A PALEY-WIENER THEOREM FOR THE ASKEY-WILSON FUNCTION TRANSFORM

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Abstract. We define an analogue of the Paley-Wiener space in the context of the Askey-Wilson function transform, compute explicitly its reproducing kernel and prove that the growth of functions in this space of entire functions is of order two and type $\ln q^{-1}$, providing a Paley-Wiener Theorem for the Askey-Wilson transform. Up to a change of scale, this growth is related to the refined concepts of exponential order and growth proposed by J. P. Ramis. The Paley-Wiener theorem is proved by combining a sampling theorem with a result on interpolation of entire functions due to M. E. H. Ismail and D. Stanton.

1. Introduction

Let $M(r; f) = \sup \{|f(z)| : |z| \leq r\}$ and consider the space $A$, constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$, satisfying

(1.1) \[ M(r; f) = O(e^{\pi r}). \]

Consider also the space $PW$ constituted by the analytic continuation to the whole complex plane of the functions $f \in L^2(\mathbb{R})$ such that, for some $u \in L^2(-\pi, \pi)$,

(1.2) \[ f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{izt} u(t) \, dt. \]

A celebrated classical theorem of Paley and Wiener says that

$A = PW$.

The growth condition (1.1) means that $f : \mathbb{C} \to \mathbb{C}$ has order one and type $\pi$ and the space $PW$ is called the Paley-Wiener space of band-limited functions; it is the reproducing kernel Hilbert space mapped via the Fourier transform into $L^2$ functions supported on the interval $[-\pi, \pi]$. See [25] for more details.
Another famous result, the Whittaker-Shannon-Kotel’nikov sampling theorem, asserts that every function in the space $PW$ admits the following representation:

\begin{equation}
    f(x) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi (x - n)}{\pi (x - n)}.
\end{equation}

As a result, research concerning extensions of the sampling theorem has been historically associated with the corresponding extensions of the Paley-Wiener theorem.

The sampling theorem is known to hold for more general transforms, including the Hankel, Dunkl and Jacobi function transforms \cite{14}, \cite{7}, \cite{26}; and the Paley-Wiener theorem is known to extend to such special function transforms \cite{10}.

Many sampling theorems have been recently considered in the $q$-case \cite{1}, \cite{2}, \cite{5}, \cite{17}. When thinking about these extensions, one should keep in mind that many of the classical $q$-functions are special cases of a very general basic hypergeometric function known as the Askey-Wilson function. This fact is known as the “Askey-Wilson transform scheme” \cite{9}.

Recently, one of us has found a sampling theorem for the Askey-Wilson function transform \cite{6}. Thus, it is natural to ask for the associated Paley-Wiener theorem for the Askey-Wilson function transform. This will be done after rephrasing the results in \cite{6} in the convenient reproducing kernel Hilbert space setting.

Recent research concerning $q$-difference equations \cite{20}, interpolation of entire functions \cite{18} and moment problems \cite{4} strongly suggests that in order to deal with basic hypergeometric functions one should use the following concepts. A function $f$ has logarithmic order $\rho$ if

\[ \limsup_{r \to +\infty} \frac{\ln \ln M(r; f)}{\ln \ln r} = \rho \]

and $f$ with logarithmic order $\rho$ has logarithmic type $c$ if

\[ \limsup_{r \to +\infty} \frac{\ln M(r; f)}{(\ln r)^\rho} = c. \]

This is because basic hypergeometric functions are of order zero and therefore require a refined concept of order to define their growth. However, we will approach the topic in a slightly different manner in this paper: Instead of considering a function in $\mu$, we will consider a function in $z = q^\mu$ and use the classical definitions of order and type in $\mu$. Looking at objects from this point of view, our Askey-Wilson Paley-Wiener space turns out to be constituted by functions of order two with type $\ln(1/q)$. This is equivalent to saying that, in the variable $z = q^\mu$, they have logarithmic order two and logarithmic type $\ln(1/q)$.

We have organized the paper in the following way. The next section reviews the definitions of the Askey-Wilson polynomials and functions and provides a short outline of the $L^2$ theory of the Askey-Wilson transform. Then, in the third section, we present a detailed study of the reproducing kernel Hilbert space which is naturally associated to the Askey-Wilson function transform (in much the same way $PW$ is associated to the Fourier transform). We compute a basis for this space as well as the explicit formula for the reproducing kernel and recover by this method the sampling theorem of \cite{6}. Finally, in the last section we prove a Paley-Wiener theorem, by describing the growth of functions in the reproducing kernel Hilbert space in terms of their order and type.
2. The Askey-Wilson function transform

2.1. The Askey-Wilson polynomials. Choose a number \( q \) such that \( 0 < q < 1 \).
The notational conventions from [12],

\[
(a; q)_0 = 1, \quad (a; q)_n = \prod_{k=1}^{n} (1 - a q^{k-1}),
\]

\[
(a; q)_\infty = \lim_{n \to \infty} (a; q)_n, \quad (a_1, ..., a_m; q)_n = \prod_{l=1}^{m} (a_l; q)_n, \quad |q| < 1,
\]

where \( n = 1, 2, \ldots \), will be used. The symbol \( r+1\phi_r \) stands for the function

\[
r_{r+1}\phi_r \left( \frac{a_1, \ldots, a_{r+1}}{b_1, \ldots, b_r} \bigg| q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_{r+1}; q)_n}{(q, b_1, \ldots, b_r; q)_n} z^n.
\]

The Askey-Wilson polynomials \( p_n(x; a, b, c, d) \), with \( x = \frac{z+z^{-1}}{2} \), are defined by (2.1)

\[
p_n(z + z^{-1}, a, b, c, d) = \frac{(ab, ac, ad; q)_n}{a^n} 4\phi_3 \left( \frac{q^{-n}, q^{n-1}abcd, az, a/z}{ab, ac, ad} \bigg| q; q \right).
\]

If \( a, b, c, d \in \mathbb{C} \) are four reals or two reals and one pair of conjugates or two
pairs of conjugates such that \( |ab|, |ac|, |ad|, |bc|, |cd| < 1 \), then the Askey-Wilson
polynomials are real-valued and their orthogonality can be written as an integral over \( x = \frac{z+z^{-1}}{2} \) in \([-1, 1]\) plus a finite sum over a discrete set with mass points outside \([-1, 1]\). This finite sum does not occur if \( |a|, |b|, |c|, |d| < 1 \). When
max \(|a|, |b|, |c|, |d| < 1 \), the Askey-Wilson polynomials satisfy the orthogonality relation

\[
\int_{-1}^{1} p_n(x; a, b, c, d) p_m(x; a, b, c, d) w(x) dx = h_n \delta_{m,n},
\]

where

\[
w(x) = \frac{(x^2, 1/x^2; q)_\infty \sin \theta}{(ax, a/x, bx, b/x, cx, c/x, dx, d/x; q)_\infty}
\]

and

\[
h_n = \frac{2\pi (abcdq^{2n}; q)_\infty (abcdq^{n-1}; q)_n}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}.
\]

The Askey-Wilson function is defined as

\[
\phi_8(z) = \frac{1}{(bc, q/ad; q)_\infty} 4\phi_3 \left( \frac{\tilde{a}/\gamma, \tilde{a}^2, az, a/z}{ab, ac, ad} \bigg| q; q \right) + \frac{(\tilde{a}/\gamma, \tilde{a}^2, q/bd, q/cd, az, a/z; q)_\infty}{(q\gamma/d, q/\gamma d, ab, ac, ad/q, qz/d, q/zd; q)_\infty} 4\phi_3 \left( \frac{q\gamma/d, q/\gamma d, qz/d, q/zd}{q/bd, q/cd, q^2/ad} \bigg| q; q \right),
\]

where

\[
\tilde{a} = \sqrt{q^{-1}abcd}, \quad \tilde{b} = ab/\tilde{a} = q\tilde{a}/cd, \quad \tilde{c} = ac/\tilde{a} = q\tilde{a}/bd, \quad \tilde{d} = ad/\tilde{a} = q\tilde{a}/bc.
\]

The function \( \phi_8 \) is introduced in [15], and it can also be defined as a single \( 8\phi_7 \)
with a very-well-poised \( 8W_7 \) structure [24]. The function \( \phi_8 \) is meromorphic in \( \gamma \).
Moreover, its poles are simple and can be removed by multiplying it by the factor \((q^ζ/d, q/γd; q)_{∞}\).

Now we will define the Askey-Wilson function transform, following the construction in \([8]\). A new weight function is defined as

\[ W(x) = \Delta(x) \Theta(x), \]

where, using the notation \(θ(x) = (x, q/x; q)_{∞}\) for the renormalized Jacobi theta function, the function \(Θ\) is defined as

\[ Θ(x) = \frac{θ(dx, d/x)}{θ(dt, dx/dx)}. \]

For generic parameters \(a, b, c, d\) such that the weight function \(W\) has simple poles, we define a measure \(v\), depending on these parameters, by

\[
\int f(x) dv(x) = \frac{K}{4π} \int \phi(γ(x) W(x) \frac{dx}{x} 
+ \frac{K}{2} ∑_{x∈D} (f(x) + f(x^{-1})) \text{ Res}_{y=x} \left( \frac{W(y)}{y} \right),
\]

where \(K\) is a constant (the exact value will not be required), \(S = S_- ∪ S_+\) is the infinite discrete set given by

\[ S_- = \{dtq^k; k ∈ Z, dtq^k < -1\}, \]

\[ S_+ = \{aq^k; k ∈ Z, aq^k > 1\}. \]

In the next sections we will often refer to the measure defined above as being of the form \(v = v_c + ν_d\), where \(v_c\) is the continuous measure

\[ dv_c(x) = Θ(x)Δ(x) dx/x \]

and \(ν_d\) is the discrete part, supported in the set \(S\).

Now, let \(L^2_+(v)\) be the Hilbert space with respect to the measure \(v\) constituted by functions \(f\) satisfying \(f(x) = f(x^{-1})\), \(ν\)-almost everywhere. The Askey-Wilson function transform is defined by

\[ (Ff)(γ) = \int f(x) ϕ(γ(x)) dv(x) \]

for compactly supported functions \(f \in L^2_+(v)\). Let \(L^2_+(v)\) be the same space with respect to the same measure, but replacing the parameters \(a, b, c, d\) by the dual parameters \(a, b, c, d\). The main result in \([8]\) states that \(F\) extends to an isometric isomorphism

\[ F : L^2_+(v) \rightarrow L^2_+(v). \]

3. The Askey-Wilson Paley-Wiener space

3.1. Reproducing kernel Hilbert spaces. We will now introduce some concepts concerning reproducing kernel Hilbert spaces. This exposition is taken from \([14]\), \([11]\) and \([21]\).

Let \(H_{rep}\) be a class of complex-valued functions, defined in a set \(X ⊂ \mathbb{C}\), such that \(H_{rep}\) is a Hilbert space. We say that \(k(γ, x)\) is a reproducing kernel of \(H_{rep}\) if \(k(γ, x) ∈ H_{rep}\) for every \(γ \in X\) and every \(f ∈ H_{rep}\) satisfies the reproducing equation

\[ f(γ) = (f(\cdot), k(\cdot, γ))_{H_{rep}}. \]
Now we will use the language in Saitoh [21] and proceed to give a brief account of the required results.

Consider a second Hilbert space, $H$. For each $t$ belonging to a domain $X$, let $K(.,t)$ belong to $H$. Then,

$$k(\gamma,x) = \langle K(\gamma,x), K(\gamma,x) \rangle_H$$

is defined on $X \times X$. Suppose that we have an isometric transformation

$$\langle Fg \rangle(\gamma) = \langle g, K(\gamma) \rangle_H$$

and denote the set of images by $F(H)$. The following theorem can be found in [21]:

**Theorem A.** If $F$ is a one-to-one isometric transformation, the kernel $k(\gamma,x)$ determines uniquely a reproducing kernel Hilbert space for which it is the reproducing kernel. This reproducing kernel Hilbert space is precisely $F(H)$, and it can have no other reproducing kernel. If $\{S_n\}$ is a basis of $F(H)$, then

$$k(\gamma,x) = \sum_n S_n(\gamma)S_n(x).$$

There is a general formulation of the sampling theorem in reproducing kernel Hilbert spaces [15]. We will use the following “orthogonal basis case”.

**Theorem B.** With the notation established earlier, we have: If there exists $\{t_n\}_{n \in \mathbb{I} \subset \mathbb{Z}}$ such that $\{K(.,t_n)\}_{n \in \mathbb{I}}$ is an orthogonal basis, we then have the sampling expansion

$$f(t) = \sum_{n \in \mathbb{I}} f(t_n) k(t,t_n)$$

in $F(H)$, pointwise over $\mathbb{I}$ and uniformly over any compact subset of $X$ for which $\|K_t\|$ is bounded.

The chief example of a reproducing kernel Hilbert space is $PW$. In this situation the reproducing kernel is the function $\sin \pi(x - \gamma)/\pi(x - \gamma)$, the sampling points are $t_n = n$ and the uniformly convergent expansion is the Whittaker-Shannon-Kotel’nikov sampling formula.

### 3.2. The Askey-Wilson function reproducing kernel

Let us look at the reproducing kernel Hilbert space associated to the Askey-Wilson function transform.

The first task is to consider a proper analogue of band-limited functions. This is done by defining a finite continuous Askey-Wilson function in much the same way it was done in [6].

We start by removing the poles of the function $\phi_{aq^\mu}$: Consider a function $u_\mu$, analytic in the variable $\mu$, defined as

$$u_\mu(x,a,b,c,d \mid q) = (\tilde{a}q^\mu; \tilde{a}q^{-\mu}; q)_\infty \phi_{aq^\mu} (e^{i\theta}), \ x = \cos \theta.$$

Then we consider what is going to be the analogue of the transform [12]: if $\max(|a|, |b|, |c|, |d|) < 1$, the finite continuous Askey-Wilson transform $\mathcal{J}$ is defined by

$$\mathcal{J}(f)(\mu) = \int_{-1}^{1} f(x)u_\mu(x,a,b,c,d \mid q) w(x,a,b,c,d \mid q) dx.$$
The continuous Askey-Wilson transform relates to the Askey-Wilson transform as follows: If \( \tilde{f} \) is the analytic function such that \( f(\cos \theta) = \tilde{f}(e^{i\theta}) \), then

\[
\mathcal{J}(f)(\mu) = \frac{4i\pi}{K} (\tilde{a}q^{\mu}; \tilde{a}q^{-\mu}; q)_{\infty} \mathcal{F} \left( \tilde{f} \right)(q^{\mu}).
\]

**Definition 1.** The Askey-Wilson Paley-Wiener space, \( \mathcal{PW}_{AW} \), is the space constituted by the analytic functions \( f \in L^2_+(v) \) such that, for some \( u \in L^2_+(w(x, a, b, c, d \mid q), dx) \),

\[
f = \mathcal{J}(u).
\]

Let us look at this particular setting from the point of view of Theorem A.

**Theorem 1.** If \( \max(|a|, |b|, |c|, |d|) < 1 \), then the set \( \mathcal{PW}_{AW} \) is a Hilbert space of entire functions with reproducing kernel \( k(\gamma, \lambda) \). The functions

\[
S_n^{(\tilde{a})}(\mu; q) = \frac{(-1)^n q^{n(n+1)} (1 - \tilde{a}^2 q^{2n}) (\tilde{a}q^{\mu}, \tilde{a}q^{-\mu}; q)_{\infty}}{(q; q)_n (a, \tilde{a}^2 q^{2n}; q)_{\infty} (1 - \tilde{a}q^{n+\mu})(1 - \tilde{a}q^{-n-\mu})}
\]

constitute an orthogonal basis of \( \mathcal{PW}_{AW} \), and the reproducing kernel is given explicitly by

\[
k(\gamma, \lambda) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})}(\gamma; q) S_n^{(\tilde{a})}(\lambda; q).
\]

**Proof.** To fulfill the conditions in Theorem A, we need to show that the finite continuous Askey-Wilson transform is a one-to-one isomorphism between \( \mathcal{A}_{AW} \) and \( \mathcal{PW}_{AW} \). To see that it is one-to-one, observe that, since, if

\[
\int_{-1}^{1} f(x)u_{\mu}(x; a, b, c, d \mid q) w(x, a, b, c, d \mid q) dx = 0, \text{ for all } \mu \in \mathbb{C},
\]

then we have, in particular, that

\[
\int_{-1}^{1} f(x)u_{n}(x; a, b, c, d \mid q) w(x, a, b, c, d \mid q) dx, \text{ for } n = 0, 1, \ldots
\]

Since for integer values of \( \mu \), \( u_{\mu} \) is a multiple of the Askey-Wilson polynomials,

\[
u_n(x; a, b, c; d) = \frac{(-1)^n q^{-n(n-1)/2}}{(ab, ac, bc; q)_n} d^{-n} p_n(x; a, b, c, d),
\]

we can use the completeness of the system of the Askey-Wilson polynomials to get \( f = 0 \). Consequently, \( \mathcal{J}(f) \) is one-to-one. From the definition,

\[
\mathcal{PW}_{AW} = \mathcal{J} \left[ L^2(\mathcal{W}(x, a, b, c, d \mid q) dx) \right].
\]

Therefore, endowing \( \mathcal{PW}_{AW} \) with the inner product

\[
\langle \mathcal{J}(f), \mathcal{J}(g) \rangle_{\mathcal{PW}_{AW}} = \int_{-1}^{1} f(x) g(x) w(x, a, b, c, d \mid q) dx,
\]

the finite Askey-Wilson transform \( \mathcal{J} \) becomes a Hilbert space isometry between \( L^2(\mathcal{W}(x, a, b, c, d \mid q) dx) \) and \( \mathcal{PW}_{AW} \).
It remains to show that the functions $S_n^{(\tilde{a})} (\mu; q)$ provide an orthogonal basis for $\mathcal{P}_{AW}$. By the definitions (3.1) and (3.2),

$$\mathcal{J}(u_n)(\mu) = \int_{-1}^{1} u_n(x)u_\mu(x) w(x, a, b, c, d \mid q) dx$$

$$= (-1)^n q^{-(n(n-1)/2)} \frac{d^n(ab, ac, bc; q)_n}{d^n(ab, ac, bc; q)_n} \int_{-1}^{1} p_n(x)u_\mu(x) w(x, a, b, c, d \mid q) dx.$$

Now we can use Proposition 6 of [6] to conclude that

$$\mathcal{J}(u_n)(\mu) = S_n^{(\tilde{a})} (\mu; q).$$

By (3.2), $\{u_n\}$ is an orthogonal basis of $L^2_w(x, a, b, c, d \mid q)dx$. Since $\mathcal{J}$ is isometric onto $\mathcal{P}_{AW}$, it follows that $S_n^{(\tilde{a})} (\mu; q)$ is a basis of $\mathcal{P}_{AW}$. □

Remark 1. The functions $S_n^{(\tilde{a})} (\gamma; q)$ play the same role in our setting as do the functions $\sin \pi (x - n) / \pi (x - n)$ in the Paley-Wiener space.

Now, Theorem 1 and Theorem B give the following sampling theorem. This has been proved in [6], but the approach with reproducing kernels provides the uniform convergence that will be used in the next section.

**Theorem 2.** For $f \in \mathcal{P}_{AW}$ we have

$$f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(\tilde{a})} (\mu; q),$$

where $S_n^{(\tilde{a})} (\mu; q)$ is given by

$$S_n^{(\tilde{a})} (\mu; q) = (-1)^n q^{\frac{n(n+1)}{2}} (\tilde{a}^2 q^{2n}) (\tilde{a}q^\mu, \tilde{a}q^{-\mu}; q)_\infty (1 - \tilde{a}q^{n+\mu}) (1 - \tilde{a}q^{n-\mu}).$$

The convergence is uniform on every compact subset of the real line.

**Proof.** Observe that from

$$S_n^{(\tilde{a})} (m; q) = \delta_{n,m},$$

we obtain

$$g(\mu, m) = \sum_{n=0}^{\infty} S_n^{(\tilde{a})} (\mu; q) S_n^{(\tilde{a})} (m; q) = S_m^{(\tilde{a})} (\mu; q).$$

Moreover,

$$g(m, m) = S_m^{(\tilde{a})} (m; q) = 1,$$

and the result follows from Theorem 1 and Theorem B. □

### 4. The Askey-Wilson Paley-Wiener Theorem

Recall that the entire function $f$ is of order $\rho$ if

$$\lim_{r \to \infty} \frac{\ln \ln(M(r; f))}{\ln r} = \rho.$$

A constant has order zero, by convention.

The entire function $f$ of positive order $\rho$ is of type $\tau$ if

$$\lim_{r \to \infty} \frac{\ln(M(r; f))}{r^\rho} = \tau.$$
Definition 2. The space $A_{AW}$, which will be the analogue of $A$ in the Askey-Wilson setting, is the space constituted of analytic functions $f$ such that $f \in L^2(w(x, a, b, c, d \mid q), dx)$ and

$$M(r; f) = O(e^{\ln(1/q)r^2}),$$

that is, of order 2 and type $\ln(1/q)$.

It is easy to see that the functions in $A_{AW}$ satisfy the conditions in [18, Theorem 3.1]. We rewrite this statement as:

**Theorem C.** Every $f \in A_{AW}$ admits the expansion

$$f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(\mu)}(\mu; q).$$

The next result is the Paley-Wiener theorem for the Askey-Wilson function transform. The cornerstone of its proof is the fact that the entire function expansion (4.1) and the sampling expansion (3.4) are exactly the same.

**Theorem 3.** If $\max(|a|, |b|, |c|, |d|) < 1$, then $A_{AW} = PW_{AW}$.

**Proof.** Take $f \in PW_{AW}$. By definition we have, for some $u \in L^2(w(x, a, b, c, d \mid q), dx)$,

$$f(\mu) = \mathcal{F}(u)(\mu) = \int_{-1}^{1} u(x) u(\mu) w(x, a, b, c, d \mid q) dx.$$

We need to study the growth of

$$M(r; u(\mu)).$$

From formula (5.4) in [24] and for $0 \leq \theta \leq \pi$, we have

$$u_r(x) = \frac{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, qe^{i\theta}/d; q)_{\infty}}{(ab, ac, bc, c^{2i\theta}; q)_{\infty}} (q^{1-r}/e^{i\theta}; q)_{\infty} (1 + o(1)), \text{ as } r \to \infty.$$

Let $-1 < \delta < 0$. Then

$$M(n + \delta; u(\mu)) = O\left((q^{1-\delta-n}/d; q)_{n}\right).$$

This implies that

$$M(n + \delta; u(\mu)) = O\left((q/d)^n q^{-n(n+1+2\delta)/2}\right).$$

Therefore,

$$\limsup_{r \to \infty} \frac{\ln \ln(M(r; u(\mu)))}{\ln r} = 2$$

and

$$\limsup_{r \to \infty} \frac{\ln(M(r; u(\mu))}{r^2} = \ln (1/q).$$

This condition implies that $u(\mu)$ is of order 2 and type at most $\ln(1/q)$. Therefore,

$$u(\mu)(x) \in A_{AW}.$$

This shows that $f \in A_{AW}$. Conversely, let $f \in A_{AW}$. By Theorem C,

$$f(\mu) = \sum_{n=0}^{\infty} f(n) S_n^{(\mu)}(\mu; q).$$
In the end of the proof of Theorem 1 we have seen that
\[ S_n^{(a)}(\mu; q) = \mathcal{F}(u_n)(\mu). \]

Then, the sampling formula of Theorem 2 can be written as
\[
f(\mu) = \sum_{n=0}^{\infty} f(n) \mathcal{F}(u_n)(\mu) \\
= \sum_{n=0}^{\infty} f(n) \int_{-1}^{1} u_n(x) u_{\mu}(x) w(x, a, b, c, d | q) dx.
\]

The uniform convergence of the sampling series allows us to interchange the integral with the sum in such a way that
\[
f(\mu) = \int_{-1}^{1} \left( \sum_{n=0}^{\infty} f(n) u_n(x) \right) u_{\mu}(x) w(x, a, b, c, d | q) dx.
\]

Then we have written \( f \) in the form
\[
f(\mu) = \mathcal{F}(u)(\mu),
\]
with
\[
u(x) = \left( \sum_{n=0}^{\infty} f(n) u_n(x) \right) \in L^2(w(x, a, b, c, d | q), dx).
\]

As a result, \( f \in \mathcal{P}W_{\text{AW}}. \)

\[ \square \]

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