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FINITELY GENERATED SUBGROUPS OF LATTICES IN $\mathrm{PSL}_2\mathbb{C}$

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Abstract. Let Γ be a lattice in $\mathrm{PSL}_2(\mathbb{C})$. The pro-normal topology on Γ is defined by taking all cosets of nontrivial normal subgroups as a basis. This topology is finer than the pro-finite topology, but it is not discrete. We prove that every finitely generated subgroup $\Delta < \Gamma$ is closed in the pro-normal topology. As a corollary we deduce that if H is a maximal subgroup of a lattice in $\mathrm{PSL}_2(\mathbb{C})$, then either H is of finite index or H is not finitely generated.

1. Introduction

In most infinite groups, it is virtually impossible to understand the lattice of all subgroups. Group theorists therefore focus their attention on special families of subgroups such as finite index subgroups, normal subgroups, or finitely generated subgroups. This paper will study the family of finitely generated subgroups of lattices in $\mathrm{PSL}_2(\mathbb{C})$. Recall that lattices in $\mathrm{PSL}_2(\mathbb{C})$ are the fundamental groups of finite volume hyperbolic 3-orbifolds. For these groups we establish a connection between finitely generated subgroups and normal subgroups. This connection is best expressed in the following topological terms.

If a family of subgroups $\mathcal{N}$ is invariant under conjugation and satisfies the condition,

$$\text{for all } N_1, N_2 \in \mathcal{N} \ \ \exists\, N_3 \in \mathcal{N} \text{ such that } N_3 \leq N_1 \cap N_2, \tag{1}$$

then one can define an invariant topology in which the given family of groups constitutes a basis of open neighborhoods for the identity element. The most famous example is the *pro-finite* topology, which is obtained by taking $\mathcal{N}$ to be the family of (normal) finite index subgroups. When the family of all nontrivial normal subgroups satisfies condition (1), we refer to the resulting topology as the *pro-normal* topology. The pro-normal topology is usually much finer than the pro-finite topology.

With this terminology we can state the main theorem.

Theorem 1.1. *Let Γ be a lattice in $\mathrm{PSL}_2(\mathbb{C})$. Then the pro-normal topology is well defined on Γ and every finitely generated subgroup $\Delta < \Gamma$ is closed in this topology. Moreover if Δ is of infinite index, then it is the intersection of open subgroups strictly containing Δ.*

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Recall that a subgroup is open in the pro-normal topology if and only if it contains a nontrivial normal subgroup, an open subgroup is automatically closed, and a group is closed in the pro-normal topology if and only if it is the intersection of open subgroups (see Remark 3.4). From this it follows that the second statement of Theorem 1.1 is stronger only if Δ is open. As we will see in the proof, "most" finitely generated subgroups are not open. Nevertheless, the stronger version is used below to prove quickly Corollary 1.2.

It is a well known question whether or not Theorem 1.1 remains true for the coarser pro-finite topology. Groups in which every finitely generated subgroup is closed in the pro-finite topology are called *locally extended residually finite*, or *LERF* for short. The list of groups that are known to be LERF is rather short. All lattices in $\mathrm{PSL}_2(\mathbb{R})$ (see [11]) are LERF. Recently the family of lattices in $\mathrm{PSL}_2(\mathbb{C})$ that were known to be LERF (see for example [6]) was significantly expanded to include all Bianchi groups, that is, lattices of the form $\mathrm{PSL}_2(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers in a totally imaginary field of degree two over $\mathbb{Q}$. This follows from the work of Agol, Long and Reid [2] combined with the recently proved tameness conjecture [1, 3]. In a completely different direction it was recently proved by Grigorchuk and Wilson that the first Grigorchuk group has the LERF property [7].

Let Γ be a group and $\Delta < \Gamma$ a subgroup. We say that Δ is a *maximal* subgroup if there is no subgroup Σ such that $\Delta \lneqq \Sigma \lneqq \Gamma$. In [9], Margulis and Soĭfer prove that every finitely generated linear group that is not virtually solvable admits a maximal subgroup of infinite index. On various occasions both Margulis and Soĭfer asked the question, in which cases can such maximal subgroups of infinite index be finitely generated? Easy examples of finitely generated maximal subgroups such as $\mathrm{PSL}_2(\mathbb{Z}) < \mathrm{PSL}_2(\mathbb{Z}[1/p])$ indicate that one should be careful about the exact phrasing of the question. Soĭfer suggested the setting of lattices in simple Lie groups, and indeed even for $\mathrm{PSL}_3(\mathbb{Z})$ the question is wide open.

Let Γ be a group and $\Delta < \Gamma$ a subgroup. We say that Δ is *pro-dense* if Δ is a proper subgroup and $\Delta N = \Gamma$ for every normal subgroup $\mathrm{id} \neq N \lhd \Gamma$. When the pro-normal topology on Γ is well defined, pro-dense subgroups are exactly the subgroups that are dense in this topology. In [5] it is proven that if Γ is a finitely generated linear group with simple Zariski closure or if it is a nonelementary hyperbolic group with no finite normal subgroups, then Γ admits a pro-dense subgroup. Note that all lattices in $\mathrm{PSL}_2(\mathbb{C})$ satisfy the first hypothesis, and uniform lattices satisfy both hypotheses.

In groups that have few quotients such as simple groups it can be relatively easy to construct pro-dense subgroups. This becomes more of a challenge in groups that have many normal subgroups, such as hyperbolic groups. It is conjectured in [5] that a pro-dense subgroup of a hyperbolic group cannot be finitely generated.

Our main theorem above answers these two questions for the special case of lattices in the group $\mathrm{PSL}_2(\mathbb{C})$.

Corollary 1.2. *Let Γ be a lattice in $\mathrm{PSL}_2(\mathbb{C})$ and $\Delta < \Gamma$ a maximal subgroup of infinite index or a pro-dense subgroup. Then Δ cannot be finitely generated.*

Proof. In both cases the only open subgroup strictly containing Δ is Γ itself. If Δ is maximal, this is true by definition, even without the openness condition. If Δ is pro-dense, then it is dense in the pro-normal topology and it cannot be contained in an open (and hence closed) proper subgroup. Thus Δ cannot be finitely generated or it would violate the conclusion of Theorem 1.1. $\square$

The main new tool used to prove Theorem 1.1 and Corollary 1.2 is the recent resolution of the Marden conjecture [1, 3]. This powerful result is very specific to hyperbolic 3-space. It therefore seems likely that new techniques will be required to tackle these questions for other simple Lie groups.

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2. Geometrically finite groups

The following proposition plays a central role in the proof. Ashot Minasyan [10] proved an analogus statement for hyperbolic groups, which implies our statement in the special case where Γ is a uniform lattice. Nevertheless the proof below works in both the uniform and nonuniform case:

Proposition 2.1. *Let $\Gamma < Isom^+(\mathbb{H}^m)$ be a lattice (for $m > 2$). Let $\Delta < \Gamma$ be a geometrically finite subgroup of infinite index. Then there exists a characteristic subgroup $id \neq N \lhd \Gamma$ such that $N \cap \Delta = id$.*

By Selberg's lemma, there exists a finite index torsion-free characteristic subgroup $\Gamma^{\mathrm{tf}} \leq \Gamma$. (First use Selberg's lemma to find an index $k < \infty$ torsion-free subgroup; then take the intersection of all index k subgroups to obtain Γ^{tf}. Let M be the hyperbolic manifold $\mathbb{H}^m/\Gamma^{\mathrm{tf}}$.) For a hyperbolic element $g \in \Gamma^{\mathrm{tf}}$, let $g^* \subset M$ denote the closed geodesic corresponding to the conjugacy class of g. Let $A_g \subset \mathbb{H}^m$ denote the axis of the isometry g. Define the torsion free subgroup $\Delta^{\mathrm{tf}} := \Delta \cap \Gamma^{\mathrm{tf}}$.

To handle lattices with torsion, it will be necessary to consider the lattice $\mathrm{Aut} := \mathrm{Aut}(\Gamma^{\mathrm{tf}}) < \mathrm{Isom}^+(\mathbb{H}^m)$, which by Mostow rigidity is a finite extension of Γ, and the finite group $\mathrm{Out} := \mathrm{Out}(\Gamma^{\mathrm{tf}})$ of outer automorphisms of Γ^{tf} acting isometrically on M. The reader is encouraged to assume on a first reading that Γ is torsion free and $\mathrm{Out}(\Gamma)$ is the trivial group.

Lemma 2.2. *There exists a primitive hyperbolic element $\gamma \in \Gamma$ whose axis A_γ cannot be translated into the convex hull of Δ by an element of Aut. Moreover, there exist constants $\eta, L > 0$ such that: for any element $\phi \in Aut$ and any hyperbolic element $\delta \in \Delta$ the intersection*

$$\mathcal{N}_\eta A_\delta \cap A_{\phi.\gamma},$$

where $\mathcal{N}_\eta A_\delta$ is a radius η neighborhood of A_δ, is a (possibly empty) geodesic segment of length less than L.

Proof. The lattice Γ is finite index in Aut. It suffices to prove the theorem in the case where $\Gamma = \mathrm{Aut}$.

Let $\mathrm{Hull} \subset \mathbb{H}^m$ be the convex hull of Δ. Consider the Γ-invariant set $\mathcal{S} = \{h.\mathrm{Hull}\}_{h\Delta \in \Gamma/\Delta}$ of isometric copies of Hull under the action of Γ. As Γ is discrete and Δ is geometrically finite, any bounded open set $\mathcal{O} \subset \mathbb{H}^m$ intersects a finite set $\mathcal{S}_{\mathcal{O}}$ of elements of $\mathcal{S}$. It also follows from geometric finiteness that the limit set $\Lambda \subset \partial\mathbb{H}^m$ of Δ has measure zero. So for any fnite set $\{h_i\}_{h_i \in \Gamma}$ the union $\bigcup_i h_i.\Lambda$ will be a closed measure zero subset of the boundary sphere $\partial\mathbb{H}^m$. We can therefore choose a bi-infinite geodesic of $\mathbb{H}^m$ passing through $\mathcal{O}$ with endpoints outside $\bigcup_{h.\mathrm{Hull} \in \mathcal{S}_{\mathcal{O}}} h.\Lambda$. By perturbing the endpoints of the geodesic slightly we can assume it is the axis of a primitive hyperbolic element $\gamma \in \Gamma$.

The second step is to prove that the constants $\eta, L > 0$ exist for the element γ. Suppose these constants do not exist. Then there exist sequences $\eta_i \searrow 0$ (assume $\eta_i < 1$), $L_i \to \infty$, elements $g_i \in \Gamma$, and hyperbolic elements $\delta_i \in \Delta$ such that

$$\mathcal{N}_{\eta_i} A_{\delta_i} \cap A_{g_i \gamma g_i^{-1}}$$

is a geodesic segment σ_i of length L_i. Let $\mathcal{F} \subset \mathbb{H}^m$ be the compact subset of a fundamental domain for Δ given by points within distance 1 from the convex hull of Δ, which cover a point in $\mathbb{H}^m/\Delta^{\mathrm{tf}}$ of injectivity radius at least $r/2$. For each i there is an $h_i \in \Delta$ such that h_i moves the midpoint of the geodesic segment σ_i into $\mathcal{F}$. Since the group Γ is discrete and $\mathcal{F} \subset \mathbb{H}^3$ is compact, we may pass to a subsequence where the segments $h_i.\sigma_i$ all lie on a common bi-infinite geodesic σ. Since $\eta_i \to 0$, the geodesic σ must lie in the convex core of Δ. But σ is the axis of a conjugate of γ (in the group Γ). This contradicts the first part of the proof and completes the proof of the lemma. □

Now fix a hyperbolic element $\gamma \in \Gamma^{\mathrm{tf}}$ satisfying Lemma 2.2. Let $\ell > 0$ be the length of the closed geodesic $\gamma^* \subset M$. Let $\mathrm{Out}.\gamma^*$ denote the immersed 1-manifold $\bigcup_{\phi \in \mathrm{Out}} \phi(\gamma^*)$. An annoying detail is that $\mathrm{Out}.\gamma^* \subset M$ may not be an embedded 1-manifold. Nonetheless, there are constants $\rho > 0$, $c \in (0,1)$, and a finite union of embedded intervals $\{I_\alpha\}_\alpha \subseteq \gamma^* \subset M$ of total length at least $c \cdot \ell$ such that: a ρ-tubular neighborhood of I_α is embedded and disjoint from a ρ-tubular neighborhood of $\phi(I_\beta)$ whenever $\phi(I_\beta) \neq I_\alpha$ as subsets of M.

Define $N_n \leq \Gamma^{\mathrm{tf}}$ to be the subgroup generated by the set

$$\bigcup_{\phi \in \mathrm{Aut}} \phi.\gamma^n.$$

Notice that N_n is a characteristic subgroup of Γ^{tf}. Since Γ^{tf} is a characteristic subgroup of Γ, N_n is also a characteristic subgroup of Γ. Note that N_n is torsion free.

We are now prepared to prove Proposition 2.1. The argument is similar to [8, 5.5F]. We will show that for a sufficiently large number n, the group $N = N_n$ has a trivial intersection with Δ.

Recall that ℓ is the length of the closed geodesic $\gamma^* \subset M$. Recall the constants η and L from Lemma 2.2. Define $\varepsilon := \min\{\rho, \eta\}$. Fix an integer n_0 satisfying

$$c\ell \cdot n_0 > 2\pi/\varepsilon + L.$$

Pick any $n > n_0$. Suppose that $\Delta \cap N_n$ is nontrivial. Since $N_n \leq \Gamma^{\mathrm{tf}}$ is torsion free and $\Delta^{\mathrm{tf}} \leq \Delta$ is a finite index subgroup, it follows that there is a nontrivial element $\delta \in \Delta^{\mathrm{tf}} \cap N_n$.

For clarity let us break the proof into two cases, depending on whether or not Γ has parabolic elements.

Case 1 (Assume that Γ has no parabolic elements). Topologically the existence of an element $\delta \in \Delta^{\mathrm{tf}} \cap N_n \leq \Gamma^{\mathrm{tf}}$ says there is a compact genus zero surface S with $(k+1)$ circular boundary components $C_0, C_1, \ldots, C_k$ and a map

$$f : (S, C_0, C_1, \ldots, C_k) \longrightarrow (M, \delta^*, \phi_1(\gamma^*), \ldots, \phi_k(\gamma^*))$$

such that $\phi_j \in \mathrm{Out}$, $f : C_0 \longrightarrow \delta^*$ is a parametrization of δ^* and $f : C_j \longrightarrow \phi_j(\gamma^*)$ $(j > 0)$ wraps $i_j \cdot n$ times around $\phi_j(\gamma^*)$ (possibly in the reverse direction) for some

$i_j \in \mathbb{N}$. Let us assume we have found such a surface S with the minimal number of boundary components.

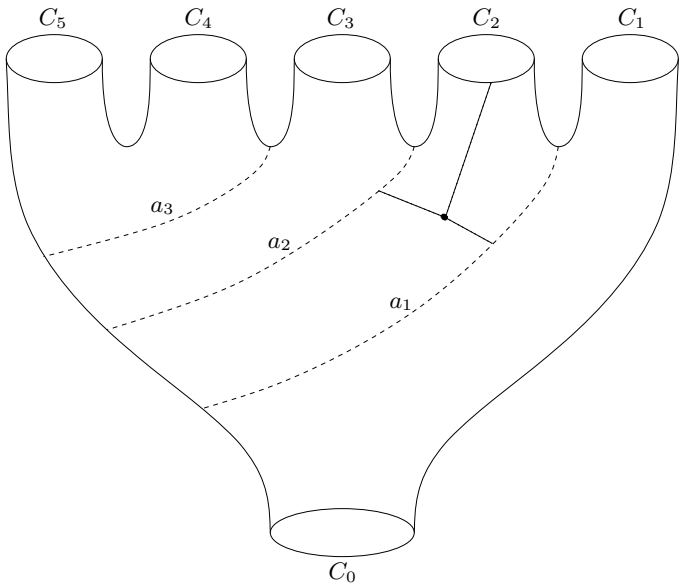

FIGURE 1. The surface S

We would now like to alter f by a homotopy rel boundary to form a pleated surface (or a minimal surface). Consider Figure 1. Add simple closed curves $\{a_1, a_2, \ldots, a_{k-2}\}$ to S as shown in the figure, forming a pants decomposition of S. If for some j the curve $f(a_j)$ were homotopically trivial, then we could cut along a_j, add a disk, and find a surface with fewer boundary components than S. Therefore by minimality, each curve $f(a_j) \subset M$ is homotopic to a closed geodesic a_j^*. By another minimality argument one can show that for $1 \leq j \leq k-1$ the curve $f(a_j)$ is not freely homotopic into any of the closed geodesics $\{\phi(\gamma^*)\}_{\phi \in \mathrm{Out}}$.

For $1 \leq j \leq k-3$ consider the pants bounded by a_j, a_{j+1}, and C_{j+1}. Add a basepoint to the jth pair of pants P_j and paths to the boundary components as shown in the figure for $j = 1$. These (together with a choice of orientation) make each boundary component a_j, a_{j+1}, and C_{j+1} a well defined element $[a_j], [a_{j+1}]$, and $[C_{j+1}]$ respectively of the fundamental group. Suppose $f_*([a_j])$ and $f_*([a_{j+1}])$ are contained in a common cyclic subgroup of $\pi_1(M)$. Then $f_*([C_{j+1}])$ would be contained in the same cyclic subgroup. In this case $f(a_j)$ would be freely homotopic into $\phi_{j+1}(\gamma^*)$, which was ruled out in the previous paragraph. Therefore $f_*([a_j])$ and $f_*([a_{j+1}])$ are not contained in a common cyclic subgroup of $\pi_1(M)$. A similar minimality argument shows that any pair of the triple a_{k-2}, C_{k-1}, and C_k is not mapped by f into a common cyclic subgroup of $\pi_1(M)$.

With this information we may "pull f tight" along the curves $a_1, a_2, \ldots, a_{k-2}$ to the closed geodesics $a_1^*, a_2^*, \ldots, a_{k-2}^* \subset M$, and then use ideal hyperbolic triangles to map each pair of pants into M as a pleated surface. (See Thurston's description of this in [13, Sec. 2]. Alternatively one can map each pair of pants to a minimal surface with geodesic boundary.) Let us denote the resulting map again by f. There is a hyperbolic metric on S with geodesic boundary components such that

f maps paths in S to paths in M of the same length. Let us now use S to denote the surface equipped with this hyperbolic metric.

Recall that on the embedded intervals $\{I_\alpha\}_\alpha \subset \gamma^*$ of total length at least $c\ell$, the ρ-tubular neighborhoods of the intervals $\{\phi(I_\alpha)\}_{\phi\in\mathrm{Out}}$ are pairwise disjoint and embedded. This implies that at least $100c$ percent of C_1 (i.e. curves of total length at least $ci_1n\ell$) is at least a distance 2ρ from $\bigcup_{i>1} C_i$.

For $0 \le i < k$ let $\sigma_i \subset S$ denote the geodesic segment in S of shortest length joining C_i to C_k. Note that by minimality, the σ_i are pairwise disjoint. Cut S along the segments $\{\sigma_i\}_{0\le i<k}$ to form the simply connected surface cut$S$. Lift f to a map

$$\widetilde{f} : \mathrm{cut}S \longrightarrow \mathbb{H}^m.$$

Let $p \in C_1$ be the point closest to C_0. By Lemma 2.2, the (possibly disconnected) geodesic segment $\widetilde{f}(C_1 - B_S(p, L))$ is entirely outside an ε-neighborhood of $\widetilde{f}(C_0)$. Therefore an ε-neighborhood of the geodesic segment $C_1 - B_S(p, L)$ is disjoint from C_0.

Perform the above procedure for each boundary component C_i $(2 \le i \le k)$, where the role of C_k above is replaced with C_1. This shows that the hyperbolic area of an ε-neighborhood of $\bigcup_{i>0} C_i$ is at least

$$(nc\ell - L) \cdot \varepsilon \cdot k.$$

By Gauss-Bonnet, the area of S is $-2\pi(2 - k - 1) = 2\pi k - 2\pi$. (In the case of a minimal surface, the area is at most this number.) Therefore

$$2\pi k \ge nc\ell\varepsilon k - L\varepsilon k$$

yielding

$$nc\ell \le 2\pi/\varepsilon + L.$$

This contradicts our choice of $n > n_0$. This proves (2) under the assumption that Γ contains no parabolic elements.

Case 2 (Γ contains parabolic elements). If δ is a parabolic isometry, then S is replaced with a genus zero surface with k boundary components and one puncture. The map f then maps the end of S out the cusp of M corresponding to δ.

Proceeding as in Case (1), add the curves a_j to S. It may happen that a curve $f(a_j)$ represents a parabolic element of Γ. In this case cut S along the first such curve a_j and map the resulting surface into M, taking the curve $f(a_j)$ out the corresponding cusp of M. Proceed with the above argument treating the new boundary curve a_j as the "last" boundary curve of the pleated surface. Redefine $k = j$, and call this Case (2b).

If δ is parabolic, then the argument of Case (1) is even easier. Intersecting C_0 is no longer a problem, and at least $100c$ percent of each C_i, $i > 0$, has an embedded tubular neighborhood of radius $\rho < \varepsilon$. So an ε-neighborhood of $\bigcup_{i>0} C_i$ has area at least $nc\ell\varepsilon k$. By Gauss-Bonnet the area of S is either $-2\pi(2-k-1)$ or $-2\pi(2-k-2)$ in Case (2b). Therefore $2\pi k \ge nc\ell\varepsilon k$, implying $nc\ell \le 2\pi/\varepsilon$. This is again a contradiction.

Finally, if δ is not parabolic and we are in Case (2b), then for $0 \le i \le k$ let $\sigma_i \subset S$ be the semi-infinite embedded geodesic hitting C_i orthogonally and heading out the cusp corresponding to a_k. Again by minimality, the σ_i are pairwise disjoint.

Cut S along the σ_i to form a simply connected surface $\mathrm{cut}S$. Lift f to a map

$$\widetilde{f} : \mathrm{cut}S \to \mathbb{H}^m.$$

For $p_i \in C_i$ the closest point to C_0, Lemma 2.2 again tells us the geodesic segment $\widetilde{f}(C_i - B_S(p_i, L))$ is entirely outside an ε-neighborhood of $\widetilde{f}(C_0)$. As before the hyperbolic area of an ε-neighborhood of $\bigcup_{i>0} C_i$ is at least $(nc\ell - L)\varepsilon k$, violating Gauss-Bonnet.

3. Finitely generated subgroups

We begin by quoting the two main theorems from hyperbolic geometry which we will use.

Theorem 3.1 (Marden Conjecture [1, 3])**.** *Let Δ be a discrete finitely generated torsion-free subgroup of $\mathrm{PSL}_2\,\mathbb{C}$. Then the hyperbolic manifold $\mathbb{H}^3/\Delta$ is topologically tame; i.e., it is homeomorphic to the interior of a compact manifold with (possibly empty) boundary.*

Theorem 3.2 (Covering Theorem [12, 4])**.** *Let $\Gamma < \mathrm{PSL}_2\,\mathbb{C}$ be a torsion-free lattice. Let $\Delta < \Gamma$ be a subgroup such that $\mathbb{H}^3/\Delta$ is topologically tame. Then either*
(1) Δ is geometrically finite or
(2) there exists a finite index subgroup $H \leq \Gamma$ and a subgroup $F \leq \Delta$ of index one or two such that H splits as a semidirect product $F \rtimes \mathbb{Z}$.

We may now combine Theorems 3.1 and 3.2 with Selberg's lemma to prove a strengthened version of the Covering Theorem for lattices with torsion.

Corollary 3.3. *Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a lattice. Let $\Delta < \Gamma$ be a finitely generated subgroup. Then either*
(1) Δ is geometrically finite or
(2) there exists a torsion-free finite index subgroup $H \leq \Gamma$ and a finite index normal subgroup $F \trianglelefteq \Delta$ such that H splits as a semidirect product $F \rtimes \mathbb{Z}$.

Proof. Assume Δ is not geometrically finite. Apply Selberg's lemma to obtain a torsion-free finite index normal subgroup $\Gamma^{\mathrm{tf}} \leq \Gamma$. The subgroup $\Delta \cap \Gamma^{\mathrm{tf}}$ is normal torsion-free and of finite index in Δ. Apply Theorems 3.1 and 3.2 to the pair Γ^{tf} and $\Delta \cap \Gamma^{\mathrm{tf}}$.

We know $\Delta \cap \Gamma^{\mathrm{tf}}$ is not geometrically finite. So there is a finite index subgroup $H \leq \Gamma^{\mathrm{tf}}$ and a subgroup $F \leq \Delta \cap \Gamma^{\mathrm{tf}}$ of index one or two such that H splits as a semidirect product $F \rtimes \mathbb{Z}$. Since its index is at most two, F is in fact a characteristic subgroup of $\Delta \cap \Gamma^{\mathrm{tf}}$, implying it is a normal subgroup of Δ. This proves the corollary. □

We are now ready to prove our main theorem, which we restate here for convenience.

Theorem 1.1. *Let Γ be a lattice in $\mathrm{PSL}_2(\mathbb{C})$. The pro-normal topology is well defined on Γ, and every finitely generated subgroup $\Delta < \Gamma$ is closed in this topology. Moreover if Δ is of infinite index, then it is the intersection of open subgroups strictly containing Δ.*

Remark 3.4. Before beginning the proof let us recall why a subgroup is closed in the pro-normal topology if and only if it is the intersection of open subgroups. One direction is obvious: an open subgroup is also closed because its complement is

a union of open cosets. Conversely, applying the definition of the topology, a set $S \subseteq \Gamma$ is open if and only if

$$S = \bigcup_{\mathrm{id} \neq N \lhd \Gamma} \bigcup_{\{\gamma \mid \gamma N \subset S\}} \gamma N.$$

(Pick an open neighborhood in S about each point in S to obtain the above union.) Thus Δ is closed if and only if it is of the form

$$\Delta = \bigcap_{\mathrm{id} \neq N \lhd \Gamma} \bigcup_{\{\gamma \mid \gamma N \cap \Delta \neq \emptyset\}} \gamma N = \bigcap_{\mathrm{id} \neq N \lhd \Gamma} \Delta N.$$

The last equality is just because

$$\bigcup_{\{\gamma \mid \gamma N \cap \Delta \neq \emptyset\}} \gamma N = \bigcup_{\{\delta \in \Delta\}} \delta N \stackrel{\text{def}}{=} \Delta N.$$

When Δ is a subgroup, ΔN is an open subgroup for every fixed $\mathrm{id} \neq N \lhd \Gamma$, which establishes the remark.

Proof. Let $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ be a lattice and $\Delta < \Gamma$ a finitely generated subgroup. We first have to prove that the pro-normal topology is well defined on Γ. In search of a contradiction, assume that N_1 and N_2 are two nontrivial normal subgroups with $N_1 \cap N_2 = \mathrm{id}$. By Borel's density theorem Γ is Zariski dense in $\mathrm{PSL}_2(\mathbb{C})$. Since $\mathrm{PSL}_2(\mathbb{C})$ is simple, N_1 and N_2 are also Zariski dense. But $[N_1, N_2] < N_1 \cap N_2 = \mathrm{id}$, and this commutativity extends to the Zariski closure, proving that $\mathrm{PSL}_2(\mathbb{C})$ is commutative, which is absurd.

When Δ is of finite index, it is clearly closed in the pro-normal topology and even in the pro-finite topology. Thus from here on we will assume that Δ is of infinite index and we will prove the second, stronger, assertion of Theorem 1.1. We distinguish between two cases:

Case 1 (Δ is geometrically finite). Recall that we are assuming that Δ is of infinite index. In this case Δ cannot contain a nontrivial normal subgroup of Γ. This is because the limit set of a geometrically finite subgroup has measure zero, whereas Γ and a nontrivial normal subgroup of Γ have the same limit set, which is the whole sphere at infinity.

We will prove that

$$\Delta = \overline{\Delta} \stackrel{\text{def}}{=} \bigcap_{\{N \lhd \Gamma, N \not\leq \Delta\}} \Delta N. \tag{2}$$

Assume by way of contradiction that $\overline{\Delta} \neq \Delta$ and let $\gamma \in \overline{\Delta} \setminus \Delta$. Applying Proposition 2.1, we find an infinite normal subgroup $N \lhd \Gamma$ such that $\Delta \cap N = \mathrm{id}$, and therefore $\Delta N = \Delta \ltimes N$. By the definition of $\overline{\Delta}$ we know that $\gamma \in \overline{\Delta} \leq \Delta N$. Let $\gamma = \delta n$ be the unique way of factoring γ into a product with $n \in N$ and $\delta \in \Delta$. Since Γ is a finitely generated linear group it is residually finite, and we can find a normal finite index subgroup $N' \lhd \Gamma$ such that $n \notin N'$. Consider the group $M = N \cap N'$. M is nontrivial and intersects Δ trivially. Using the definition of $\overline{\Delta}$ again, $\gamma \in \overline{\Delta} \leq \Delta M$, so we can write $\gamma = \delta' m$ with $\delta' \in \Delta, m \in M$. But by construction, $n \notin M$, so $n \neq m$. This contradicts the uniqueness of the factorization $\gamma = \delta n$.

Case 2 (Δ is geometrically infinite). In this case we argue that Δ is pro-finitely closed in Γ, i.e. that it is the intersection of subgroups of finite index. This will prove the theorem because every finite index subgroup contains a normal subgroup (making it open in the pro-normal topology) and because Δ, being of infinite index, cannot be equal to a finite index subgroup.

Let us find groups H and F as in Corollary 3.3. Let $G = \langle H, \Delta \rangle$ be the group generated by H and Δ. Since G is of finite index in Γ, it is enough to show that Δ is closed in the pro-finite topology on G. Factoring out by the normal subgroup $F \lhd G$, it is enough to show that the finite subgroup Δ/F is pro-finitely closed in G/F. Note that G/F is virtually cyclic and in particular it is residually finite.

It remains only to observe that any finite subgroup is closed in the pro-finite topology on a residually finite group. Indeed in such a group, by definition, the trivial subgroup is closed and hence so is every singleton. A finite group, as a finite union of singletons, must therefore also be closed. □

Remark 3.5. One can also define the pro-characteristic topology as the topology generated by all the nontrivial characteristic subgroups of Γ. In general this topology is coarser than the pro-normal topology but still finer than the pro-finite topology. Going over the proof it is not difficult to see that we have in fact proved the stronger statement that every finitely generated subgroup $\Delta < \Gamma$ is closed in the pro-characteristic topology on Γ. But this is of no consequence because the two topologies actually coincide for lattices in $\mathrm{PSL}_2(\mathbb{C})$. Indeed assume that $N \lhd \Gamma$ is a normal subgroup and consider the characteristic subgroup

$$\bigcap_{\phi \in \mathrm{Out}(\Gamma)} \phi N.$$

By Mostow rigidity $\mathrm{Out}(\Gamma)$ is finite. Using the fact that the pro-normal topology is well defined, expressed in equation (1), such a subgroup cannot be trivial. Hence every normal subgroup is open in the pro-characteristic topology.

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