition 2.2 actually shows the following result:

Remark 2.4. Let M be a connected CW complex such that $\pi_k(X) = 0$ for k odd and $\pi_k(M) \cong \mathbb{Z}$ for k positive even integers. Then

$$\operatorname{rank}(H_k(M)) = \begin{cases} p(k), & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd,} \end{cases}$$

where $p(k)$ represents the number of all possible partitions of k.

Examples of such a CW complex M include the infinite product $\prod_{i=1}^{\infty} K(\mathbb{Z}, 2i)$ (with the weak topology) of Eilenberg-MacLane spaces and $BU = \lim_{q \to \infty} BU_q$. Although the homotopy types of these topological spaces are different, their corresponding Betti numbers coincide.

Now we will compute Betti numbers of $C_{4,2}(\mathbb{P}^n)$ $(n \geq 9)$ in a different way. Since $C_{p,2}(\mathbb{P}^n) - \mathrm{SP}^2(G(p+1, n+1)) = Q_{p,n}$, we have $H_i(C_{p,2}(\mathbb{P}^n), \mathrm{SP}^2(G(p+1, n+1)) \cong H_i^{BM}(Q_{p,n})$ for all i, where H_i^{BM} denotes the Borel-Moore homology. Let $A_{p,n}$ be the fiber bundle over $G(p+2, n+1)$ whose fiber is the space of all quadric hypersurfaces in $\mathbb{P}^{p+1}$ and let $B_{p,n}$ be the fiber bundle over $G(p+2, n+1)$ whose fiber is the space of pairs of hyperplanes in $\mathbb{P}^{p+1}$. From the definition of $Q_{p,n}$, we have $H_i(A_{p,n}, B_{p,n}) \cong H_i^{BM}(Q_{p,n})$ for all i. In particular,

$$H_i(C_{4,2}(\mathbb{P}^n), \mathrm{SP}^2(G(5, n+1)) \cong H_i(A_{4,n}, B_{4,n}) \tag{2.3}$$

for $i \geq 0$ and $n \geq 9$.

Lemma 2.5. *Let $A_{4,n}, B_{4,n}$ be defined as above,*

$$\beta_i(\mathrm{SP}^2(G(5, n+1))) = \begin{cases} 1, 1, 3, 5, 11, & \textit{for } i = 0, 2, 4, 6, 8; \\ 0, & \textit{for } i \textit{ odd;} \end{cases}$$

$$\beta_i(A_{4,n}) = \begin{cases} 1, 2, 4, 7, 12, & \textit{for } i = 0, 2, 4, 6, 8; \\ 0, & \textit{for } i \textit{ odd;} \end{cases}$$

$$\beta_i(B_{4,n}) = \begin{cases} 1, 2, 5, 9, 17, & \textit{for } i = 0, 2, 4, 6, 8; \\ 0, & \textit{for } i \textit{ odd.} \end{cases}$$

Proof. To show the first formula, we note that all the odd Betti numbers of $G(5, n+1)$ are zero and the first five even Betti numbers of $G(5, n+1)$ are given by

$$\beta_i(G(5, n+1)) = 1, 1, 2, 3, 5 \text{ for } i = 0, 2, 4, 6, 8.$$

Therefore all the odd Betti numbers of $\mathrm{SP}^2(G(5, n+1))$ vanish and the first five even Betti numbers of $\mathrm{SP}^2(G(5, n+1))$ are given by (a special case of MacDonald's formula [M])

$$\beta_i(\mathrm{SP}^2(G(5, n+1))) = 1, 1, 3, 5, 11 \text{ for } i = 0, 2, 4, 6, 8.$$

To show the second formula, we note that $A_{4,n}$ is a fiber bundle over $G(6, n+1)$ with fibers the space of all quadric hypersurfaces in $\mathbb{P}^5$; i.e., fibers are isomorphic to $\mathbb{P}^{20}$. By Lemma 2.1, all the odd Betti numbers of $A_{4,n}$ vanish since both $G(6, n+1)$ and $\mathbb{P}^{20}$ only carry non-vanishing even Betti numbers. Again, by Lemma 2.1,

$$\beta_{2k}(A_{4,n}) = \bigoplus_{i+j=k} \beta_{2i}(G(6, n+1)) \cdot \beta_{2j}(\mathbb{P}^{20}). \tag{2.4}$$

The first five even Betti numbers of $G(6, n+1)$ are given by

$$\beta_i(G(6, n+1)) = 1, 1, 2, 3, 5 \text{ for } i = 0, 2, 4, 6, 8.$$

Hence from equation (2.4), we get the first five even Betti numbers of $\widetilde{Y}$:

$$\beta_i(A_{4,n}) = 1, 2, 4, 7, 12 \text{ for } i = 0, 2, 4, 6, 8. \tag{2.5}$$

To show the third formula, we note that $B_{4,n}$ is a fiber bundle over $G(6, n+1)$ with fibers the space of pairs of hyperplanes in $\mathbb{P}^{p+1}$; i.e., fibers are isomorphic to $\mathrm{SP}^2(\mathbb{P}^5)$. By Lemma 2.1, all the odd Betti numbers of $B_{4,n}$ vanish and the even Betti numbers of $B_{4,n}$ are given by the formula

$$\beta_{2k}(B_{4,n}) = \bigoplus_{i+j=k} \beta_{2i}(G(6, n+1)) \cdot \beta_{2j}(\mathrm{SP}^2(\mathbb{P}^5)). \tag{2.6}$$

The first five Betti numbers of $\mathrm{SP}^2(\mathbb{P}^5)$ are given as follows (cf. [M]):

$$\beta_i(\mathrm{SP}^2(\mathbb{P}^5)) = 1, 1, 2, 2, 3 \text{ for } i = 0, 2, 4, 6, 8.$$

Therefore the five Betti numbers of Z are given by the formula

$$\beta_i(B_{4,n}) = 1, 2, 5, 9, 17 \text{ for } i = 0, 2, 4, 6, 8. \tag{2.7}$$

□

Proposition 2.6. *The relations among the first 10 Betti numbers of $C_{4,2}(\mathbb{P}^n)$ $(n \geq 9)$ are given as follows:*

$$\beta_{2i}(C_{4,2}(\mathbb{P}^n)) - \beta_{2i+1}(C_{p,2}(\mathbb{P}^n)) = 1, 1, 2, 3, 6 \quad \textit{for } i = 0, 1, 2, 3, 4.$$

In particular, $\beta_8(C_{4,2}(\mathbb{P}^n)) \geq 6$.

Proof. Set $M = C_{4,2}(\mathbb{P}^n)$ and $X = \mathrm{SP}^2 G(5, n+1)$. From the long exact sequence of homology groups for the pair (M, X), we have

$$\cdots \to H_i(X) \to H_i(M) \to H_i(M, X) \to H_{i-1}(X) \to \cdots . \tag{2.8}$$

Since $H_{2i-1}(X) = 0$ for all i, equation (2.8) breaks into exact sequences

$$0 \to H_{2i+1}(M) \to H_{2i+1}(M, X) \to H_{2i}(X) \to H_{2i}(M) \to H_{2i}(M, X) \to 0. \tag{2.9}$$

Set $Y = A_{4,n}$ and $Z = B_{4,n}$. From the long exact sequence of homology groups for the pair (Y, Z), we have

$$\cdots \to H_i(Z) \to H_i(Y) \to H_i(Y, Z) \to H_{i-1}(Z) \to \cdots . \tag{2.10}$$

Since $H_{2i-1}(Y) = 0$ and $H_{2i-1}(Z) = 0$ for all i, equation (2.10) breaks into exact sequences

$$0 \to H_{2i+1}(Y, Z) \to H_{2i}(Z) \to H_{2i}(Y) \to H_{2i}(Y, Z) \to 0. \tag{2.11}$$

From equations (2.3), (2.9) and (2.11), we have

$$\beta_{2i+1}(M) - \beta_{2i}(Z) + \beta_{2i}(Y) + \beta_{2i}(X) - \beta_{2i}(M) = 0$$

i.e.,

(2.12)

$$\beta_{2i+1}(C_{4,2}(\mathbb{P}^n)) - \beta_{2i}(B_{4,n}) + \beta_{2i}(A_{4,n}) + \beta_{2i}(\mathrm{SP}^2 G(5, n+1)) - \beta_{2i}(C_{4,2}(\mathbb{P}^n)) = 0.$$

Now the proposition follows from equation (2.12) and Lemma 2.5. □

The contradiction between Corollary 2.3 and Proposition 2.6 comes from the assumption that the answer to Question 1.1 for $d = 2$ is affirmative. Therefore the answer to Question 1.1 for $d = 2$ is negative; i.e., the induced map $\pi_*(\mathcal{D}(d)) \to \pi_*(\mathcal{D}(\infty))$ by inclusion is not always injective for $d = 2$. This completes the proof of Theorem 1.2.

Remark 2.7. We actually used the assumption that $\pi_*(\mathcal{D}(2)) \cong \pi_*(\mathcal{D}(\infty))$ are isomorphisms as abstract groups for $k \leq 9$ in the proof of Theorem 1.2. Hence $\pi_*(\mathcal{D}(2))$ is not isomorphic to $\pi_*(\mathcal{D}(\infty))$ for all $*$ as abstract abelian groups.

3. Proof of Theorem 1.3

In this section we will prove Theorem 1.3. The method comes from Lawson's proof of the Complex Suspension Theorem [L1]; i.e., the complex suspension to the space of p-cycles yields a homotopy equivalence to the space of $(p+1)$-cycles. Here we briefly review the general construction of such a homotopy equivalence. For details, the reader is referred to [L1], [L2] and [F].

Fix a hyperplane $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ and a point $\mathbb{P}^0 \in \mathbb{P}^{n+1} - \mathbb{P}^n$. For any non-negative integer p and d, set

$$T_{p+1,d}(\mathbb{P}^{n+1}) := \left\{c = \sum n_i V_i \in C_{p+1,d}(\mathbb{P}^{n+1}) | \dim(V_i \cap \mathbb{P}^n) = p,\ \forall i\right\}$$

(when $d = 0$, $C_{p,0}(\mathbb{P}^n)$ is defined to be the empty cycle).

Proposition 3.1 ([L1])**.** *The set $T_{p+1,d}(\mathbb{P}^{n+1})$ is Zariski open in $C_{p+1,d}(\mathbb{P}^{n+1})$. Moreover, $T_{p+1,d}(\mathbb{P}^{n+1})$ is homotopy equivalent to $C_{p,d}(\mathbb{P}^n)$. In particular, their corresponding homotopy groups are isomorphic, i.e.,*

$$\pi_*(T_{p+1,d}(\mathbb{P}^{n+1})) \cong \pi_*(C_{p,d}(\mathbb{P}^n)). \tag{3.1}$$

Fix a linear embedding $\mathbb{P}^{n+1} \subset \mathbb{P}^{n+2}$ and two points $x_0, x_1 \in \mathbb{P}^{n+2} - \mathbb{P}^{n+1}$. Each projection $p_i : \mathbb{P}^{n+2} - \{x_0\} \to \mathbb{P}^{n+1}$ $(i = 0, 1)$ gives us a holomorphic line bundle over $\mathbb{P}^{n+1}$.

Let $D \in C_{n+1,e}(\mathbb{P}^{n+2})$ be an effective divisor of degree e in $\mathbb{P}^{n+2}$ such that x_0, x_1 are not in D. Any effective cycle $c \in C_{p+1,d}(\mathbb{P}^{n+1})$ can be lifted to a cycle with support in D, defined as follows:

$$\Psi_D(c) = (\Sigma_{x_0} c) \cdot D.$$

The map $\Psi(c, D) := \Psi_D$ is a continuous map in the variables c and D. Hence we have a continuous map $\Phi_D : C_{p+1,d}(\mathbb{P}^{n+1}) \to C_{p+1,de}(\mathbb{P}^{n+2} - \{x_0, x_1\})$. The composition of Φ_D with the projection $(p_0)_*$ is $(p_0)_* \circ \Phi_D = e$ (multiplication by the integer e in the monoid, $e \cdot c = c + \cdots + c$ for e times). The composition of Φ_D with the projection $(p_1)_*$ gives us a transformation of cycles in $\mathbb{P}^{n+1}$ which makes most of them intersect properly to $\mathbb{P}^n$. To see this, we consider the family of divisors tD, $0 \le t \le 1$, given by scalar multiplication by t in the line bundle $p_0 : \mathbb{P}^{n+2} - \{x_0\} \to \mathbb{P}^{n+1}$.

Assume x_1 is not in tD for all t. Then the above construction gives us a family transformation

$$F_{tD} := (p_1)_* \circ \Psi_{tD} : C_{p+1,d}(\mathbb{P}^{n+1}) \to C_{p+1,de}(\mathbb{P}^{n+1})$$

for $0 \le t \le 1$. Note that $F_{0D} \equiv d$(multiplication by d).

The question is for a fixed c, which divisors $D \in C_{n+1,e}(\mathbb{P}^{n+2})$ (x_0 is not in D and x_1 is not in $\bigcup_{0 \le t \le 1} tD$) have the property that

$$F_{tD}(c) \in T_{p+1,de}(\mathbb{P}^{n+1})$$

for all $0 < t \le 1$.

Set $B_c := \{D \in C_{n+1,e}(\mathbb{P}^{n+2}) | F_{tD}(c)$ is not in $T_{p+1,de}(\mathbb{P}^{n+1})$ for some $0 < t \le 1\}$, i.e., all degree e divisors on $\mathbb{P}^{n+2}$ such that some component of

$$(p_1)_* \circ \Psi_{tD}(c) \subset \mathbb{P}^n$$

for some $t > 0$.

Proposition 3.2 ([L1])**.** *For $c \in C_{p+1,d}(\mathbb{P}^{n+1})$, $\operatorname{codim}_{\mathbb{C}} B_c \ge \binom{p+e+1}{e}$.*

In this construction, if we take $e = 1$, then F_{tD} maps $C_{p+1,d}(\mathbb{P}^{n+1})$ to itself, i.e.,

$$F_{tD} := (p_1)_* \circ \Psi_{tD} : C_{p+1,d}(\mathbb{P}^{n+1}) \to C_{p+1,d}(\mathbb{P}^{n+1}).$$

Moreover, the image of F_{tD} is in the Zariski open subset $T_{p+1,d}(\mathbb{P}^{n+1})$ if D is not B_c. We can find such a D if $\mathrm{codim}_{\mathbb{C}} B_c \geq \binom{p+1+1}{1} = p+2$ is positive.

Suppose now that $f : S^k \to C_{p+1,d}(\mathbb{P}^{n+1})$ is a continuous map for $0 < k \leq 2p+2$. We may assume that f is piecewise linear up to homotopy. Then the map f is homotopic to a map $S^k \to T_{p+1,d}(\mathbb{P}^{n+1})$. To see this, we consider the family

$$F_{tD} \circ f : S^k \to C_{p+1,d}(\mathbb{P}^{n+1}), \quad 0 \leq t \leq 1,$$

where D lies outside the union $\bigcup_{x \in S^k} B_{f(x)}$. This is a set of real codimension larger than or equal to $2(p+2) - (k+1)$. Therefore, $2(p+2) - (k+1) \geq 1$, i.e., $k \leq 2p+2$, so such a D exists. This proves that the map $i_* : \pi_k(T_{p+1,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p+1,d}(\mathbb{P}^{n+1}))$ induced by inclusion $i : T_{p+1,d}(\mathbb{P}^{n+1})) \hookrightarrow C_{p+1,d}(\mathbb{P}^{n+1})$ is *surjective* if $k \leq 2p+2$.

Similarly, suppose that $g : (D^{k+1}, S^k) \to (C_{p+1,d}(\mathbb{P}^{n+1}), T_{p+1,d}(\mathbb{P}^{n+1}))$ is a pair of continuous maps for $0 < k \leq 2p+1$. Then the map g can be deformed through a map of pairs to $\tilde{g} : (D^{k+1}, S^k) \to (T_{p+1,d}(\mathbb{P}^{n+1}), T_{p+1,d}(\mathbb{P}^{n+1}))$ if $2(p+2) - (k+2) \geq 1$, i.e., $k \leq 2p+1$. This proves that the map $i_* : \pi_k(T_{p+1,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p+1,d}(\mathbb{P}^{n+1}))$ induced by inclusion $i : T_{p+1,d}(\mathbb{P}^{n+1})) \hookrightarrow C_{p+1,d}(\mathbb{P}^{n+1})$ is *injective* if $k \leq 2p+1$.

Therefore,

$$\pi_k(T_{p+1,d}(\mathbb{P}^{n+1})) \overset{i_*}{\cong} \pi_k(C_{p+1,d}(\mathbb{P}^{n+1})) \tag{3.2}$$

for $0 \leq k \leq 2p+1$.

The combination of equations (3.1) and (3.2) gives us the following result:

Proposition 3.3. *The complex suspension $\Sigma : C_{p,d}(\mathbb{P}^n) \to C_{p+1,d}(\mathbb{P}^{n+1})$ induces an isomorphism*

$$\Sigma_* : \pi_k(C_{p,d}(\mathbb{P}^n)) \cong \pi_k(C_{p+1,d}(\mathbb{P}^{n+1})) \tag{3.3}$$

for $0 \leq k \leq 2p+1$.

As a corollary, we get the simply connectedness of $C_{p,d}(\mathbb{P}^n)$, which has been obtained using general position arguments by Lawson ([L1], the proof of Lemma 2.6):

Corollary 3.4 ([L1]). *The Chow variety $C_{p,d}(\mathbb{P}^n)$ is simply connected for integers $p, d, n \geq 0$.*

Proof. Since $C_{0,d}(\mathbb{P}^n)$ can be identified with the d-th symmetric product $\mathrm{SP}^d(\mathbb{P}^n)$ of $\mathbb{P}^n$ and $\mathrm{SP}^d(\mathbb{P}^n)$ is path connected, we have $\pi_0(C_{0,d}(\mathbb{P}^n)) = 0$ for all $d, n \geq 0$. Repeating using equation (3.3), we know $\pi_0(C_{p,d}(\mathbb{P}^n)) = 0$ for all $p, d, n \geq 0$. Moreover, since $\mathrm{SP}^d(\mathbb{P}^n)$ is simply connected for all $d, n \geq 0$, we have $\pi_1(C_{0,d}(\mathbb{P}^n)) = 0$ for all $d, n \geq 0$. Repeating using equation (3.3), we get

$$\pi_1(C_{p,d}(\mathbb{P}^n)) \cong \pi_1(C_{p-1,d}(\mathbb{P}^{n-1})) \cong \cdots \cong \pi_1(C_{0,d}(\mathbb{P}^{n-p})) = 0$$

for all $p, d, n \geq 0$. □

Now we study the connectedness of maps induced by the inclusion $i : \mathbb{P}^n \hookrightarrow \mathbb{P}^{n+1}$.

Proposition 3.5. *For any integer $d \geq 1$, the inclusion $i : C_{p,d}(\mathbb{P}^n) \hookrightarrow C_{p,d}(\mathbb{P}^{n+1})$ induces an isomorphism*

$$\pi_k(C_{p,d}(\mathbb{P}^n)) \overset{i_*}{\cong} \pi_k(C_{p,d}(\mathbb{P}^{n+1})) \tag{3.4}$$

for $0 \leq k \leq 2(n-p)$.

Remark 3.6. By using Proposition 3.5, we give another possibly more elementary proof of Corollary 3.4. If $n = p$, then $C_{p,d}(\mathbb{P}^n)$ is a point and so it is simply connected. If $n = p+1$, then $C_{p,d}(\mathbb{P}^n) \cong \mathbb{P}^{\binom{n+d}{d}-1}$ so it is simply connected. If $n-p \geq 2$, then $\pi_k(C_{p,d}(\mathbb{P}^n)) \cong \pi_k(C_{p,d}(\mathbb{P}^{n-1})) \cong \cdots \cong \pi_k(C_{p,d}(\mathbb{P}^{p+1})) = 0$ for $k \leq 1$ by using Proposition 3.5 and so $C_{p,d}(\mathbb{P}^n)$ is simply connected.

Proposition 3.5 can be used to compute the second homotopy group of Chow varieties.

Corollary 3.7. *For $d \geq 1$ and $n > p \geq 0$, we have $\pi_2(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z}$ and hence $H_2(C_{p,d}(\mathbb{P}^n)) \cong \mathbb{Z}$.*

Proof. Replacing π_k by π_2 in Remark 3.6 yields the proof of the first statement. The second statement is a result of the first statement, Corollary 3.4 and the Hurewicz isomorphism theorem. $\square$

Lawson's idea in the proof of the Complex Suspension Theorem in [L1] can be used to prove Proposition 3.5.

For any non-negative integer p and d, set

$$U_{p,d}(\mathbb{P}^{n+1}) := \left\{c = \sum n_i V_i \in C_{p,d}(\mathbb{P}^{n+1}) | \mathbb{P}^0 \text{ is not in } \bigcup_i V_i\right\}.$$

Proposition 3.5 follows directly from the combination of Lemmas 3.8 and 3.9 below:

Lemma 3.8. *$U_{p,d}(\mathbb{P}^{n+1})$ is homotopy equivalent to $C_{p,d}(\mathbb{P}^n)$. In particular, their corresponding homotopy groups are isomorphic, i.e.,*

$$\pi_*(U_{p,d}(\mathbb{P}^{n+1})) \cong \pi_*(C_{p,d}(\mathbb{P}^n)). \tag{3.5}$$

Proof. Let $p_0 : \mathbb{P}^{n+1} - \mathbb{P}^0 \to \mathbb{P}^n$ be the canonical projection away from $\mathbb{P}^0 \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Then p_0 induces a deformation retract from $U_{p,d}(\mathbb{P}^{n+1})$ to $C_{p,d}(\mathbb{P}^n)$.

To see this, note that p_0 is a holomorphic line bundle and let $F_t : (\mathbb{P}^{n+1} - \mathbb{P}^0) \times \mathbb{C} \to \mathbb{P}^{n+1} - \mathbb{P}^0$ denote the scalar multiplication by $t \in \mathbb{C}$ in this bundle. This map F_t is holomorphic (in fact, algebraic) and satisfies $F_1 = id_{\mathbb{P}^{n+1}-\mathbb{P}^0}$ and $F_0 = p_0$. Hence F_t induces a family of continuous maps $(F_t)_* : U_{p,d}(\mathbb{P}^{n+1}) \to C_{p,d}(\mathbb{P}^n)$. Therefore, $(p_0)_*$ is a deformation retraction. $\square$

Lemma 3.9. *The inclusion $i : U_{p,d}(\mathbb{P}^{n+1}) \hookrightarrow C_{p,d}(\mathbb{P}^{n+1})$ is $2(n-p)$-connected.*

Proof. By definition, it is enough to show that the induced maps on homotopy groups

$$i_* : \pi_k(U_{p,d}(\mathbb{P}^{n+1})) \to \pi_k(C_{p,d}(\mathbb{P}^{n+1}))$$

are isomorphisms for $k \leq 2(n-p)$. Let $f : S^k \to C_{p,d}(\mathbb{P}^{n+1})$ be a continuous map for $k \leq 2(n-p)$. We may assume f to be piecewise linear up to homotopy. Then f is homotopic to a map $S^k \to U_{p,d}(\mathbb{P}^{n+1})$. To see this, we first note that the union

$$\bigcup_{x \in S^k} f(x)$$

is a set of real codimension $\geq 2(n+1) - 2p - k \geq 2 > 0$. So we can find a point $Q \in \mathbb{P}^{n+1} - \mathbb{P}^n$ such that Q is not in $\bigcup_{x \in S^k} f(x)$. Let G_t be a family of automorphism of $\mathbb{P}^{n+1}$ mapping $\mathbb{P}^0$ to Q but preserving $\mathbb{P}^n$. Composing with the automorphism G_t, we obtain the family $G_t \circ f : S^k \to C_{p,d}(\mathbb{P}^{n+1})$ such that $G_0 \circ f = f$ and $G_1 \circ f : S^k \to U_{p,d}(\mathbb{P}^{n+1})$. Hence i_* is *surjective* for $k \leq 2(n-p)$.

Similarly, suppose g is a map of pairs $g : (D^{k+1}, S^k) \to (C_{p,d}(\mathbb{P}^{n+1}), U_{p,d}(\mathbb{P}^{n+1}))$. Then the map can be deformed through a map of pairs to one with image in $U_{p,d}(\mathbb{P}^{n+1})$ if $k \leq 2(n-p)$. Therefore, i_* is *injective* for $k \leq 2(n-p)$. $\square$

The proof of Theorem 1.3. By Proposition 3.5, $\pi_k(C_{p,d}(\mathbb{P}^n))$ is stable when $n \to \infty$. By the combination of equations (3.3) and (3.4), we have the isomorphism

$$\pi_k(C_{p,d}(\mathbb{P}^n)) \cong \lim_{m,q \to \infty} \pi_k(C_{p+q,d}(\mathbb{P}^{n+m+q})$$

for $0 \leq k \leq 2p+1$ and $k \leq 2(n-p)$. This completes the proof of Theorem 1.3. $\square$

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