A PALEY-WIENER THEOREM
FOR THE ASKEY-WILSON FUNCTION TRANSFORM

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Abstract. We define an analogue of the Paley-Wiener space in the context of the Askey-Wilson function transform, compute explicitly its reproducing kernel and prove that the growth of functions in this space of entire functions is of order two and type $\ln q^{-1}$, providing a Paley-Wiener Theorem for the Askey-Wilson transform. Up to a change of scale, this growth is related to the refined concepts of exponential order and growth proposed by J. P. Ramis. The Paley-Wiener theorem is proved by combining a sampling theorem with a result on interpolation of entire functions due to M. E. H. Ismail and D. Stanton.

1. Introduction

Let $M(r; f) = \sup \{|f(z)| : |z| \leq r\}$ and consider the space $A$, constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$, satisfying

\begin{equation}
M(r; f) = O(e^{\pi r}). \tag{1.1}
\end{equation}

Consider also the space $PW$ constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$ such that, for some $u \in L^2(-\pi, \pi)$,

\begin{equation}
f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{izt} u(t) \, dt. \tag{1.2}
\end{equation}

A celebrated classical theorem of Paley and Wiener says that

$A = PW$.

The growth condition (1.1) means that $f : \mathbb{C} \rightarrow \mathbb{C}$ has order one and type $\pi$ and the space $PW$ is called the Paley-Wiener space of band-limited functions; it is the reproducing kernel Hilbert space mapped via the Fourier transform into $L^2$ functions supported on the interval $[-\pi, \pi]$. See [25] for more details.

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Another famous result, the Whittaker-Shannon-Kotel’nikov sampling theorem, asserts that every function in the space PW admits the following representation:

\[
f(x) = \sum_{n=-\infty}^{\infty} f(n) \sin \frac{\pi(x - n)}{\pi(x - n)}.
\]

As a result, research concerning extensions of the sampling theorem has been historically associated with the corresponding extensions of the Paley-Wiener theorem.

The sampling theorem is known to hold for more general transforms, including the Hankel, Dunkl and Jacobi function transforms [14], [7], [26]; and the Paley-Wiener theorem is known to extend to such special function transforms [10].

Many sampling theorems have been recently considered in the \(q\)-case [1], [2], [5], [17]. When thinking about these extensions, one should keep in mind that many of the classical \(q\)-functions are special cases of a very general basic hypergeometric function known as the Askey-Wilson function. This fact is known as the “Askey-Wilson transform scheme” [9].

Recently, one of us has found a sampling theorem for the Askey-Wilson function transform [6]. Thus, it is natural to ask for the associated Paley-Wiener theorem. It is the purpose of this paper to address this question, providing a Paley-Wiener theorem for the Askey-Wilson function transform. This will be done after rephrasing the results in [6] in the convenient reproducing kernel Hilbert space setting.

Recent research concerning \(q\)-difference equations [20], interpolation of entire functions [18] and moment problems [4] strongly suggests that in order to deal with basic hypergeometric functions one should use the following concepts. A function \(f\) has logarithmic order \(\rho\) if

\[
\limsup_{r \to +\infty} \frac{\ln \ln M(r; f)}{\ln \ln r} = \rho
\]

and \(f\) with logarithmic order \(\rho\) has logarithmic type \(c\) if

\[
\limsup_{r \to +\infty} \frac{\ln M(r; f)}{(\ln r)^\rho} = c.
\]

This is because basic hypergeometric functions are of order zero and therefore require a refined concept of order to define their growth. However, we will approach the topic in a slightly different manner in this paper: Instead of considering a function in \(\mu\), we will consider a function in \(z = q^\mu\) and use the classical definitions of order and type in \(\mu\). Looking at objects from this point of view, our Askey-Wilson Paley-Wiener space turns out to be constituted by functions of order two with type \(\ln(1/q)\). This is equivalent to saying that, in the variable \(z = q^\mu\), they have logarithmic order two and logarithmic type \(\ln(1/q)\).

We have organized the paper in the following way. The next section reviews the definitions of the Askey-Wilson polynomials and functions and provides a short outline of the \(L^2\) theory of the Askey-Wilson transform. Then, in the third section, we present a detailed study of the reproducing kernel Hilbert space which is naturally associated to the Askey-Wilson function transform (in much the same way \(PW\) is associated to the Fourier transform). We compute a basis for this space as well as the explicit formula for the reproducing kernel and recover by this method the sampling theorem of [6]. Finally, in the last section we prove a Paley-Wiener theorem, by describing the growth of functions in the reproducing kernel Hilbert space in terms of their order and type.
2. The Askey-Wilson Function Transform

2.1. The Askey-Wilson polynomials. Choose a number $q$ such that $0 < q < 1$. The notational conventions from [12],

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n, \quad (a_1, \ldots, a_m; q)_n = \prod_{l=1}^{m} (a_l; q)_n, \quad |q| < 1,$$

where $n = 1, 2, \ldots$, will be used. The symbol $r_{+1}\phi_r$ stands for the function

$r_{+1}\phi_r \left( \frac{a_1, \ldots, a_{r+1}}{b_1, \ldots, b_r} \mid q, z \right) = \sum_{n=0}^{\infty} (a_1, \ldots, a_{r+1}; q)_n q^n z^n$.

The Askey-Wilson polynomials $p_n(x; a, b, c, d)$, with $x = \frac{z + z^{-1}}{2}$, are defined by (2.1)

$$p_n \left( \frac{z + z^{-1}}{2}; a, b, c, d \right) = \frac{(ab, ac, ad; q)_n}{a^n} 4\phi_3 \left( \frac{q^{-n}, q^{n-1}abcd, az, a/z}{ab, ac, ad} \mid q; q \right).$$

If $a, b, c, d \in \mathbb{C}$ are four reals or two reals and one pair of conjugates or two pairs of conjugates such that $|ab|, |ac|, |ad|, |bc|, |cd| < 1$, then the Askey-Wilson polynomials are real-valued and their orthogonality can be written as an integral over $x = \frac{z + z^{-1}}{2} \in [-1, 1]$ plus a finite sum over a discrete set with mass points outside $[-1, 1]$. This finite sum does not occur if $|a|, |b|, |c|, |d| < 1$. When $\max(|a|, |b|, |c|, |d|) < 1$, the Askey-Wilson polynomials satisfy the orthogonality relation

$$\int_{-1}^{1} p_n(x; a, b, c, d) p_m(x; a, b, c, d) w(x) dx = h_n \delta_{m,n},$$

where

$$w(x) = \frac{(x^2, 1/x^2; q)_\infty \sin \theta}{(ax, a/x, bx, b/x, cx, c/x, dx, d/x; q)_\infty}$$

and

$$h_n = \frac{2\pi (abcdq^{2n}; q)_\infty (abcdq^{n-1}; q)_n}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}.$$
Moreover, its poles are simple and can be removed by multiplying it by the factor $(q^d/d,q/\gamma d;q)_\infty$.

Now we will define the Askey-Wilson function transform, following the construction in [8]. A new weight function is defined as

$$W(x) = \Delta(x) \Theta(x),$$

where, using the notation $\theta(x) = (x,q/x;q)_\infty$ for the renormalized Jacobi theta function, the function $\Theta$ is defined as

$$\Theta(x) = \frac{\theta(dx,d/x)}{\theta(dt,dt/x)}.$$

For generic parameters $a,b,c,d$ such that the weight function $W$ has simple poles, we define a measure $v$, depending on these parameters, by

$$\int f(x) dv(x) = \frac{K}{4\pi} \int f(x) \phi_\gamma(x) W(x) dx,$$

$$+ \frac{K}{2} \sum_{x \in D} (f(x) + f(x^{-1})) \text{Res}_{y=x} \left( \frac{W(y)}{y} \right),$$

where $K$ is a constant (the exact value will not be required), $S = S_- \cup S_+$ is the infinite discrete set given by

$$S_- = \{ dtq^k; k \in \mathbb{Z}, dtq^k < -1 \},$$

$$S_+ = \{ aq^k; k \in \mathbb{Z}, aq^k > 1 \}.$$

In the next sections we will often refer to the measure defined above as being of the form $v = v_c + v_d$, where $v_c$ is the continuous measure

$$dv_c(x) = \Theta(x) \Delta(x) dx/x$$

and $v_d$ is the discrete part, supported in the set $S$.

Now, let $L^2_+(v)$ be the Hilbert space with respect to the measure $v$ constituted by functions $f$ satisfying $f(x) = f(x^{-1})$, $v$-almost everywhere. The Askey-Wilson function transform is defined by

$$(\mathcal{F}f)(\gamma) = \int f(x) \phi_\gamma(x) dv(x)$$

for compactly supported functions $f \in L^2_+(v)$. Let $L^2_+(\bar{v})$ be the same space with respect to the same measure, but replacing the parameters $a,b,c,d$ by the dual parameters $\hat{a}, \hat{b}, \hat{c}, \hat{d}$. The main result in [8] states that $\mathcal{F}$ extends to an isometric isomorphism

$$\mathcal{F} : L^2_+(v) \to L^2_+(\bar{v}).$$

3. The Askey-Wilson Paley-Wiener Space

3.1. Reproducing kernel Hilbert spaces. We will now introduce some concepts concerning reproducing kernel Hilbert spaces. This exposition is taken from [14], [11] and [21].

Let $H_{\text{rep}}$ be a class of complex-valued functions, defined in a set $X \subset \mathbb{C}$, such that $H_{\text{rep}}$ is a Hilbert space. We say that $k(\gamma,x)$ is a reproducing kernel of $H_{\text{rep}}$ if $k(\gamma,x) \in H_{\text{rep}}$ for every $\gamma \in X$ and every $f \in H_{\text{rep}}$ satisfies the reproducing equation

$$f(\gamma) = \langle f(\cdot), k(\cdot, \gamma) \rangle_{H_{\text{rep}}}.$$
Now we will use the language in Saitoh [21] and proceed to give a brief account of the required results.

Consider a second Hilbert space, $H$. For each $t$ belonging to a domain $X$, let $K(\cdot, t)$ belong to $H$. Then,

$$k(\gamma, x) = \langle K(\cdot, \gamma), K(\cdot, x) \rangle_H$$

is defined on $X \times X$. Suppose that we have an isometric transformation

$$(Fg)(\gamma) = \langle g, K(\cdot, \gamma) \rangle_H$$

and denote the set of images by $F(H)$. The following theorem can be found in [21]:

**Theorem A.** If $F$ is a one-to-one isometric transformation, the kernel $k(\gamma, x)$ determines uniquely a reproducing kernel Hilbert space for which it is the reproducing kernel. This reproducing kernel Hilbert space is precisely $F(H)$, and it can have no other reproducing kernel. If $\{S_n\}$ is a basis of $F(H)$, then

$$k(\gamma, x) = \sum_n S_n(\gamma)S_n(x).$$

There is a general formulation of the sampling theorem in reproducing kernel Hilbert spaces [15]. We will use the following “orthogonal basis case”.

**Theorem B.** With the notation established earlier, we have: If there exists $\{t_n\}_{n \in \mathbb{Z}}$ such that $\{K(\cdot, t_n)\}_{n \in \mathbb{Z}}$ is an orthogonal basis, we then have the sampling expansion

$$f(t) = \sum_{n \in \mathbb{I}} f(t_n) \frac{k(t, t_n)}{k(t_n, t_n)}$$

in $F(H)$, pointwise over $\mathbb{I}$ and uniformly over any compact subset of $X$ for which $\|K_t\|$ is bounded.

The chief example of a reproducing kernel Hilbert space is $PW$. In this situation the reproducing kernel is the function $\sin \pi (x - \gamma) / \pi (x - \gamma)$, the sampling points are $t_n = n$ and the uniformly convergent expansion is the Whittaker-Shannon-Kotel’nikov sampling formula.

### 3.2. The Askey-Wilson function reproducing kernel

Let us look at the reproducing kernel Hilbert space associated to the Askey-Wilson function transform.

The first task is to consider a proper analogue of band-limited functions. This is done by defining a finite continuous Askey-Wilson function in much the same way it was done in [6].

We start by removing the poles of the function $\phi_{\tilde{a}q^\mu}$: Consider a function $u_\mu$, analytic in the variable $\mu$, defined as

$$u_\mu(x, a, b, c, d | q) = (\tilde{a}q^\mu; \tilde{a}q^{-\mu}; q)_\infty \phi_{\tilde{a}q^\mu}(e^{i\theta}), x = \cos \theta.$$ 

Then we consider what is going to be the analogue of the transform [12]: if $\max(|a|, |b|, |c|, |d|) \leq 1$, the finite continuous Askey-Wilson transform $\mathcal{J}$ is defined by

$$\mathcal{J}(f)(\mu) = \int_{-1}^1 f(x)u_\mu(x; a, b, c, d | q) w(x, a, b, c, d | q) dx.$$
The continuous Askey-Wilson transform relates to the Askey-Wilson transform as follows: If \( \hat{f} \) is the analytic function such that \( f(\cos \theta) = \hat{f}(e^{i\theta}) \), then
\[
\mathcal{F}(f)(\mu) = \frac{4i\pi}{K}(\hat{a}q^\mu; \hat{a}q^{-\mu}; q)_{\infty}\mathcal{F}\left(\frac{\hat{f}}{\Theta}\right)(\hat{a}q^\mu).
\]

**Definition 1.** The Askey-Wilson Paley-Wiener space, \( \mathcal{PW}_{AW} \), is the space constituted by the analytic functions \( f \in L^2_p(v) \) such that, for some \( u \in \mathcal{L}^2(w(x, a, b, c, d \mid q), dx) \),
\[
f = \mathcal{F}(u).
\]

Let us look at this particular setting from the point of view of Theorem A.

**Theorem 1.** If \( \max(|a|, |b|, |c|, |d|) < 1 \), then the set \( \mathcal{PW}_{AW} \) is a Hilbert space of entire functions with reproducing kernel \( k(\gamma, \lambda) \). The functions
\[
S_n^{(\bar{a})}(\mu; q) = \frac{(-1)^n}{(q; q)_n (a, \hat{a}^2 q^{n}; q)_{\infty}} (1 - \hat{a} q^\mu)(1 - \hat{a} q^{-\mu})
\]
constitute an orthogonal basis of \( \mathcal{PW}_{AW} \), and the reproducing kernel is given explicitly by
\[
k(\gamma, \lambda) = \sum_{n=0}^{\infty} S_n^{(\bar{a})}(\gamma; q) S_n^{(\bar{a})}(\lambda; q).
\]

**Proof.** To fulfill the conditions in Theorem A, we need to show that the finite continuous Askey-Wilson transform is a one-to-one isomorphism between \( \mathcal{A}_{AW} \) and \( \mathcal{PW}_{AW} \). To see that it is one-to-one, observe that, since, if
\[
\int_{-1}^{1} f(x)u_{\mu}(x; a, b, c, d \mid q) w(x, a, b, c, d \mid q)dx = 0, \text{ for all } \mu \in \mathbb{C},
\]
then we have, in particular, that
\[
\int_{-1}^{1} f(x)u_{n}(x; a, b, c, d \mid q) w(x, a, b, c, d \mid q)dx, \text{ for } n = 0, 1, \ldots
\]
Since for integer values of \( \mu \), \( u_{\mu} \) is a multiple of the Askey-Wilson polynomials,
\[
u_n(x; a, b, c, d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab, ac, bc; q)_n} d^{-n} p_n(x; a, b, c, d),
\]
we can use the completeness of the system of the Askey-Wilson polynomials to get \( f = 0 \). Consequently, \( \mathcal{F}(f) \) is one-to-one. From the definition,
\[
\mathcal{PW}_{AW} = \mathcal{F}\left[L^2(w(x, a, b, c, d \mid q)dx)\right].
\]
Therefore, endowing \( \mathcal{PW}_{AW} \) with the inner product
\[
\langle \mathcal{F}(f), \mathcal{F}(g) \rangle_{\mathcal{PW}_{AW}} = \int_{-1}^{1} f(x)g(x)w(x, a, b, c, d \mid q)dx,
\]
the finite Askey-Wilson transform \( \mathcal{F} \) becomes a Hilbert space isometry between \( L^2(w(x, a, b, c, d \mid q)dx) \) and \( \mathcal{PW}_{AW} \).
It remains to show that the functions $S_n^{(\tilde{a})} (\mu; q)$ provide an orthogonal basis for $\mathcal{PW}_{AW}$. By the definitions (3.1) and (3.2),
\[
\mathcal{J}(u_n)(\mu) = \int_{-1}^{1} u_n(x) u_\mu(x) w(x, a, b, c, d \mid q) dx
\]
\[
= (-1)^n q^{-n(n-1)/2} \frac{d^n(ab, ac, bc; q)_n}{(ab, ac, bc; q)_n} \int_{-1}^{1} p_n(x) u_\mu(x) w(x, a, b, c, d \mid q) dx.
\]
Now we can use Proposition 6 of [6] to conclude that
\[
\mathcal{J}(u_n)(\mu) = S_n^{(\tilde{a})} (\mu; q).
\]
By (3.2), \{u_n\} is an orthogonal basis of $L^2(\mathcal{W}(x, a, b, c, d \mid q) dx)$. Since $\mathcal{J}$ is isometric onto $\mathcal{PW}_{AW}$, it follows that $S_n^{(\tilde{a})} (\mu; q)$ is a basis of $\mathcal{PW}_{AW}$.

**Remark 1.** The functions $S_n^{(\tilde{a})} (\gamma; q)$ play the same role in our setting as do the functions $\sin (\pi (x - n) / \pi (x - n))$ in the Paley-Wiener space.

Now, Theorem 1 and Theorem B give the following sampling theorem. This has been proved in [6], but the approach with reproducing kernels provides the uniform convergence that will be used in the next section.

**Theorem 2.** For $f \in \mathcal{PW}_{AW}$ we have
\[
f(\mu) = \sum_{n=0}^{\infty} f (n) S_n^{(\tilde{a})} (\mu; q),
\]
where $S_n^{(\tilde{a})} (\mu; q)$ is given by
\[
S_n^{(\tilde{a})} (\mu; q) = (-1)^n q^{n(n+1)/2} \frac{(1 - \tilde{a}^2 q^{2n}) (\tilde{a}q^\mu, \tilde{a}q^{-\mu}; q)_\infty}{(q; q)_n (a, \tilde{a}^2 q^n; q)_\infty (1 - \tilde{a}q^n) (1 - \tilde{a}q^{-n})}.
\]
The convergence is uniform on every compact subset of the real line.

**Proof.** Observe that from
\[
S_n^{(\tilde{a})} (m; q) = \delta_{n,m},
\]
we obtain
\[
g(\mu, m) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})} (\mu; q) S_n^{(\tilde{a})} (m; q) = S_m^{(\tilde{a})} (\mu; q).
\]
Moreover,
\[
g(m, m) = S_m^{(\tilde{a})} (m; q) = 1,
\]
and the result follows from Theorem 1 and Theorem B.

4. **The Askey-Wilson Paley-Wiener theorem**

Recall that the entire function $f$ is of order $\rho$ if
\[
\lim_{r \to \infty} \frac{\ln \ln (M(r; f))}{\ln r} = \rho.
\]
A constant has order zero, by convention.

The entire function $f$ of positive order $\rho$ is of type $\tau$ if
\[
\lim_{r \to \infty} \frac{\ln (M(r; f))}{r^\rho} = \tau.
\]
Definition 2. The space \( \mathcal{A}_{AW} \), which will be the analogue of \( \mathcal{A} \) in the Askey-Wilson setting, is the space constituted of analytic functions \( f \) such that \( f \in L^2(w(x, a, b, c, d | q), dx) \) and
\[
M(r; f) = O(e^{\ln(1/q) r^2}),
\]
that is, of order 2 and type \( \ln(1/q) \).

It is easy to see that the functions in \( \mathcal{A}_{AW} \) satisfy the conditions in [18, Theorem 3.1]. We rewrite this statement as:

**Theorem C.** Every \( f \in \mathcal{A}_{AW} \) admits the expansion
\[
(4.1) \quad f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(\bar{a})}(\mu; q).
\]

The next result is the Paley-Wiener theorem for the Askey-Wilson function transform. The cornerstone of its proof is the fact that the entire function expansion (4.1) and the sampling expansion (3.4) are exactly the same.

**Theorem 3.** If \( \max(|a|, |b|, |c|, |d|) < 1 \), then \( \mathcal{A}_{AW} = PW_{AW} \).

**Proof.** Take \( f \in PW_{AW} \). By definition we have, for some \( u \in L^2(w(x, a, b, c, d | q), dx) \),
\[
f(\mu) = \mathcal{J}(u)(\mu) = \int_{-1}^{1} u(x) u_{\mu}(x) w(x, a, b, c, d | q) dx.
\]

We need to study the growth of \( M(r; u_{\mu}) \).

From formula (5.4) in [24] and for \( 0 \leq \theta \leq \pi \), we have
\[
 u_r(x) = \frac{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, qe^{i\theta}/d; q)_{\infty}}{(ab, ac, bc, ce^{2i\theta}; q)_{\infty}} (q^{1-r}/e^{i\theta} d; q)_{\infty} [1 + o(1)], \quad \text{as} \quad r \to \infty.
\]

Let \( -1 < \delta < 0 \). Then
\[
 M(n + \delta; u_{\mu}) = O \left( (q^{1-\delta-n}/d; q)_{\infty} \right).
\]

This implies that
\[
 M(n + \delta; u_{\mu}) = O \left( (q/d)^n q^{-n(n+1+2\delta)/2} \right).
\]

Therefore,
\[
 \limsup_{r \to \infty} \frac{\ln \ln(M(r; u_{\mu}))}{\ln r} = 2
\]
and
\[
 \limsup_{r \to \infty} \frac{\ln(M(r; u_{\mu}))}{r^2} = \ln(1/q).
\]

This condition implies that \( u_{\mu} \) is of order 2 and type at most \( \ln(1/q) \). Therefore,
\[
 u_{\mu}(x) \in \mathcal{A}_{AW}.
\]

This shows that \( f \in \mathcal{A}_{AW} \). Conversely, let \( f \in \mathcal{A}_{AW} \). By Theorem C,
\[
 f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(\bar{a})}(\mu; q).
\]
In the end of the proof of Theorem 1 we have seen that
\[ S^{(a)}_n(\mu; q) = J(\mu). \]
Then, the sampling formula of Theorem 2 can be written as
\[ f(\mu) = \sum_{n=0}^{\infty} f(n) J(\mu), \]
\[ \int_{-1}^{1} u_n(x) u_\mu(x) w(x, a, b, c, d | q) dx. \]

The uniform convergence of the sampling series allows us to interchange the integral with the sum in such a way that
\[ f(\mu) = \int_{-1}^{1} \left( \sum_{n=0}^{\infty} f(n) u_n(x) \right) u_\mu(x) w(x, a, b, c, d | q) dx. \]

Then we have written \( f \) in the form
\[ f(\mu) = J(u)(\mu), \]
with
\[ u(x) = \left( \sum_{n=0}^{\infty} f(n) u_n(x) \right) \in L^2(w(x, a, b, c, d | q), dx). \]

As a result, \( f \in \mathcal{P}W_{AW}. \)

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