ON THE GLAUBERMAN CORRESPONDENT OF A BLOCK

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Abstract. In this paper, we analyze the compatibility of Fong’s reduction and the Glauberman correspondence of characters and then clarify that the $p$-solvable hypothesis in a paper of Harris and Linckelmann is not necessary.

Let $O$ be a complete discrete valuation ring with an algebraically closed residue field $k$ of characteristic $p$ and a quotient field $K$ of characteristic 0. In addition, $K$ is also assumed to be big enough for all finite groups that we consider below. Let $G$ be a finite group. We denote by $\text{Irr}(G)$ the set of all irreducible characters of $G$. Let $A$ be another finite group acting on $G$. Then clearly $A$ acts on $\text{Irr}(G)$. The following theorem is due to Glauberman [4] and relates the $A$-fixed set $(\text{Irr}_K(G))^A$ with the set $\text{Irr}_K(C_G(A))$.

Theorem 1. For any $A$-group $G$ with $A$ solvable and $\lvert G \rvert$ and $\lvert A \rvert$ coprime, there is a bijection $\pi(G, A) : \text{Irr}_K(G)^A \to \text{Irr}_K(C_G(A))$ such that

1.1. For any normal subgroup $B$ of $A$, the bijection $\pi(G, B)$ maps $(\text{Irr}_K(G))^A$ to $(\text{Irr}_K(C_G(B)))^A$, and we have $\pi(G, A) = \pi(C_G(B), A/B) \circ \pi(G, B)$ on $(\text{Irr}_K(G))^A$.

1.2. If $A$ is a $q$-group for some prime $q$, for any $\chi \in (\text{Irr}_K(G))^A$ the corresponding irreducible character $\pi(G, A)(\chi)$ of $C_G(A)$ is the unique irreducible constituent of $\text{Res}^G_{C_G(A)}(\chi)$ occurring with a multiplicity prime to $q$.

2. Continue to keep the notation in Theorem 1. Let $b$ be a block idempotent of $OG$. We denote by $\text{Irr}(G, b)$ the set of all irreducible characters of $G$ provided by some $KGb$-module. If $A$ centralizes a defect group $P$ of $b$ by [3] Prop. 1 and Th. 1], $A$ stabilizes all characters of $\text{Irr}(G, b)$ and there is a unique block idempotent $c$ of $O_{C_G(A)}$ such that $\text{Irr}(C_G(A), c) = \pi(G, A)(\text{Irr}(G, b))$; moreover $b$ and $c$ are perfectly isometric (refer to [1]). Such a block idempotent $c$ is called the Glauberman correspondent of $b$ (see [9]). A perfect isometry between blocks is considered to be the character-theoretic ‘shadow’ of a derived equivalence. So it is very interesting to ask whether there is such a derived equivalence between a block and its Glauberman correspondent. More strongly, with the $p$-solvable hypothesis, Harris and Linckelmann proved that a block and its Glauberman correspondent are Morita equivalent. In this paper, we analyze the compatibility of Fong’s reduction and the Glauberman correspondence of characters and then clarify that the $p$-solvable hypothesis in [5] is not necessary.
Theorem 3. Let $G$ be an $A$-group with $A$ solvable and $|G|$ and $|A|$ be co-prime. Let $b$ be a block idempotent of $OG$ with a defect group $P$ centralized by $A$ and let $c$ be the Glauberman correspondent of $b$ (refer to [9]). If $G = O_P(G)C_G(A)$, then there is an indecomposable $O(G \times C_G(A))$-module $M$ having the following properties:

3.1. The $p$-subgroup $\Delta(P) = \{(x, x)|x \in P\}$ of $G \times C_G(A)$ is a vertex of $M$.

3.2. $M$ induces a Morita equivalence between $OGb$ and $O C_G(A)c$, and the bijection between $\operatorname{Irr}(G, b)$ and $\operatorname{Irr}(C_G(A), c)$ induced by this Morita equivalence coincides with the Glauberman correspondence from $\operatorname{Irr}(G, b)$ to $\operatorname{Irr}(C_G(A), c)$.

4. Firstly we introduce some notation and terminology used in this paper. Let $S$ be a set and $H$ be a group acting on $S$. For any $h \in H$ and $s \in S$, we write the action of $h$ on $s$ as $h \cdot s$. Let $A$ be an $O$-algebra with the identity element. We denote by $A^*$ the multiplicative group of all invertible elements of $A$. Let $K$ be a finite group. A group $\hat{K}$ is an $O^*$-group with the $O^*$-quotient $\hat{K}$ if there is an injective group homomorphism $\rho : O^* \rightarrow Z(\hat{K})$ such that $\hat{K}/\rho(O^*) \cong K$.

Then $\hat{K}$ can be regarded as a central extension of $K$ by $O$, and, in this case, we denote by $O, \hat{K}$ the twisted group algebra corresponding to this central extension (see [10]) for any $\lambda \in O^*$ and $x \in \hat{K}$, we write the product $\rho(\lambda)x$ as $\lambda \cdot x$ for convenience. For any subgroup $L$ of $K$, we denote by $\hat{L}$ the inverse image of $L$ through the canonical surjective homomorphism $\hat{K} \rightarrow K$; then $\hat{L}$ is an $O^*$-group with the $O^*$-quotient $\hat{L}$. The group $\hat{K}$ with another injective homomorphism $O^* \rightarrow Z(\hat{K}), \lambda \mapsto \rho(\lambda^{-1})$ is also an $O^*$-group with the $O^*$-quotient $\hat{K}$. We call this $O^*$-group the opposite $O^*$-group of $\hat{K}$ and denote it by $\hat{K}^o$. Let $\hat{H}$ be an $O^*$-group with the $O^*$-quotient $\hat{H}$. A homomorphism $\theta : \hat{K} \rightarrow \hat{H}$ is called a homomorphism of $O^*$-groups if for any $\lambda \in O^*$ and $x \in \hat{K}$, we have $\theta(\lambda \cdot x) = \lambda \cdot \theta(x)$. Below, we will occasionally use the induction of interior $K$-algebras, the Brauer quotients and the Brauer homomorphisms; for their definitions and corresponding notation, readers can refer to [10]; by the word “character”, we sometimes indicate a character on a group and sometimes indicate a character on an algebra, but that does not cause any confusion.

Next we begin to analyze the compatibility of Fong’s reduction and the Glauberman correspondence of characters. Let $G$ be a finite group, $b$ be a block idempotent of $OG$ and $P$ be a defect group of $b$. Let $A$ be another finite group acting on $G$. We assume that $|G|$ and $|A|$ are coprime, $O_P(G) \cap C_G(A) = O_P(C_G(A))$, $A$ is cyclic with the order a power of some prime $q$ and $A$ centralizes $P$. Let $c$ be the Glauberman correspondent of $b$. Set $C = C_G(A)$ and let $R$ be a Sylow $p$-subgroup of $C$ containing $P$.

5. Let $I$ be the set of all block idempotents $f$ of $OQ_P(G)$ such that $bf \neq 0$. The action of $A$ on $O_P(G)$ induces an action of $A$ on $OQ_P(G)$, which again induces an action of $A$ on $I$. We consider the action of the semidirect $G \rtimes A$ defined by the equality $g \cdot f = gfg^{-1}$, where $g \in G$ and $f \in I$, together with the action of $A$ on $I$. By this action, it is well known that $G$ acts transitively on $I$; moreover since $|G|$ and $|A|$ are co-prime, by [3] Lemma 13.8 and Cor. 13.9, the set $I^A$ of $A$-fixed points in $I$ is non-empty and $C_G(A)$ acts transitively on $I^A$. We denote by $K$ the stabilizer of $f$ in $G$ and set $Tr^G_K(f) = \sum_{x \in G/K} xfx^{-1}$, where $G/K$ is a set of representatives of right cosets of $K$ in $G$. Clearly $ff^x = 0$ for any $x \in G - K$;
defined by these two actions of 

\[3, \text{Lemma 13.8 and Cor. 13.9}\], the 

\[J\]

on 

\[\mathbb{Z}\]

is an isomorphism. Obviously \(\mathbb{Z}\) and \(\mathbb{Z}\) are co-prime, by \[3\] Lemma 13.8 and Cor. 13.9, the \(J^A\) of all \(A\)-stable elements of \(J\) is non-empty and \(C_K(A)\) acts transitively on \(J^A\). Similarly \(b\) also has \(A\)-stable defect groups and \(C_G(A)\) acts transitively on all these defect groups. By the isomorphism (6.2), all defect groups of \(d\) are also defect groups of \(b\). Since \(P\) is a defect group centralized by \(A\), adjusting \(K\) and \(d\) by suitable \(C_G(A)\)-conjugation, we can assume that \(P \leq K\) and \(P\) is a defect group of \(d\). Let \(e\) and \(g\) be the Glauberman correspondents of \(d\) and \(f\) respectively. Since we assume \(O_{p'}(G) \cap C_G(A) = O_{p'}(C_G(A))\), \(g\) is actually a block idempotent of \(O_{p'}(C_G(A))\).

**Proposition 8.** Keep the notation and hypotheses as above. Then the stabilizer of \(\hat{g}\) under the \(C_G(A)\)-conjugation is equal to \(C_K(A)\), \(ge = e\) and there is an isomorphism

\[(8.1) \quad \text{Ind}_{C_K(A)}^{C_G(A)}(OC_K(A)e) \cong OC_G(A)c.\]

**Proof.** It is trivial to see that \[5\] Th. 5.1 still holds without the \(p\)-solvable condition.

Now we assume that the block idempotent \(f\) of \(O_{p'}(G)\) is stabilized by \(G \rtimes A\); then \(K = G, d = b, e = c\) and \(C_K(A) = C_G(A)\).

9. Obviously \(G\) acts on the full matrix algebra \(O_{p'}(G)f\) over \(O\), and by the Skolem-Noether theorem, there exists a group homomorphism

\[\rho : G \rightarrow \text{Aut}(O_{p'}(G)f) \cong (O_{p'}(G)f)^*/O^*.\]

We denote by \(\hat{G}\) the set of all elements \((x, s)\) such that \(\rho(x)\) is the image of \(s\) in \((O_{p'}(G)f)^*/O^*,\) where \(s \in (O_{p'}(G)f)^*\) and \(x \in G\). \(\hat{G}\) is a group with respect to the multiplication \((x, s)(x', s') = (xx', ss')\). Endowed with the homomorphism \(O^* \rightarrow \hat{G}\) mapping \(\lambda\) onto \((1, \lambda)\), \(\hat{G}\) becomes an \(O^*\)-group with the \(O^*\)-quotient \(G\). There is an injective group homomorphism \(O_{p'}(G) \rightarrow \hat{G}, x \mapsto (x, xb)\), whose image is normal in \(\hat{G}\) and intersects \(O^*\) trivially. Later we always identify \(O_{p'}(G)\) with this image through the injective homomorphism.
10. We set
\[ \hat{G} = G/O_p'(G) \quad \text{and} \quad \hat{G} = \hat{G}/O_p'(G). \]
For any \((x, s) \in \hat{G}\), we denote by \([x, s]\) the image of \((x, s)\) in \(\hat{G}\). With the obvious injective homomorphism \(O^* \to \hat{G}/O_p'(G)\) mapping \(\lambda\) onto \((1, \lambda)\), \(\hat{G}\) is an \(O^*\)-group with the \(O^*\)-quotient \(\hat{G}\). The actions of \(A\) on \(\hat{O}_p'(G)\) and \(\hat{G}\) induce an action of \(A\) on \(\hat{G}\) defined by the equality
\[ a \cdot (x, s) = (a \cdot x, a \cdot s) \]
for any \(a \in A\) and \((x, s) \in \hat{G}\). Obviously \(O_p'(G)\) is an \(A\)-stable normal subgroup of \(\hat{G}\) and thus the action of \(A\) on \(\hat{G}\) induces an action of \(A\) on \(\hat{G}\). Set \(m = |G|/O_p'(G)\) and denote by \(\mu_m\) the subgroup of \(O^*\) of all \(m\)-th roots of unity. For any \(x \in G\), choose \(s_x \in (O_p'(G)f)^*\) such that \((x, s_x) \in \hat{G}\) and \(\det(s_x) = 1\). Set \(G' = \{(x, \lambda s_x) | \lambda \in \mu_m, x \in G\}\).

**Lemma 11.** Keep the notation and hypotheses as above. Then \(G'\) is an \(A\)-stable finite subgroup of \(\hat{G}\) such that \(\hat{G} = O^*G'\) and \(O_p'(G) \leq G'\); moreover, the exponent of \(G'\) divides \(m\).

**Proof.** This lemma easily follows from the proof of \([5\text{ I}.4.5\text{ (i)}]\). \(\square\)

12. Since \(\hat{O}_p'(G)f\) is a full matrix algebra over \(O\) and its rank over \(O\) is prime to \(p\), the restriction to \(R\) of the homomorphism \(\rho\) can be lifted to a unique homomorphism \(\sigma : R \to (\hat{O}_p'(G)f)^*\) such that \(\det(\sigma(u)) = 1\) for any \(u \in R\) (refer to \([7\text{ para. 6.2}]\)). By this homomorphism, we define a new homomorphism \(R \to \hat{G}, u \mapsto (u, \sigma(u))\), which is injective and whose image intersects \(O^*\) trivially. We identity \(R\) with its image through this homomorphism. We claim that
\[ R \subset C_G'(A). \]

Firstly, since \(\hat{O}_p'(G)f\) is a full matrix algebra over \(O\), by the Skolem-Noether theorem, for any \(a \in A\) and \(u \in R\), \(a(\sigma(u))\) is equal to some conjugate of \(\sigma(u)\) in \(\hat{O}_p'(G)f\) and thus \(\det(a(\sigma(u))) = 1\); then it follows from the uniqueness of the homomorphism \(\sigma\) that \(a(\sigma(u)) = \sigma(u)\) for any \(a \in A\) and \(u \in R\). This shows that \(A\) centralizes the subgroup \(R\) of \(\hat{G}\). Secondly it is clear that \(s_u\sigma(u)^{-1}\) is an invertible element of \(O\) with order dividing \(m\). Therefore there is some \(\lambda \in \mu_m\) such that \(s_u = \lambda\sigma(u)\) and then \((u, \sigma(u)) \in G'\). The claim is proved.

13. By \([5\text{ Th. 4.4}]\), there is an algebra isomorphism
\[ O_Gf \cong \hat{O}_p'(G)f \otimes_O \hat{O}^\diamond \]
mapping \(xf\) onto \(s \otimes (x, s)\) for any \(x \in G\). Since \(fb = b\), this isomorphism determines a central primitive idempotent \(\tilde{b}\) of \(O \cdot \hat{G}^\diamond\) such that
\[ O_Gb \cong \hat{O}_p'(G)f \otimes_O \hat{O}^\diamond \hat{b}. \]

Moreover the image of \(P\) in \(\hat{G}\) is a defect group of \(\tilde{b}\) (refer to \([8\text{ Cor. 6.6}]\)). Since \(P\) and the image of \(P\) in \(\hat{G}\) are isomorphic, we identify them. We let \(\alpha\) be the unique character of \(\hat{O}_p'(G)f\) and denote by \(\text{Irr}(\hat{G}^\diamond, \tilde{b})\) the set of characters of all
irreducible \( \mathcal{K} \otimes_{\mathcal{O}} \hat{\mathcal{O}} \tilde{b} \)-modules. Then by the isomorphism (13.2), we obtain a bijection

\[
(13.3) \quad \text{Irr}(G, b) \to \text{Irr}(\hat{G}^0, \tilde{b}), \chi \mapsto \tilde{\chi}
\]
such that for any \( x \in G \), \( \chi(x) = \alpha(s)\tilde{\chi}(x, s) \).

14. By paragraph 9 applied to \( C \), \( \mathcal{O}O_{p'}(C) \) and \( g \), we obtain a group homomorphism

\[
g : C \to \text{Aut}(\mathcal{O}O_{p'}(C)g) \cong (\mathcal{O}O_{p'}(C)g)^*/\mathcal{O}^*.
\]

By this homomorphism, we construct an \( \mathcal{O}^* \)-group \( \tilde{C} \) with the \( \mathcal{O}^* \)-quotient \( C \), which is the set of all elements \( (y, t) \) such that \( g(y) \) is the image of \( t \) in \( (\mathcal{O}O_{p'}(C)g)^*/\mathcal{O}^* \), where \( t \in (\mathcal{O}O_{p'}(C)g)^* \) and \( y \in C \). Just as \( \hat{G} \) in paragraph 9, \( \tilde{C} \) has three subgroups: the first is the normal group \( \{(y, ty) | y \in \mathcal{O}p'(C)\} \) and we identity this normal subgroup with \( \mathcal{O}p'(C) \); the second is the subgroup \( \mathcal{C}' \) consisting of all elements \( (y, \lambda t_y) \), where \( \lambda \in \mathcal{O}_m \), \( y \in \mathcal{C} \) and \( t_y \) belongs to \( (\mathcal{O}O_{p'}(C)g)^* \) such that \( (y, t_y) \in \mathcal{C} ' \) and \( \det(t_y) = 1 \); the third is the \( \mathcal{O} \)-group \( R = \{ (u, \varsigma(u)) | u \in \mathcal{R} \} \), where \( \varsigma : \mathcal{R} \to (\mathcal{O}O_{p'}(C)g)^* \) is the unique lifting of the restriction of \( g \) to \( \mathcal{R} \) such that \( \det(\varsigma(u)) = 1 \) for any \( u \in \mathcal{R} \). Note that the exponent of \( \mathcal{C}' \) also divides \( m \) (refer to Lemma 11) and that \( \mathcal{R} \subset \mathcal{C}' \) (refer to (12.1)). We set

\[
\mathcal{C} = C/\mathcal{O}O_{p'}(C) \quad \text{and} \quad \mathcal{C} \mathcal{C} = \mathcal{C}/\mathcal{O}O_{p'}(C)
\]

and denote by \( (x, t) \) the image of \( (x, t) \) in \( \mathcal{C} \) for any \( (x, t) \in \mathcal{C} \). Then \( \mathcal{C} \) is an \( \mathcal{O}^* \)-group with \( \mathcal{O}^* \)-quotient \( \mathcal{C} \).

15. By [5] Th. 4.4], there is an algebra isomorphism

\[
(15.1) \quad \mathcal{O}Cg \cong \mathcal{O}O_{p'}(C)g \otimes_{\mathcal{O}} \mathcal{O}_{\mathcal{C}} \hat{\mathcal{C}}^{0}
\]

mapping \( xg \) onto \( t \otimes (x, t) \) for any \( x \in \mathcal{C} \). Since \( gc = c \), this isomorphism determines a central primitive idempotent \( \tilde{c} \) of \( \mathcal{O}_{\mathcal{C}} \hat{\mathcal{C}}^{0} \) such that

\[
(15.2) \quad \mathcal{O}Cc \cong \mathcal{O}O_{p'}(C)g \otimes_{\mathcal{O}} \mathcal{O}_{\mathcal{C}} \hat{\mathcal{C}}^{0} \tilde{c}.
\]

Let \( \beta \) be the unique character of \( \mathcal{O}O_{p'}(C)g \) and denote by \( \text{Irr}(\hat{\mathcal{C}}^{0}, \tilde{c}) \) the set of characters of all irreducible \( \mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}_{\mathcal{C}} \hat{\mathcal{C}}^{0} \tilde{c} \). By the isomorphism (15.2), we obtain a bijection

\[
(15.3) \quad \text{Irr}(C, c) \to \text{Irr}(\hat{\mathcal{C}}^{0}, \tilde{c}), \psi \mapsto \tilde{\psi}
\]
such that for any \( x \in \mathcal{C} \), \( \psi(x) = \beta(t)\tilde{\psi}(x, t) \).

**Lemma 16.** Keep the notation and hypotheses as above. Then \( C_{G'}(A) = \hat{C} \) and \( C_{G'}(A) \) is a subgroup of \( \hat{C} \) such that \( \mathcal{O}^*C_{G'}(A) = \hat{C} \).

**Proof.** Take \( x \in \mathcal{C} \) and let \( K \) be the set of all the inverse images of \( x \) through the canonical surjective homomorphism \( G' \to G \). Then since it follows from Lemma 11 that \( G' \) is \( A \)-stable, \( A \) stabilizes \( K \). Consider the action of \( \mu_m \times A \) on \( K \) defined by the left multiplication of \( \mu_m \) on \( K \) and the above \( A \)-action on \( K \) together. Since \( \mu_m \) acts regularly on \( K \) and \( A \) and \( \mu_m \) have co-prime order, by [3] Lemma 13.8 and Cor. 13.9], \( A \) has to stabilize some element of \( K \) and \( \mu_m \) regularly acts on these \( A \)-stabilized elements; in particular, \( A \) stabilizes all elements of \( K \). This shows that \( C_{G'}(A) \) is a subgroup of \( \hat{C} \) such that \( \mathcal{O}^*C_{G'}(A) = \hat{C} \). Thus \( C_{G'}(A) = \hat{C} \). \( \square \)
The following two lemmas will be used to analyze the local structure of the \(O(G \times C_G(A))\)-module in Theorem 3 inducing Morita equivalence between \(Ogb\) and \(O_{C_G}(A)c\). For the compatibility of Fong’s reduction with the Glauberman correspondence of characters (see Proposition 23 below), they can be skipped for now.

**Lemma 17.** Keep the notation and hypotheses as above. There is an isomorphism \(\Phi : C_{G'}(A) \cong C'\), which preserves \(R\) and \(O_{p'}(C)\) element-wise and makes the following diagram commutative:

\[
\begin{align*}
1 & \to \mu_m \to C_{G'}(A) \to C \to 1 \\
1 & \to \mu_m \to C' \to C \to 1.
\end{align*}
\]

**Proof.** We choose a local subring \(O'\) of \(K\) obtained by adding a primitive \(|O_{p'}(G)|\)-th root of unity to the ring of rational integers \(\mathbb{Z}\) and localizing \(\mathbb{Z}[\xi]\) at a maximal ideal containing \(q\); then \(|O_{p'}(G)|\) is invertible in \(O'\) and the residue field \(k'\) of \(O'\) is equal to \(\mathbb{F}_q(\xi)\) and has characteristic \(q\). Let \(\alpha\) be the unique character of \(\mathcal{O}O_{p'}(G)f\). Then for any \(a \in \mathcal{O}O_{p'}(G)f\), by the Fourier inversion formula, we have

\[
a = \frac{\alpha(1)}{|O_{p'}(G)|} \sum_{z \in O_{p'}(G)} \alpha(az^{-1})z.
\]

Whenever \(a\) takes \(f\) or \(s_x\) for \(x \in C\), \(az^{-1}\) has finite order dividing \(m\). Therefore \(\alpha(az^{-1}) \in \mathbb{Z}[\xi], f \in (\mathcal{O}'O_{p'}(G))^A\) and \(s_x \in (\mathcal{O}'O_{p'}(G))^A\) for any \(x \in C\). Furthermore \(s_x \in (\mathcal{O}'O_{p'}(G)f)^A\) for any \(x \in C\) since \(s_x f = s_x\). Similarly \(g \in \mathcal{O}'O_{p'}(C)\) and \(t_y \in \mathcal{O}'O_{p'}(C)g\) for any \(y \in C\).

We denote by \(\tilde{g}\) the image of \(g\) in \(k'O_{p'}(C)\). Since \(br_A(s_x)\) and \(br_A(t_x)\) both act on \(k'O_{p'}(C)\tilde{g}\) as \(x\) for any \(x \in C\), there is a unique \(\nu_x \in k'^*\) such that \(br_A(s_x) = \nu_x br_A(t_x)\). Moreover since \(br_A(s_x)\) and \(br_A(t_x)\) both have order dividing \(m\), \(\nu_x\) has finite order dividing \(m\). Therefore we can uniquely lift \(\nu_x\) to an invertible element of \(O'\) with the same order as \(\nu_x\); we still denote this invertible element by \(\nu_x\). Then by the proof of [4.5.8], we can prove that there is an isomorphism of \(O'\)-groups

\[
\tilde{C} \cong \tilde{C}
\]

which maps \((x, t_x)\) onto \((x, \nu_x s_x)\) for any \(x \in C\) and \((x, y)\) onto \((x, f)\) for any \(x \in O_{p'}(C)\). In particular, the isomorphism (17.2) induces an isomorphism \(C_{G'}(A) \cong C'\). The latter preserves \(O_{p'}(C)\) element-wise and fulfilling (17.1), but unfortunately it does not preserve \(R\) element-wise. So we have to adjust the isomorphism (17.2).

Since \(R \subset C_{G'}(A)\) and \(R \subset C'\) (refer to (12.1) and paragraph 14), without loss of generality, we take \(t_u\) to be \(\zeta(u)\) and \(s_u\) to be \(\sigma(u)\) for any \(u \in R\). Then the isomorphism \(\tilde{C} \cong \tilde{C}\) above implies that for any \(u, v \in R, \nu_u \nu_v = \nu_{uv}\). This shows that the function \(\nu : R \rightarrow \mathbb{Z}[\xi], u \mapsto \nu_u\) is a linear character of \(R\). We claim that \(\nu\) is \(C\)-stable. By the uniqueness of \(\sigma\) and \(\zeta\), we have \(x \sigma(u)x^{-1} = \sigma(xux^{-1})\) and \(x \zeta(u)x^{-1} = \zeta(xux^{-1})\) for any \(x \in C\) and \(u \in R\) such that \(xux^{-1} \in R\). Then we have

\[
\nu(xux^{-1}) = br_A(\sigma(xux^{-1}))br_A(\zeta(xux^{-1})) = br_A(x \sigma(u)x^{-1}) br_A(x \zeta(u)x^{-1}) = \nu(u).
\]

The claim is proved. □
Notice that $R$ is a Sylow $p$-subgroup of $C$. Then by [5 Prop. 2.6], $\nu$ can be extended to a linear character of $C$ and we still denote this character by $\nu$. By using $\nu$, we define a map

$$\tilde{C} \to \tilde{C}, \ (x, t_x) \mapsto \nu(x^{-1})\mu_x(x, s_x).$$

Then it is easily checked that this map is an isomorphism of $O^*$-groups and its inverse induces the desired isomorphism $\Phi : C_G(A) \cong C'$ in the lemma.

18. We define a function $\alpha'$ on $G'$ by setting

$$\alpha'(x, s) = \alpha(s)$$

for any $(x, s) \in G'$. Then $\alpha'$ is an irreducible character of $G'$ extending $\alpha$ through the inclusion $O_{\nu}(G) \hookrightarrow G'$. Since $O_{\nu}(G)f$ is a full matrix algebra over $O$, by the Skolem-Noether theorem, for any $(x, s) \in G'$ and $a \in A$, $\alpha(s)$ is equal to some conjugate of $s$ in $O_{\nu}(G)f$. Therefore $\alpha'(a(x, s)) = \alpha'(x, s)$ and $\alpha'$ is $A$-stable. Similarly a function $\beta'$ on $C'$ defined by

$$\beta'(y, t) = \beta(t)$$

for any $(y, t) \in C'$ is also an irreducible character of $C'$, which extends $\beta$ through the inclusion $I_{\nu}(G) \hookrightarrow C'$.

Lemma 19. Keep the notation and hypotheses as above. Then there is a linear character $\gamma' : C' \to O$ such that the following hold:

19.1. $\mu_m \subset \ker(\gamma')$ and $\im(\gamma') \subset \mu_m$.

19.2. The map $\Theta : C_{G'}(A) \to C', (t, y) \mapsto \gamma'(\Phi(t, y))\Phi(t, y)$ is an isomorphism preserving $O_{\nu}(C)$ elementwise.

19.3. $\pi(G', A)(\alpha') \circ \Theta^{-1} = \beta'$.

Proof. By [5 Th. 2.3], $\pi(G', A)(\alpha') \circ \Phi^{-1}$ is an irreducible character of $C'$ and is an extension of $\beta$. Therefore there is a linear character $\gamma' : C' \to O$ of $C'$ such that $\pi(G', A)(\alpha') \circ \Phi^{-1} = \gamma' \beta'$ and $O_{\nu}(C) \subset \ker(\gamma)$. Take $\xi \in \mu_m$. Since

$$\pi(G', A)(\alpha') \circ \Phi^{-1}(1, jG) = \xi(\pi(G', A)(\beta') \circ \Phi^{-1})(1, g) = \xi \beta(1) = \beta'(\xi g, 1),$$

$\gamma'(\xi) = 1$ and $\mu_m \subset \ker(\gamma')$. The inclusion $\im(\gamma') \subset \mu_m$ is obvious since the exponent of $C'$ divides $m$. So up to now, 19.1 is proved. The latter two can be trivially verified.

Lemma 20. Keep the notation and hypotheses as above. Then $C_G(A) = C_G(A)O_{\nu}(G)/O_{\nu}(G)$ and we have an isomorphism

$$(20.1) \quad C_G(A) \cong \tilde{C}, \ xO_{\nu}(G) \mapsto xO_{\nu}(C).$$

Proof. Obviously $C_G(A)O_{\nu}(G)/O_{\nu}(G) \subset C_G(A)$. Let $xO_{\nu}(G) \in C_G(A)$. We consider the $O_{\nu}(G) \rtimes A$ on $O_{\nu}(G)\times O_{\nu}(G)$ defined by the left multiplication of $O_{\nu}(G)$ on $xO_{\nu}(G)$ and the obvious action of $A$ on $xO_{\nu}(G)$ induced by the action of $A$ on $G$. Since $A$ and $O_{\nu}(G)$ have co-prime order and $O_{\nu}(G)$ acts regularly on $xO_{\nu}(G)$, by [3 Lemma 13.8 and Cor. 13.9], the set of all $A$-stable elements in $xO_{\nu}(G)$ is non-empty and $C_{O_{\nu}(G)}(A)$ acts transitively on it. This shows that $C_G(A) \subset C_G(A)O_{\nu}(G)/O_{\nu}(G)$. Consequently $C_G(A)O_{\nu}(G)/O_{\nu}(G) = C_G(A)$.

By this equality and the assumption that $C_G(A) \cap O_{\nu}(G) = O_{\nu}(C)$, it is trivial to see that (20.1) is an isomorphism.
Below, we always identify $C_{\hat{G}}(A)$ with $\hat{C}$ through the isomorphism (20.1) and thus $\hat{C}$ is a subgroup of $\hat{G}$.

**Lemma 21.** Keep the notation and hypotheses as above. Then we have $C_{\hat{G}}(A) = \hat{C}$.

**Proof.** Set $\hat{G}' = G'/O_{\rho'}(G)$. By Lemma 11, $\hat{G}'$ is a subgroup of $\hat{G}$ such that $O^*\hat{G}' = \hat{G}$. Then replacing $\hat{G}$ by $\hat{G}'$ and $G'$ by $\hat{G}'$ in the proof of Lemma 16, we can prove this lemma. \[\square\]

**Lemma 22.** Keep the notation and hypotheses as above and denote by

$$\Gamma : \hat{C} \cong \tilde{C}$$

the isomorphism induced by $\Theta$ (see Lemma 19). Then $\Gamma$ induces an isomorphism of $O^*$-groups

$$\hat{\Gamma} : \hat{\tilde{C}} \cong \tilde{\hat{C}}.$$

**Proof.** Since $\Gamma$ maps $\hat{C} \cap O_{\rho'}(G)$ onto $O_{\rho'}(C)$, it induces an isomorphism of $O^*$-groups

$$\hat{\Gamma} : \hat{C}/O_{\rho'}(C) \cong \hat{\tilde{C}}/O_{\rho'}(C) \cong \tilde{C}/O_{\rho'}(C) = \tilde{\hat{C}}.$$

By Lemma 16, $\tilde{C} = C_{\hat{G}}(A)$; therefore $\hat{C}/O_{\rho'}(C) = C_{\hat{G}}(A)/O_{\rho'}(G) \subset C_{\hat{G}}(A)$. On the other hand, it follows from Lemmas 20 and 21 that $\hat{C}/O_{\rho'}(G)$ and $C_{\hat{G}}(A)$ both are $O^*$-groups with the $O^*$-quotient $\hat{C}$. Therefore the above inclusion $C_{\hat{G}}(A)/O_{\rho'}(G) \subset C_{\hat{G}}(A)$ should be an equality. Then $\hat{\Gamma}$ is the desired isomorphism. \[\square\]

Obviously the group isomorphisms $\Gamma$ and $\hat{\Gamma}$ induce isomorphisms

$$O, \hat{C} \cong O, \hat{\tilde{C}} \text{ and } O, \tilde{\hat{C}} \cong O, \tilde{\hat{C}}.$$  

We still denote these isomorphisms by $\Gamma$ and $\hat{\Gamma}$ respectively.

**Proposition 23.** Keep the notation and hypotheses as above. Then there is a bijection $\pi(\hat{G}^o, A) : \text{Irr}(\hat{G}^o, \tilde{b}) \to \text{Irr}(\tilde{G}^o, \hat{c})$ such that

23.1. For any $\hat{\chi} \in \text{Irr}(\hat{G}^o, \tilde{b})$, $\pi(\hat{G}^o, A)(\hat{\chi}) \circ \Gamma^{-1}$ is the unique irreducible constituent of the restriction of $\hat{\chi}$ to $K \otimes O, \hat{\tilde{G}}^o$, which occurs with a multiplicity prime to $q$.

23.2. If $\chi \in \text{Irr}(G, b)$ corresponds to $\tilde{\chi} \in \text{Irr}(\tilde{b})$ through the bijection (13.3), then $\pi(\hat{G}^o, A)(\chi)$ corresponds to $\pi(\hat{G}^o, A)(\tilde{\chi})$ through the bijection (15.3).

**Proof.** In this proof, we will identify $\hat{C}$ with $\tilde{C}$ through $\Gamma$ and $\hat{\hat{C}}$ with $\tilde{\hat{C}}$ through $\hat{\Gamma}$.

By our hypothesis, the map $\pi(G, A, b) : \text{Irr}(G, b) \to \text{Irr}(C, c)$, $\chi \mapsto \pi(G, A)(\chi)$ is a bijection. We denote by $\pi(\hat{G}^o, A)$ the composition of the inverse of the bijection (13.3), $\pi(G, A, b)$ and the bijection (15.3). Take $\hat{\chi} \in \text{Irr}(G, b)$ and set $\psi = \pi(G, A)(\chi)$; then $\pi(\hat{G}^o, A)(\hat{\chi}) = \hat{\psi}$. Then in order to prove the proposition above, it suffices to show that $\hat{\psi}$ is the unique irreducible constituent of the restriction of $\hat{\chi}$ to $K \otimes O, \hat{\tilde{G}}^o$, which occurs with a multiplicity prime to $q$.

We inflate $\chi$ to $\chi'$ through the canonical surjective homomorphism $G' \to G$ and $\psi$ to $\psi'$ through the canonical surjective homomorphism $C' \to C$. Since $\pi(G, A)(\chi) = \psi$, $\pi(G', A)(\chi') = \psi'$. We define a function $\chi'$ on $G'$ by setting $\chi'(x, s) = \chi((x, s))$ for any $(x, s) \in G'$. Then $\chi'$ is an irreducible character of $G'$ and it follows from the
isomorphism (13.3) that $\chi'(x, s) = \alpha'(x, s) \tilde{\chi}'(x, s)$ for any $(x, s) \in G'$. Moreover since $\chi'$ and $\alpha'$ are A-stable, so is $\tilde{\chi}'$. Similarly the function $\tilde{\psi}'$ on $C'$ by defining $\tilde{\psi}'(y, t) = \tilde{\psi}'(y, t)$ for any $(y, t) \in C'$ is also an irreducible character of $C'$ and fulfills the equality $\psi'(y, t) = \beta'(y, t) \tilde{\psi}'(y, t)$ for any $(y, t) \in C'$.

We decompose the restriction $\text{Res}^{C'}_{C^0}(\alpha')$ as the sum $\sum_{1 \leq i \leq n} \alpha'_i$ of irreducible characters of $C'$ and the restriction $\text{Res}^{C'}_{C^0}(\chi')$ as the sum $\sum_{1 \leq j \leq m} \tilde{\chi}'_j$. Then $\text{Res}^{C'}_{C^0}(\chi') = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} \alpha'_i \tilde{\chi}'_j$. Now we claim that if $\psi'$ is an irreducible constituent of some $\alpha'_i \tilde{\chi}'_j$, then

$$\alpha'_i = \beta' \quad \text{and} \quad \tilde{\chi}'_j = \tilde{\psi}'_i.$$  

Since $\pi(G', A)(\chi') = \psi'$ and $\pi(G', A)(\alpha') = \beta'$, this implies that the multiplicity of $\tilde{\psi}'_i$ in the restriction $\text{Res}^{C'}_{C^0}(\chi')$ is prime to $q$. In particular, the multiplicity of $\tilde{\psi}'_i$ in the restriction of $\tilde{\chi}$ to $K \otimes O \tilde{\mathcal{C}}^0$ is prime to $q$; the uniqueness of $\tilde{\psi}'_i$ follows from [2 Th. 6.13].

Suppose $\alpha'_i \neq \beta'$. Since $g$ is a central idempotent of $OC'$ and $\psi'$ a character of $C'$ provided by some irreducible $KC'g$-module (refer to the isomorphisms (15.1) and (15.2)), $\alpha'_i(g) \tilde{\chi}'_j(g) = 0$ and thus $\alpha'_i(g) \neq 0$. But since $\alpha'_i$ is irreducible, $\alpha'_i$ is a character of some irreducible $KC'g$-module and the restriction $\text{Res}^{C'}_{OC'}(\alpha'_i)$ is some multiple of $\beta$. On the other hand, since $\pi(G', A)(\alpha') = \beta'$, there are $\alpha'_{i_1}, \alpha'_{i_2}, \cdots, \alpha'_{i_l}$ among $\alpha'_1, \alpha'_2, \ldots, \alpha'_n$ such that $\alpha'_{i_1} = \alpha'_{i_2} = \cdots = \alpha'_{i_l} = \beta'$, where $l$ is the multiplicity of $\beta'$ in the restriction of $\alpha'$ to $C'$. Therefore since

$$\text{Res}^{C'}_{OC'}(\text{Res}^{C'}_{C^0}(\alpha')) = \text{Res}^{C'}_{OC'}(\text{Res}^{C'}_{OC'}(\alpha')) = \text{Res}^{C'}_{OC'}(\alpha'),$$

the multiplicity of $\beta$ in $\text{Res}^{C'}_{OC'}(\alpha')$ is greater than the sum of $l$ with the multiplicity of $\beta$ in $\text{Res}^{C'}_{OC'}(\alpha'_i)$ and thus strictly exceeds $l$. But the multiplicity of $\beta$ in $\text{Res}^{C'}_{OC'}(\alpha')$ is $l$. So a contradiction is produced and $\alpha'_i = \beta'$.

Note that $\tilde{\chi}'_j$ is an irreducible constituent of $\text{Res}^{C'}_{C^0}(\tilde{\chi}')$ and that $\tilde{\chi}$ is a character of an irreducible $K \otimes O \tilde{\mathcal{C}}^0$-module. Therefore there is a character $\tilde{\chi}_j$ of some irreducible $K \otimes O \tilde{\mathcal{C}}^0$-module such that for any $(x, s) \in C'$, $\tilde{\chi}_j(x, s) = \tilde{\chi}_j(x, s)$. Then by the isomorphism (15.1), $\alpha'_i \tilde{\chi}'_j$ is an irreducible character of $G'$. So $\psi' = \alpha'_i \tilde{\chi}'_j$ and $\tilde{\chi}'_j = \tilde{\psi}'_i$.

24. A proof of Theorem 2. In order to avoid unnecessary repetition, we continue to keep all the notation from paragraph 5 to paragraph 23. First we consider the case that the block idempotent $f$ of $OC^0(G)$ (see paragraph 2.1) is not stabilized by $G$. Then we have isomorphisms (see (6.2) and (8.1))

$$\text{Ind}^G_K(OC^0d) \cong OGb \quad \text{and} \quad \text{Ind}^{OC^0(A)}_{OC^0(\hat{A})}(OC_K(A)e) \cong OC_G(A)e.$$  

Suppose that the $OC_K(A \times K)$-module $N$ induces a Morita equivalence between $OC^0d$ and $OC_K(A)e$, which induces the Glauberman correspondence of characters from $\text{Irr}(K, d)$ to $\text{Irr}(C_K(A), e)$. Then it is trivial to see that the $OC_G(A \times G)$-module $M = OC_K(A \times G) \otimes _{OC_K(A \times K)} N$ induces a Morita equivalence between $OGb$ and $OC_G$. Take $\chi \in \text{Irr}(K, d)$. Let $W$ be a $KKd$-module providing the
character $\chi$. Since

$$(\mathcal{K} \otimes_O M) \otimes_{KG} (KG \otimes_{KK} W) \cong (KKG(A) \otimes_{KK} (\mathcal{K} \otimes_O N) \otimes_{KK} KG) \otimes_{KG}(KG \otimes_{KK} W) \cong KK_{C,C}(A) (\mathcal{K} \otimes_O N) \otimes_{KK} KG$$

and the character of $(\mathcal{K} \otimes_O N) \otimes_{KK} W$ is $\pi(K,A)(\chi)$, then by [5], Th. 2.3, the Morita equivalence induced by $M$ induces the Glauberman correspondence of characters from $\text{Irr}(G, b)$ to $\text{Irr}(C, c)$. Here we note that $N$ and $M$ have common vertexes and source modules.

Therefore in order to prove Theorem 2, we can assume without loss of generality that $f$ is stabilized by $G$. Then we have the algebra isomorphisms (see (13.2) and (15.2))

$$\mathcal{O}Gb \cong \mathcal{O}O_{\mathcal{O}}(G)f \otimes_O O_{\mathcal{C}} \hat{G}^\circ \hat{b} \quad \text{and} \quad \mathcal{O}Cc \cong \mathcal{O}O_{\mathcal{O}}(G)g \otimes_O O_{\mathcal{C}} \hat{C}^\circ \hat{c},$$

group isomorphisms (see Lemma 22)

$$\Gamma : \hat{C} \cong \hat{\hat{C}} \quad \text{and} \quad \hat{\Gamma} : \hat{\hat{C}} \cong \hat{C}$$

and isomorphisms of algebras

$$\Gamma : O_{\mathcal{C}} \hat{C} \cong O_{\mathcal{C}} \hat{\hat{C}} \quad \text{and} \quad \hat{\Gamma} : O_{\mathcal{C}} \hat{\hat{C}} \cong O_{\mathcal{C}} \hat{C}.$$ 

Let $i$ be a primitive idempotent of $\mathcal{O}O_{\mathcal{O}}(G)f$ and $j$ be a primitive idempotent of $\mathcal{O}O_{\mathcal{O}}(C)g$. It is well known that the $O_{\mathcal{C}} \hat{C}^\circ \hat{b} \otimes_O \mathcal{O}Gb$-module $i\mathcal{O}O_{\mathcal{O}}(G) \otimes_O O_{\mathcal{C}} \hat{C}^\circ \hat{b}$ induces a Morita equivalence between $\mathcal{O}Gb$ and $O_{\mathcal{C}} \hat{C}^\circ \hat{b}$, which induces the bijection (13.3) of characters, and that the $O_{\mathcal{C}}G \otimes_O O_{\mathcal{C}} \hat{C}^\circ \hat{c}$-module $\mathcal{O}O_{\mathcal{O}}(C)j \otimes_O O_{\mathcal{C}} \hat{C}^\circ \hat{c}$ induces a Morita equivalence between $O_{\mathcal{C}} \hat{C}^\circ \hat{c}$ and $O_{\mathcal{C}} \hat{C}$, which induces the inverse of the bijection (15.3) of characters. Moreover since we assume $G = O_{\mathcal{O}}(G)C_{\mathcal{O}}$, $\hat{\hat{C}} = \hat{C}$ and then it follows from Proposition 23 that $\hat{\Gamma}$ maps $O_{\mathcal{C}} \hat{C}^\circ \hat{b}$ onto $O_{\mathcal{C}} \hat{C}^\circ \hat{c}$ isomorphically; in this case, it is clear that the $O_{\mathcal{C}} \hat{C}^\circ \otimes_O O_{\mathcal{C}} \hat{C}$-module $1_{\mathcal{C}} \hat{b}$, which is defined by the equality $((1 \otimes x) \otimes (1 \otimes y)) \cdot a = \hat{\Gamma}^{-1}(1 \otimes x)a(1 \otimes y^{-1})$ for any $x \in \hat{\hat{C}}$, $y \in \hat{C}$ and $a \in O_{\mathcal{C}} \hat{C}^\circ \hat{b}$, induces a Morita equivalence between $O_{\mathcal{C}} \hat{C}^\circ \hat{b}$ and $O_{\mathcal{C}} \hat{C}^\circ \hat{c}$. Then we can easily see that the $O(C \times G)$-module

$$(\mathcal{O}O_{\mathcal{O}}(C)j \otimes_O O_{\mathcal{C}} \hat{C}^\circ \hat{c}) \otimes_{O_{\mathcal{C}} \hat{C}} (i\mathcal{O}O_{\mathcal{O}}(G) \otimes_O 1_{\mathcal{C}} \hat{G} \hat{b}) \cong \mathcal{O}O_{\mathcal{O}}(C)j \otimes_O \hat{\Gamma}^{-1}(O_{\mathcal{C}} \hat{C}^\circ \hat{b}) \otimes_O i\mathcal{O}O_{\mathcal{O}}(G)$$

induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}Cc$ (note that $O(C \times G) \cong O_{\mathcal{C}} \otimes_O O_{\mathcal{O}}$). Moreover by Proposition 23, the bijection of characters induced by this Morita equivalence coincides with the Glauberman correspondence of characters between $\text{Irr}(G, b)$ and $\text{Irr}(C, c)$.

Finally, employing the fifth paragraph in the proof of 1.1 in the reduced case in [5], we can prove that $\mathcal{O}O_{\mathcal{O}}(C)j \otimes_O \hat{\Gamma}^{-1}(O_{\mathcal{C}} \hat{C}^\circ \hat{b}) \otimes_O i\mathcal{O}O_{\mathcal{O}}(G)$ has $D(P)$ as a vertex.
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