ANOTHER PROOF FOR THE REMOVABLE SINGULARITIES OF THE HEAT EQUATION

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Abstract. We give two different simple proofs for the removable singularities of the heat equation in $(\Omega \setminus \{x_0\}) \times (0, T)$, where $x_0 \in \Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 3$. We also give a necessary and sufficient condition for removable singularities of the heat equation in $(\Omega \setminus \{x_0\}) \times (0, T)$ for the case $n = 2$.


It is interesting to find a necessary and sufficient condition for the solutions of the equations to have removable singularities. In $[8]$ S.Y. Hsu proved the following theorem.

Theorem 1. Let $n \geq 3$ and let $0 \in \Omega \subset \mathbb{R}^n$ be a domain. Suppose $u$ is a solution of the heat equation

$$u_t = \Delta u$$

in $(\Omega \setminus \{0\}) \times (0, T)$. Then $u$ has removable singularities at $\{0\} \times (0, T)$ if and only if for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $B_{R_0}(0) \subset \Omega$ depending on $t_1$, $t_2$ and $\delta$, such that

$$|u(x, t)| \leq \delta |x|^{2-n}$$

for any $0 < |x| \leq R_0$ and $t_1 \leq t \leq t_2$.

The proof in $[8]$ is based on the Green function estimates of $[9]$ and a careful analysis of the behavior of the solution near the singularities using the Duhamel principle. In this paper we will use the Schauder estimates for the heat equation $[2]$, $[12]$, and the technique of $[1]$ and $[7]$ to give two different simple proofs of the above result. We also obtain the following result for the solution of the heat equation in two dimensions.

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Theorem 2. Let $0 \in \Omega \subset \mathbb{R}^2$ be a domain. Suppose $u$ is a solution of the heat equation in $(\Omega \setminus \{0\}) \times (0, T)$. Then $u$ has removable singularities at $\{0\} \times (0, T)$ if and only if for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $B_{R_0}(0) \subset \Omega$ depending on $t_1$, $t_2$ and $\delta$ such that

$$
|u(x, t)| \leq \delta (\log(1/|x|))^{-1}
$$

for any $0 < |x| \leq R_0$ and $t_1 \leq t \leq t_2$.

Remark 3. Note that the function $\log |x|$ satisfies the heat equation in $(\mathbb{R}^2 \setminus \{0\}) \times (0, \infty)$, but it has non-removable singularities on $\{0\} \times (0, \infty)$ and it does not satisfy (3). Hence (3) is sharp.

We start with some definitions. For any set $A$ we let $\chi_A$ be the characteristic function of the set $A$. Let $0 \in \Omega \subset \mathbb{R}^n$ be a bounded domain. We say that a solution $u$ of the heat equation (1) in $(\Omega \setminus \{0\}) \times (0, T)$ has removable singularities at $\{0\} \times (0, T)$ if there exists a classical solution $v$ of (1) in $\Omega \times (0, T)$ such that $u = v$ in $(\Omega \setminus \{0\}) \times (0, T)$. For any $R > 0$ let $B_R = B_R(0) = \{x: |x| < R\} \subset \mathbb{R}^n$.

Proof of Theorem 1. Suppose $u$ has removable singularities in $\{0\} \times (0, T)$. By the same argument as in the proof in section 3 of [5], for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $\overline{B}_{R_0} \subset \Omega$ depending on $t_1$, $t_2$ and $\delta$ such that (2) holds.

Suppose (2) holds. Then for any $0 < t_1 < t_2 < T$ and $\delta \in (0, 1)$ there exists $\overline{B}_{R_0} \subset \Omega$ depending on $t_1$, $t_2$ and $\delta$ such that (2) holds for any $0 < |x| \leq R_0$ and $t_1 \leq t \leq t_2$.

For any $0 < |x| \leq R_0$, let

$$
w(y, s) = u(|x| y, |x|^2 s) \quad \forall 0 < |y| \leq R_0/|x|, t_1/|x|^2 \leq s \leq t_2/|x|^2.
$$

Then $w$ is a solution of (1) in $(\overline{B}_1 \setminus \{0\}) \times (|x|^{-2} t_1, |x|^{-2} t_2)$. By (2),

$$
|w(y, s)| \leq \delta (|x| |y|)^{2-n} \quad \forall 0 < |y| \leq R_0/|x|, t_1/|x|^2 \leq s \leq t_2/|x|^2.
$$

Let $t_1 < t_3 < t_2$. Then

$$
t_3/|x|^2 - t_1/|x|^2 \geq t_3 - t_1/R_0^2.
$$

By the parabolic Schauder estimates [2], [12], (5) and (6), there exists a constant $C_1 > 0$ such that

$$
|\nabla w(y, s)| \leq C_1 \sup_{1/2 \leq |z| \leq 1} w(z, \tau) \leq C_2 \delta |x|^{2-n}
$$

holds for any $2/3 \leq |y| \leq 3/4$, $t_3/|x|^2 \leq s \leq t_2/|x|^2$, where $C_2 = 2^{n-2} C_1$. By (4) and (7),

$$
|\nabla u(z, t)| \leq C_2 \delta |x|^{1-n} \quad \forall |z| = 3/4 |x|, 0 < |x| \leq R_0, t_3 \leq t \leq t_2
$$

holds for any $2/3 \leq |x| \leq 3/4$, $t_3/|x|^2 \leq s \leq t_2/|x|^2$, where $C_2 = 2^{n-2} C_1$. By (4) and (8),

$$
|\nabla u(z, t)| \leq C_2 \delta |z|^{1-n} \quad \forall |z| = 3/4 R_0, t_3 \leq t \leq t_2.
$$

Let $R_1 = 3/(4R_0)$. We will now use a modification of the proof of Lemma 2.3 of [11] and Lemma 2.1 of [7] to complete the argument. We will first show that $u$ satisfies
(1) in $\Omega \times (t_1, t_2)$ in the distribution sense. Since $u$ satisfies (1) in $(\Omega \setminus \{0\}) \times (0, T)$, for any $0 < \varepsilon < R_1$ and $\eta \in C_0^\infty(\Omega \times (0, T))$ we have

$$
\int_{\Omega \setminus B_\varepsilon} u \eta \, dx \bigg|_{t_3}^{t_2} = \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} u \eta \, dx dt - \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dx dt\]

$$

\begin{align*}
&- \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt,
\end{align*}

(9)

where $\partial u / \partial n$ is the derivative of $u$ with respect to the unit outward normal at $\partial B_\varepsilon$. By (8),

$$
\limsup_{\varepsilon \to 0} \left| \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt \right| \leq C_2 \delta (t_2 - t_3) |\partial B_1| \|\eta\|_{L^\infty}.
$$

Since $\delta > 0$ is arbitrary, it follows that

$$
\lim_{\varepsilon \to 0} \int_{t_3}^{t_2} \int_{\partial B_\varepsilon} \eta \frac{\partial u}{\partial n} \, d\sigma dt dt dx dt = 0.
$$

By (8) and the Lebesgue dominated convergence theorem,

$$
\lim_{\varepsilon \to 0} \int_{t_3}^{t_2} \int_{\Omega \setminus B_\varepsilon} \nabla u \cdot \nabla \eta \, dx dt = \int_{t_3}^{t_2} \int_\Omega \nabla u \cdot \nabla \eta \, dx dt.
$$

Letting $\varepsilon \to 0$ in (9), by (10) and (11) it follows that

$$
\int_{t_3}^{t_2} \int_\Omega u \eta \, dx dt = \int_{t_3}^{t_2} \int_\Omega u \eta \, dx dt
$$

(12)

Hence $u$ is a distribution solution of (1) in $\Omega \times (t_1, t_2)$. By (2), for any $1 \leq p < \frac{n}{n-2}$ there exists a constant $C'_p > 0$ such that

$$
\sup_{t_1 \leq t \leq t_2} \int_{B_{R_0}} u(x, t)^p \, dx \leq C'_p.
$$

By (12) and (13) and an argument similar to the proof of [11] and section 1 of [10], $u \in L^\infty_{loc}(B_{R_0} \times (t_1, t_2))$. We now let $v$ be the solution of

$$
\begin{align*}
v_t - \Delta v &= 0 \quad \text{in } B_{R_1} \times (t_3, t_2), \\
\frac{\partial v}{\partial n}(x, t) &= \frac{\partial u}{\partial n}(x, t) \quad \text{on } \partial B_{R_1} \times (t_3, t_2), \\
v(x, t_3) &= u(x, t_3) \quad \text{in } B_{R_1}.
\end{align*}
$$

(14)

For any $0 \leq h \in C_0^\infty(B_{R_1})$ and $t_3 < t \leq t_2$ let $\eta$ be the solution of

$$
\begin{align*}
\eta_t + \Delta \eta &= 0 \quad \text{in } B_{R_1} \times (t_3, t), \\
\frac{\partial \eta}{\partial n}(x, t) &= 0 \quad \text{on } \partial B_{R_1} \times (t_3, t), \\
\eta(x, t) &= h(x) \quad \text{in } B_{R_1}.
\end{align*}
$$

(15)

By the maximum principle,

$$
0 \leq \eta \leq \|h\|_{L^\infty} \quad \text{in } B_{R_1} \times (t_3, t).
$$

(16)
Then by (14) and (15),
\begin{equation}
\int_{B_{R_1} \setminus B_1} (u - v) \eta \, dx \bigg|_{t_3}^{t} = \int_{t_3}^{t} \int_{B_{R_1} \setminus B_1} [(u - v) \eta_t + (u - v) \eta] \, dx \, dt \\
= \int_{t_3}^{t} \int_{B_{R_1} \setminus B_1} [(u - v) \eta_t + \Delta(u - v) \eta] \, dx \, dt \\
= \int_{t_3}^{t} \int_{B_{R_1} \setminus B_1} (u - v) (\eta_t + \Delta \eta) \, dx \, dt \\
- \int_{t_3}^{t} \int_{\partial B_1} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt + \int_{t_3}^{t} \int_{\partial B_1} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt \\
= - \int_{t_3}^{t} \int_{\partial B_1} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt + \int_{t_3}^{t} \int_{B_{R_1} \setminus B_1} (u - v) \eta_t \, dx \, dt \\
- \int_{t_3}^{t} \int_{\partial B_1} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt \\
- \int_{t_3}^{t} \int_{\partial B_1} \eta \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt.
\end{equation}

By (2),
\begin{equation}
\left| \int_{t_3}^{t} \int_{\partial B_1} (u - v) \frac{\partial \eta}{\partial n} \, d\sigma \, dt \right| \leq C \varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0.
\end{equation}

By (8) and (16),
\begin{equation}
\limsup_{\varepsilon \to 0} \left| \int_{t_3}^{t} \int_{\partial B_1} \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt \right| \leq C \delta.
\end{equation}

Since \( \delta > 0 \) is arbitrary, by (19) it follows that
\begin{equation}
\lim_{\varepsilon \to 0} \left| \int_{t_3}^{t} \int_{\partial B_1} \frac{\partial}{\partial n} (u - v) \, d\sigma \, dt \right| = 0.
\end{equation}

Letting \( \varepsilon \to 0 \) in (17), by (18) and (20),
\begin{equation}
\int_{B_{R_1}} (u - v)(x, t) h(x) \, dx = \int_{B_{R_1}} (u - v)(x, t_3) \eta(x, t_3) \, dx = 0.
\end{equation}

We now choose a sequence of functions \( h_i \in C_0^\infty(B_{R_1}) \) converging to \( \chi\{u>v\} \) a.e. \( x \in B_{R_1} \) as \( i \to \infty \). Putting \( h = h_i \) in (21) and letting \( i \to 0 \),
\begin{equation}
\int_{B_{R_1}} (u - v)_+(x, t) \, dx = 0 \quad \forall t_3 < t \leq t_2.
\end{equation}

By interchanging the roles of \( u \) and \( v \) we get
\begin{equation}
\int_{B_{R_1}} (v - u)_+(x, t) \, dx = 0 \quad \forall t_3 < t \leq t_2.
\end{equation}

Hence by (22) and (23),
\begin{equation}
\int_{B_{R_1}} |v - u|(x, t) \, dx = 0 \quad \forall t_3 < t \leq t_2
\end{equation}
\( \Rightarrow \) \( u(x, t) = v(x, t) \) \( \forall 0 < |x| \leq R_1, t_3 < t \leq t_2 \).

Hence \( u \) has removable singularities on \( \{0\} \times (t_3, t_2) \). Since \( 0 < t_1 < t_3 < t_2 < T \) is arbitrary, \( u \) has removable singularities on \( \{0\} \times (0, T) \) and the theorem follows. \( \square \)

**Proof of Theorem 2.** Theorem 2 follows by an argument very similar to the proof of Theorem 1 but with (3) replacing (2) in the argument. \( \square \)
An alternate proof of Theorems 1 and 2. We will show that when (2) (respectively (3)) holds, then \( u \) has removable singularities at \( \{0\} \times (0,T) \). Suppose (2) holds if \( n \geq 3 \) and (3) holds if \( n = 2 \). We first observe that by the previous argument, for any \( 0 < t_1 < t_2 < T \), \( u \) satisfies (12) and \( u \in L^\infty_{\text{loc}}(\Omega \times (0,T)) \). Let \( \overline{B}_{R_1} \subset \Omega \) and let \( w \) be the solution of
\[
\begin{cases}
  w_t = \Delta w & \text{in } B_{R_1} \times (t_1, t_2), \\
  w = u & \text{on } \overline{B}_{R_1} \times \{t_1\} \cup \partial B_{R_1} \times (t_1, t_2).
\end{cases}
\]

By the maximum principle,
\[
\|w\|_{L^\infty} \leq \|u\|_{L^\infty(B_{R_1} \times (t_1, t_2))} < \infty.
\]

For any \( \varepsilon > 0 \), let
\[
w_\varepsilon = \begin{cases}
  w - u + \varepsilon|x|^{2-n} & \text{if } n \geq 3, \\
  w - u + \varepsilon \log(R_1/|x|) & \text{if } n = 2.
\end{cases}
\]

Then \( w_\varepsilon \) satisfies
\[
\begin{cases}
  w_{\varepsilon, t} = \Delta w_\varepsilon & \text{in } (B_{R_1} \setminus \{0\}) \times (t_1, t_2), \\
  w_\varepsilon \geq u & \text{on } \partial B_{R_1} \times (t_1, t_2) \cup \overline{B}_{R_1} \times \{t_1\}.
\end{cases}
\]

By (2), (3), and (25) there exists a constant \( 0 < r_0 < R_1 \) such that
\[
w_\varepsilon \geq 0 \quad \text{on } \partial B_{r_1} \times [t_1, t_2]
\]
for all \( 0 < r_1 \leq r_0 \). By the maximum principle in \( (B_{R_1} \setminus B_{r_1}) \times (t_1, t_2) \),
\[
w_\varepsilon \geq 0 \quad \text{in } (B_{R_1} \setminus B_{r_1}) \times (t_1, t_2)
\]
\[
\Rightarrow \quad \begin{cases}
  w - u + \varepsilon|x|^{2-n} \geq 0 & \forall r_1 \leq |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{if } n \geq 3, \\
  w - u + \varepsilon \log(R_0/|x|) \geq 0 & \forall r_1 \leq |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{if } n = 2
\end{cases}
\]

\[
\Rightarrow \quad w \geq u \quad \forall 0 < |x| \leq R_1, t_1 \leq t \leq t_2 \quad \text{as } r_1 \to 0, \varepsilon \to 0.
\]

Similarly, by considering the function
\[
v_\varepsilon = \begin{cases}
  w - u - \varepsilon|x|^{2-n} & \text{if } n \geq 3, \\
  w - u - \varepsilon \log(R_1/|x|) & \text{if } n = 2
\end{cases}
\]
and applying the maximum principle and letting \( \varepsilon \to 0 \), we get
\[
w \leq u \quad \forall 0 < |x| \leq R_1, t_1 \leq t \leq t_2.
\]

By (26) and (27) we get (24), and Theorem 1 and Theorem 2 follow. \( \square \)

References


