PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 7, July 2010, Pages 2591–2599
S 0002-9939(10)10326-8
Article electronically published on March 8, 2010

AUTOMORPHISM GROUPS OF SMALL SIMPLE GROUPS OF FINITE MORLEY RANK

OLIVIER FRÉCON

(Communicated by Julia Knight)

Abstract. If G is a minimal connected simple group of finite Morley rank with a nontrivial Weyl group, then its connected definable automorphism groups are inner.

1. Introduction

Morley rank is an ordinal-valued abstract dimension notion that arose in model theory in the analysis of $\aleph_1$-categorical structures. The Zariski dimension of an algebraic variety when the ground field is algebraically closed is a very relevant example in general, and for this paper in particular.

This paper is about *groups of finite Morley rank*, groups having as Morley rank a finite ordinal, in other words a natural number. The main example of such a group is *an algebraic group over an algebraically closed field*. From the viewpoint of model theory, an algebraic group is a group G *definable* in an algebraically closed field K using the language of fields. In this case, G, its definable subsets and definable automorphisms are exactly the *constructible* ones in the sense of algebraic geometry; the Morley rank is exactly the Zariski dimension.

Examples of *non*-algebraic groups of finite Morley rank are abundant, since the class of groups of finite Morley rank is closed under finite direct sums. Such examples are nevertheless not simple, and it is not known whether any simple non-algebraic group of finite Morley rank exists. Indeed, the main conjecture in the area, the Cherlin-Zil'ber Algebraicity Conjecture, states that an infinite simple group of finite Morley rank is isomorphic as an abstract group to an algebraic group over an algebraically closed field.

One line of research motivated by this conjecture has focused on proving appropriate analogues in groups of finite Morley rank of known classical theorems about algebraic groups over algebraically closed fields. In this article, we consider the automorphism groups of simple groups of finite Morley rank. It is well-known that the group of inner automorphisms of an algebraic group G over an algebraically closed field K is of finite index in the group of algebraic automorphisms of G. Thus, any connected definable (in K) group of automorphisms of G is inner. An appropriate analogue of this result for groups of finite Morley rank is desirable since it would

Received by the editors October 14, 2009, and, in revised form, November 8, 2009, November 23, 2009, and November 29, 2009.

2000 *Mathematics Subject Classification.* Primary 03C60, 20F28; Secondary 20A15, 20E36.

Key words and phrases. Groups of finite Morley rank, Weyl group, Carter subgroups.

substantially improve the understanding of the general structure of groups of finite Morley rank (see Corollary 3.2). In this vein, we prove the following result, where the *Weyl group* is defined in Definition 1.2 and where the term *connected* for a group G of finite Morley rank means that G is equal to its *connected component* G°, which is the smallest definable subgroup of finite index.

Theorem 3.1. *Let G be a* minimal connected simple group *with a nontrivial Weyl group. Then each connected definable automorphism group of G is inner.*

As is clear by its statement, the main objects of study in Theorem 3.1 are *minimal connected simple groups of finite Morley rank*. These are the infinite simple groups of finite Morley rank, all of whose proper definable connected subgroups are solvable. Their analysis is motivated by several reasons. First, the only known strategies for classifying simple groups of finite Morley rank are inductive, and minimal connected simple groups are unavoidable for initiating any inductive argument. Second, there exist powerful methods for analyzing these simple groups, as witnessed by the numerous results proven in [AB08, ABF09, Bur07, BC09, BD09, Del08, Fré08]. The present paper improves the following theorem from [AB08]:

Fact 1.1 ([AB08, Theorem 4]). Let $\hat{G} = G \rtimes A$ be a centerless group of finite Morley rank, where

- G is a definable minimal connected simple subgroup, and
- A is a connected definable abelian subgroup.

Then the following two properties are contradictory:

- the Weyl group of G is nontrivial;
- $\hat{G}/G$ has nontrivial divisible torsion.

For studying the simple groups of finite Morley rank, we have neither root systems nor conjugacy of Borel subgroups, and the analysis of the automorphisms of simple algebraic groups is based on these notions. However, there are analogues of algebraic tori, Cartan subgroups and Weyl group for groups of finite Morley rank, and we recall below the respective definitions.

Definition 1.2.

- A *decent torus* is a definable divisible abelian group of finite Morley rank with no proper definable torsion-free quotient, and the main result concerning these groups is the conjugacy of maximal ones in any group of finite Morley rank [Che05].
- The *Carter subgroups* of group of finite Morley rank are the definable connected nilpotent subgroups of finite index in their normalizers. For this paper, the main results concerning them are their existence in any group G of finite Morley rank [FJ05] and their conjugacy when G is minimal connected simple [Fré08, Theorem 1.2].
- The *Weyl group* of a minimal connected simple group G of finite Morley rank is defined in this paper to be $N_G(C)/C$ up to isomorphism where C is any Carter subgroup of G. It is denoted $W(G)$. Thanks to the main result of [Fré08], the notion and notation are robust.

It is worth making some remarks about the proposed definitions of the Weyl group in a group of finite Morley rank. Our definition comes from [Jal06], but

several alternatives have been proposed in the literature, each having its value in a given context:

- [Del08] $W(G) := N_G(S)/C_G(S)^\circ$, where S is any nontrivial maximal *2-torus*; that is, any nontrivial maximal abelian divisible 2-subgroup;
- [BC09] $W(G) := N_G(T)/C_G(T)^\circ$, where T is any maximal decent torus;
- [AB08, BD09] $W(G) := N_G(T)/C_G(T)$, where T is any maximal decent torus.

Actually, all these definitions are equivalent in minimal connected simple groups as implied by the following fact and by the connectedness of the centralizers of decent tori [AB08, Theorem 1]:

Fact 1.3 ([ABF09, Proposition 2.3 and Corollary 2.5]). Let G be a minimal connected simple group of finite Morley rank. We consider a Carter subgroup C of G and a maximal decent torus T of G. Then $N_G(C)/C$ and $N_G(T)/C_G(T)$ are isomorphic.

Moreover, if G has a nontrivial maximal abelian divisible p-subgroup S for a prime p, then $N_G(C)/C \simeq N_G(T)/C_G(T)$ is isomorphic to $N_G(S)/C_G(S)^\circ$, too.

The previous fact essentially shows that if the Weyl group of a minimal connected simple group is nontrivial, then there is a nontrivial decent torus. Fact 1.3, a weakened version of which was proven in [BD09, §2.1], is a consequence of various results in [FJ05] and [Fré08].

In groups of finite Morley rank, a recurrent problem concerns the possible absence of torsion, that is of elements of finite order. Unlike the case of algebraic groups, there exist torsion-free nonnilpotent groups of finite Morley rank [BHMW09], and it is not known whether there exists a torsion-free *simple* group of finite Morley rank. The existence of such a simple group would also imply that of a connected minimal simple one. A torsion-free connected minimal simple group would not have a nontrivial Weyl group. This is a very pathological situation in that known techniques, generally based on torsion, are no longer effective.

This paper is a continuation of numerous recent works concerning torsion elements in (minimal connected simple) groups of finite Morley rank such as [AB08, ABF09, BC09, BD09, Che05]. We analyze only the situation where the Weyl group is nontrivial. In fact, this hypothesis may be viewed as follows. Simple algebraic groups satisfy the following two properties:

- the Weyl group is nontrivial;
- any connected definable automorphism group is inner.

But neither of these hypotheses is known to be true for simple groups of finite Morley rank. In this paper, we will show that the first of these two conditions implies the second in the minimal simple case.

2. Background and particular cases

In this section, we recall some known results and we study some particular cases of the main result (Lemmas 2.5 and 2.8). Notation will be as in [BN94], which is also our basic reference for general results about groups of finite Morley rank.

2.1. **Carter subgroups and generalized centralizers.** In the introduction, we mentioned that Carter subgroups exist in any group G of finite Morley rank and

that they are conjugate when G is either solvable or minimal connected simple. The following results about Carter subgroups will be useful here.

Fact 2.1. In any group G of finite Morley rank, the following three assertions hold:

(1) [FJ05] Each decent torus of G is contained in a Carter subgroup of G.
(2) [Fré08, Theorem 1.2] If G is a minimal connected simple group, its Carter subgroups are conjugate.
(3) [Fré00, Théorème 1.1] If G is connected and solvable, any Carter subgroup of G is self-normalizing.

In this paper we also make essential use of the *generalized centralizers* of [Fré00]. This notion is crucial for solvable groups of finite Morley rank, and it is strongly related to Carter subgroups.

In any group G and for any subset X of G, we denote by $E_G(X)$ the set of the elements $g \in G$ such that, for each $x \in X$, $[g,_n x] = 1$ for some $n \in \mathbb{N}$, where $[g,_0 x] = g$ and $[g,_{i+1} x] = [[g,_i x], x]$ for each $i \in \mathbb{N}$. This subset $E_G(X)$ is called *a generalized centralizer* of X in G.

We summarize the properties of these subsets in Fact 2.2 below. We recall that the *Fitting subgroup* $F(G)$ of any group G is the subgroup generated by all its normal nilpotent subgroups. This subgroup is definable and nilpotent in any group of finite Morley rank [BN94, Theorem 7.3].

Fact 2.2. Let G be a solvable connected group of finite Morley rank, and let X be a subset of G generating a nilpotent subgroup.

(1) [Fré00, Corollaire 5.17] $E_G(X)$ is a definable connected subgroup, and X is contained in the Fitting subgroup $F(E_G(X))$ of $E_G(X)$.
(2) [Fré00, Proposition 7.2 and Corollaire 7.4] $E_G(X) \cap H$ is connected for each normal connected definable subgroup H of G.

2.2. **Frattini Argument.** The *Frattini Argument* is a classical and very useful simple principle in group theory. In this paper it is applied several times in different ways, and here we give its general version.

Fact 2.3 (Frattini Argument). Let H be a subgroup of a group G which acts on a set E. If the action of H on E is transitive, then $G = G_x H$ for each $x \in E$, where G_x denotes the stabilizer of x in G.

2.3. **The main theorem for groups of odd type.** A particular case of our main theorem follows from the structure theory for certain minimal connected simple groups of finite Morley rank (Fact 2.4), namely those of *odd type*. We will treat this case in Lemma 2.5 below.

Groups of odd type are defined as follows. In the context of groups of finite Morley rank, for each prime p, a maximal locally finite p-subgroup is said to be a *Sylow p-subgroup*. These subgroups are known to be conjugate as soon as the ambient group is solvable [BN94, Theorem 9.35], or when $p = 2$ [BN94, Theorem 10.11]. Moreover, for each prime p, the *connected component* of each Sylow p-subgroup of a group of finite Morley rank is a central product of a p-unipotent subgroup by a p-torus. This follows from [BN94, Corollaries 5.38, 6.12 and 6.20], where the *connected component* H° of a nonnecessarily definable subgroup H of G is the smallest subgroup of finite index and of the form $H \cap D$ for a definable subgroup D of G. Then a group of finite Morley rank is said to be *of odd type* if its Sylow 2-subgroups

are infinite and if they contain no nontrivial 2-unipotent subgroup. In this case, there is a positive integer n such that each maximal 2-torus of G is isomorphic to a direct product of n copies of $\mathbb{Z}_{p^\infty}$, and n is called the *Prüfer rank* of G.

The minimal connected simple groups of finite Morley rank of odd type have been analyzed in [Del08], and we summarize the main theorem concerning them.

Fact 2.4 ([Del08, Théorème-Synthèse]). Let G be a minimal connected simple group of finite Morley rank and of odd type. Let S be a maximal 2-torus of G, and let $C := C_G(S)^\circ$. Then the Prüfer rank of G is at most two, and one has the following two possibilities:

- The Prüfer rank of G is one:
  - If C is not a Borel subgroup of G, then G is of the form $\mathrm{PSL}_2(K)$ with K an algebraically closed field of characteristic different from two.
  - If C is a Borel subgroup of G and if the Weyl group $W(G)$ of G is nontrivial, then $W(G)$ has order two, C is a 2-divisible abelian Carter subgroup of G, and $N_G(C) = C \rtimes \langle i \rangle$ for an involution i inverting C. Furthermore, all the involutions of G are conjugate.
- The Prüfer rank of G is two, and its Weyl group has order three.

We notice that *Borel subgroups* of a group of finite Morley rank are defined to be its maximal solvable connected definable subgroups.

Lemma 2.5. *Let G be a minimal connected simple group of finite Morley rank and of odd type. If its Weyl group has even order, then each connected definable automorphism group of G is inner.*

Proof. We denote by $\tilde{G}$ the (definable connected) group of inner automorphisms of G. We assume toward a contradiction that G has a connected definable automorphism group $\tilde{A}$ not contained in $\tilde{G}$. Then the group $\tilde{G}\tilde{A}$ is connected and definable, and it contains $\tilde{G}$ strictly. Moreover, there is a definable embedding from G to $\tilde{G}\tilde{A}$ with image $\tilde{G} < \tilde{G}\tilde{A}$, and this proves the existence of a connected definable group $\hat{G}$ containing strictly G as a normal subgroup and satisfying $C_{\hat{G}}(G) = 1$.

By Fact 2.4, the Weyl group of G has order two, G has Prüfer rank one, and if S denotes a maximal 2-torus of G, then $C := C_G(S)^\circ$ is a 2-divisible abelian Carter subgroup of G, and $N_G(C) = C \rtimes \langle i \rangle$ for an involution i inverting C. Furthermore, all the involutions of G are conjugate, so $C_G(i)^\circ$ is conjugate with C. In particular, $C_G(i)$ contains a unique maximal 2-torus T and i belongs to T.

By conjugacy of Carter subgroups in G (Fact 2.1 (2)) and by the Frattini Argument (Fact 2.3), we find $\hat{G} = GN_{\hat{G}}(C)$. Moreover, $N_{\hat{G}}(C)$ normalizes $N_G(C) = C \rtimes \langle i \rangle$, so it normalizes the coset iC too. But i inverts the 2-divisible abelian subgroup C, hence C acts transitively by conjugation on iC. Now, again by the Frattini Argument (Fact 2.3), we obtain $N_{\hat{G}}(C) = CC_{N_{\hat{G}}(C)}(i)$ and $\hat{G} = GC_{N_{\hat{G}}(C)}(i)$. In particular, $D := C_{N_{\hat{G}}(C)}(i)^\circ$ is an infinite connected definable subgroup of $\hat{G}$ normalizing C and centralizing i.

Since D normalizes C, it normalizes S, and it even centralizes S by rigidity of tori [BN94, Theorem 6.16]. In the same way, since D centralizes i, it normalizes $C_G(i)$ and T, so it centralizes T. Now $C_G(D)^\circ$ is a connected definable solvable subgroup containing S and $i \in T$. But G is of odd type, so by [BN94, Corollary 6.20] and by the connectedness of Sylow 2-subgroups in connected solvable groups [BN94, Theorem 9.29], the subgroup S is a Sylow 2-subgroup of $C_G(D)^\circ$. Consequently,

since $i \in C_G(D)^\circ$ is a 2-element normalizing S, we obtain $i \in S \leq C$, contradicting our choice of i. This finishes the proof. $\square$

2.4. **Weyl group.** In the context of minimal connected simple groups of finite Morley rank with a nontrivial Weyl group, the torsion elements play a central role, and the Weyl group is at the core of any analysis of the torsion. We will need the following crucial property proved in [BC09], and which provides a starting point for the subsequent general studies on the Weyl group in [BD09] and in [ABF09]. We recall that, for any prime p, a *p-unipotent subgroup* is a connected definable nilpotent p-subgroup of bounded exponent.

Fact 2.6 ([BC09, Corollary 5.3]). Let G be a minimal connected simple group of finite Morley rank. Suppose the Weyl group is nontrivial and has odd order, with p the smallest prime divisor of its order. Then any p-element representing an element of order p in $W(G)$ centralizes a nontrivial p-unipotent subgroup.

We will also need a small part of the *Bender Method* [Bur07], together with a substantial result from [AB08], as follows.

Fact 2.7. In any minimal connected simple group G of finite Morley rank, the following two assertions are satisfied:

(1) [Bur07, Corollary 2.2] If B_1 and B_2 are two distinct Borel subgroups of G, then $F(B_1) \cap F(B_2)$ is torsion-free.
(2) [AB08, Corollary 4.4] If $x \in G$ has finite order, then it is contained in any Borel subgroup containing $C_G(x)^\circ$.

Fact 2.7 (1) implies that, for each prime p, any p-unipotent subgroup of G is contained in a unique Borel subgroup. As we will see momentarily, this then leads directly to a situation in which Fact 2.7 (2) applies.

Lemma 2.8. *Let G be a minimal connected simple group of finite Morley rank, C a Carter subgroup of G, and p the smallest prime divisor of the order of $W(G)$. If C has a nontrivial p-element, then G is of odd type and $W(G)$ has even order.*

Proof. Let $x \in G$ be a p-element representing an element of order p in $N_G(C)/C$. Let T be the maximal p-torus of C. Then T is central in C [BN94, Corollary 6.12]. We show that neither C nor $C_G(C_T(x))$, if T is nontrivial, contains a nontrivial p-unipotent subgroup. Supposing the contrary, we denote by U_0 the maximal p-unipotent subgroup of C (resp. of $C_G(C_T(x))$ if T is nontrivial). Since the element x normalizes C and T, it also normalizes U_0, and $C_{U_0}(x)$ is infinite by [BN94, Corollary 6.20]. Now, by Fact 2.7 (1), there is a unique Borel subgroup B_x containing $C_{U_0}(x)^\circ$, and Fact 2.7 (2) gives $x \in B_x$. In particular, B_x contains C (resp. B_x contains $C_G(C_T(x))^\circ \geq C$), and Fact 2.1 (3) yields $x \in N_{B_x}(C) = C$, contradicting our choice of x. Hence C (resp. $C_G(C_T(x))$ if T is nontrivial) has no nontrivial p-unipotent subgroup, and T is nontrivial by [BN94, Corollary 6.20] and by the connectedness of Sylow p-subgroups in connected nilpotent groups [BN94, Corollary 6.14]. In particular $C_G(C_T(x))$ has no nontrivial p-unipotent subgroup.

We claim that $W(G)$ is of even order. Otherwise, x would centralize a nontrivial p-unipotent subgroup (Fact 2.6) that would be normalized by $C_T(x)$. Then $C_T(x)$ would centralize a nontrivial p-unipotent subgroup by [BN94, Corollary 6.20], contradicting the previous paragraph.

We show that, as desired, G is of odd type. Otherwise, by the structure of Sylow 2-subgroups [BN94, Corollary 6.22], the subgroup $C_T(x) \leq T$ would centralize a p-unipotent subgroup, contradicting the fact that $C_G(C_T(x))$ has no nontrivial p-unipotent subgroup. □

The previous argument could be simplified, with a stronger conclusion, by using the main result of [BD09, Theorem 4.1]. Indeed, that result says that in any minimal connected simple group G of finite Morley rank with a nontrivial Weyl group and for each prime $p \neq 2$ dividing the order of $W(G)$ there is no nontrivial p-torus in G. We prefer to give a proof independent of this theorem, since its proof depends on the very deep [ABC08, Main Theorem], on [Del08, Théorème-Synthèse] (see also Fact 2.4), and on the cyclicity of the Weyl group [BD09, Theorem 4.1].

3. Main theorem

Now we are ready to prove the main result of this paper in full generality.

Theorem 3.1. *Let G be a minimal connected simple group with a nontrivial Weyl group. Then each connected definable automorphism group of G is inner.*

Proof. Suppose the contrary. As in the proof of Lemma 2.5, this implies the existence of a connected definable group $\hat{G}$ containing G strictly as a normal subgroup, and satisfying $C_{\hat{G}}(G) = 1$. Moreover, we may assume that G is a maximal proper definable connected subgroup of $\hat{G}$, so $\hat{G}/G$ is abelian by the Reineke Theorem [BN94, Theorem 6.4].

We consider the smallest prime divisor p of the order of $W(G)$, and a Carter subgroup C of G (Fact 2.1 (1)). Then we find a p-element x in $N_G(C)\backslash C$. Moreover, Lemmas 2.5 and 2.8 show that C has no nontrivial p-element. In particular, since each decent torus is contained in a Carter subgroup (Fact 2.1 (1)), the conjugacy of Carter subgroups (Fact 2.1 (2)) implies that G has no nontrivial p-torus.

We show that there is an infinite abelian connected definable subgroup D of $\hat{G}$ normalizing C and centralizing x. By conjugacy of the Carter subgroups in G (Fact 2.1 (2)), the Frattini Argument (Fact 2.3) gives $\hat{G} = GN_{\hat{G}}(C) = GN_{\hat{G}}(C)^\circ$. Let S be a Sylow p-subgroup of $N_G(C)$ containing x. Since $N_G(C)/C$ is finite, it is centralized by $N_{\hat{G}}(C)^\circ$, so $N_{\hat{G}}(C)^\circ$ normalizes CS. Since C and CS/C are nilpotent, CS is solvable and its Sylow p-subgroups are conjugate by [BN94, Theorem 9.35]. Moreover, since S is a p-subgroup, the Sylow p-subgroups of CS are C-conjugate, and the Frattini Argument gives $N_{\hat{G}}(C)^\circ = CN_{N_{\hat{G}}(C)^\circ}(S)$. Hence we obtain $\hat{G} = GN_{N_{\hat{G}}(C)^\circ}(S)$ and, since $\hat{G}/G$ is infinite, $N_{N_{\hat{G}}(C)^\circ}(S)$ is infinite too. Furthermore, since C has no nontrivial p-element by the previous paragraph, S is finite and $N_{N_{\hat{G}}(C)^\circ}(S)$ is definable. Now the Reineke Theorem [BN94, Theorem 6.4] provides an infinite abelian definable subgroup D in $N_{N_{\hat{G}}(C)^\circ}(S)$, and we may choose D to be connected. Then, since S is finite, D centralizes S, and we find $D \leq C_{\hat{G}}(x)$, as desired.

We consider a Borel subgroup B_x containing $C_G(x)^\circ$. Then B_x contains x by Fact 2.7 (2). Moreover, since G has no nontrivial p-torus, the connectedness of Sylow p-subgroups in connected solvable groups [BN94, Theorem 9.29] implies that the Sylow p-subgroups of B_x are p-unipotent. Therefore B_x has a unique Sylow p-subgroup U, and $C_U(x)^\circ$ is a nontrivial p-unipotent subgroup by [BN94, Corollary 6.20]. Now Fact 2.7 (1) says that B_x is the unique Borel subgroup of G containing $C_G(x)^\circ$. In particular, D normalizes B_x and U.

We show that B_x contains $C_G(D)^\circ$. Since D normalizes U, the set UD is a solvable group. Then, by Fact 2.2 (1), the set $E := E_{UD}(D)$ is a connected definable subgroup of UD, and D is contained in $F(E)$. Moreover, the subgroup $E \cap U$ is connected by Fact 2.2 (2). Hence, since $x \in U$ centralizes D, the group $E \cap U \leq F(E)$ is infinite, and $Z(F(E)) \cap U$ is infinite too. Now $C_G(D)^\circ \geq Z(F(E))^\circ$ contains a nontrivial p-unipotent subgroup of U, and Fact 2.7 (1) gives $C_G(D)^\circ \leq B_x$.

Let T_C be the maximal decent torus of C. Since each decent torus is contained in a Carter subgroup of G (Fact 2.1 (1)), the conjugacy of Carter subgroups in G (Fact 2.1 (2)) shows that T_C is a maximal decent torus of G. Then, since the Weyl group of G is nontrivial, Fact 1.3 shows that T_C is nontrivial. But D normalizes C, so D normalizes T_C and D centralizes T_C by [BN94, Theorem 6.16]. Hence T_C is contained in $C_G(D)^\circ \leq B_x$. Now we have $x \in N_{B_x}(T_C) \leq E_{B_x}(T_C)$. Moreover, T_C is contained in $F(E_{B_x}(T_C))$ by Fact 2.2 (1). Therefore T_C is the maximal decent torus of $F(E_{B_x}(T_C))$, and it is normal in $E_{B_x}(T_C)$. Since $E_{B_x}(T_C)$ is definable and connected by Fact 2.2 (1), we obtain $x \in N_G(T_C)^\circ$. This proves that x belongs to $N_{N_G(T_C)^\circ}(C)$, and Fact 2.1 (3) provides $x \in C$, contradicting our choice of x. This finishes our proof. □

Now, with an eye toward possible applications to the Algebraicity Conjecture, we derive Corollary 3.2. We recall that, in any group of finite Morley rank, the subgroup generated by all the normal solvable subgroups is definable and solvable [BN94, Theorem 7.3]: this subgroup is called the *radical*.

Corollary 3.2. *Let G be a connected group of finite Morley rank, and let R be its radical. Assume that each normal connected definable simple section H/K of G is a minimal connected simple group with a nontrivial Weyl group. Then G/R is a direct product of definable simple groups.*

Proof. We may assume $R = 1$. Then, since G is connected, G has no normal nontrivial finite subgroup. Let S be the subgroup of G generated by its normal definable simple subgroups. We have $S = S_1 \times \cdots \times S_n$ for an integer n and n definable G-normal simple subgroups $S_1, \ldots, S_n$. By the previous theorem, the quotient $G/C_G(S)$ is covered by S, and we obtain $G = S \times C_G(S)$. Thus we may assume that $C_G(S)$ is infinite. But $C_G(S)$ has no normal nontrivial abelian subgroup since $R = 1$, hence $C_G(S)$ contains a G-normal definable simple subgroup by the analysis of minimal normal subgroups of groups of finite Morley rank [BN94, Theorem 7.8 (*iii*)], contradicting the choice of S. □

Acknowledgements

The author would like to thank Tuna Altınel for his encouragement and for his helpful conversations and comments. Also, he thanks the referee for valuable suggestions that contributed to the improvement of the paper.

References

[AB08] T. Altınel and J. Burdges, *On analogies between algebraic groups and groups of finite Morley rank,* J. London Math. Soc., **78** (2008), 213–232. MR2427061 (2009m:20039)

[ABC08] T. Altınel, A. V. Borovik and G. Cherlin, *Simple Groups of Finite Morley Rank,* Amer. Math. Soc., Providence, RI, 2008. MR2400564 (2009a:20046)

[ABF09] T. Altınel, J. Burdges and O. Frécon, *Borel subgroups of small simple groups of finite Morley rank,* in preparation.

[BHMW09] A. Baudisch, M. Hils, A. Martin-Pizarro and F. O. Wagner, *Die Böse Farbe,* J. Inst. Math. Jussieu, **8** (2009), 415–443. MR2516302

[BN94] A. V. Borovik and A. Nesin, *Groups of Finite Morley Rank,* Oxford Science Publications, New York, 1994. MR1321141 (96c:20004)

[Bur07] J. Burdges, *The Bender method in groups of finite Morley rank,* J. Algebra, **312** (2007), 33–55. MR2320445 (2008e:20042)

[BC09] J. Burdges and G. Cherlin, *Semisimple torsion in groups of finite Morley rank,* submitted.

[BD09] J. Burdges and A. Deloro, *Weyl groups of small groups of finite Morley rank,* Israel J. Math., to appear.

[Che05] G. Cherlin, *Good tori in groups of finite Morley rank,* J. Group Theory, **8** (2005), 613–621. MR2165294 (2006e:03045)

[Del08] A. Deloro, *Groupes simples connexes minimaux non-algébriques de type impair,* J. Algebra, **319** (2008), 1636–1684. MR2383061 (2009a:20048)

[Fré00] O. Frécon, *Sous-groupes anormaux dans les groupes de rang de Morley fini résolubles,* J. Algebra, **229** (2000), 118–152. MR1765797 (2001e:20031)

[Fré08] O. Frécon, *Conjugacy of Carter subgroups in groups of finite Morley rank,* J. Math Logic, **8** (2008), 41–92.

[FJ05] O. Frécon and É. Jaligot, *The existence of Carter subgroups in groups of finite Morley rank,* J. Group Theory, **8** (2005), 623–633. MR2165295 (2006e:20002)

[Jal06] É. Jaligot, *Generix never gives up,* J. Symbolic Logic, **71** (2006), 599–610. MR2225896 (2007b:20065)

Laboratoire de Mathématiques et Applications, Université de Poitiers, Téléport 2–BP 30179, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France

E-mail address: `olivier.frecon@math.univ-poitiers.fr`