A CORRECTION TO

“ADJUGATES IN BANACH ALGEBRAS”

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ABSTRACT. Let $A$ be a semisimple unital Banach algebra. We show that $\text{rank}_A(ab) = \text{rank}_A(ba)$ for all $a, b \in A$ if and only if $\text{soc}(A)$ is contained in the center of $A$, and $ab \in \text{soc}(A)$ implies $ba \in \text{soc}(A)$ for all $a, b \in A$. This corrects an erroneous statement in R.E. Harte and C. Hernández, Adjugates in Banach algebras, Proc. Amer. Math. Soc. 134(5) (2005), 1397–1404.

If $A$ is a semisimple Banach algebra with identity element $1$, then the rank of an element $a \in A$ is defined by

\[ \text{rank}_A(a) = \sup_{x \in A} \#\sigma'(x) = \sup_{x \in A} \#\sigma_A'(ax) \leq \infty. \]

Here $\sigma_A(x)$ denotes the spectrum of $x \in A$, $\sigma_A'(x) = (\sigma_A(x) \setminus \{0\})$ and $\#\sigma'(x)$ is the number of elements in $\sigma'(x)$. It can be shown that the set of finite rank elements of $A$ coincides with the socle of $A$ (denoted $\text{soc}(A)$).

Theorem 0.3 of this paper corrects the erroneous claim in [4] (see (2.5)) that $\text{rank}_A(ab) = \text{rank}_A(ba)$ for all $a, b \in A$. The difficulty here stems from the fact that Jacobson’s lemma ([1, p. 33]) is generally speaking not valid for permutations of more than two elements. So although $\sigma_A'(ab) = \sigma_A'(ba)$ for all $a, b \in A$, it is not necessarily true that $\text{rank}_A(ab) = \text{rank}_A(ba)$ for all $a, b \in A$. This is already evident from the fact that in the operator case the standard rank, $\text{rank}(T) = \dim \mathcal{R}(T)$, is equivalent to the spectral rank (1). We start with two lemmas. The first, Lemma 0.1, is an extension of Aupetit and Mouton’s diagonalization theorem.

**Lemma 0.1.** Let $A$ be a semisimple Banach algebra and $0 \neq a \in \text{soc}(A)$. If there exists $y \in A$ commuting with $a$ such that $\text{rank}_A(a) = \#\sigma_A'(ya)$, then there exist $n = \text{rank}_A(a)$ mutually orthogonal minimal projections $p_1, \ldots, p_n$ and non-zero scalars $\alpha_1, \ldots, \alpha_n$ (not necessarily distinct) such that

\[ a = \sum_{j=1}^{n} \alpha_j p_j. \]

**Proof.** Suppose $\text{rank}_A(a) = \#\sigma_A'(ya) = n$ and that $ya = ay$. We first show that this hypothesis implies that we can actually take $y$ invertible. If $a$ is invertible, then for $b \in A$ arbitrary observe that

\[ \#\sigma_A'(b) = \#\sigma_A'(a(a^{-1}b)) \leq \text{rank}_A(a) = n, \]

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which shows that every element of $A$ has a finite spectrum. Consequently, if $a$ is invertible, the Hirschfeld–Johnson criterion implies that $A$ is finite dimensional, and so the Wedderburn–Artin theorem forces $y$ to be invertible. On the other hand, if $0 \in \sigma_A(a)$, then $0 \in \sigma_A(ax)$ for all $x \in A$ because $a$, and hence also $ax$, is a left topological divisor of zero. Since the function $\lambda \mapsto a(\lambda - y)$ is analytic from $\mathbb{C}$ into $A$, and $0 \in \sigma_A(a(\lambda - y))$ for all $\lambda \in \mathbb{C}$ the scarcity theorem [1] Theorem 3.4.25] says that $\{ \lambda \in \mathbb{C} : \#\sigma'_{\lambda}(a(\lambda - y)) < n \}$ is discrete in $\mathbb{C}$. Hence we can find $\lambda$ in the resolvent set of $y$ such that $\#\sigma'_{\lambda}(a(\lambda - y)) = n = \text{rank}_A(a)$. So without loss of generality we may assume $y \in A^{-1}$. By Aupetit and Mouton’s diagonalization theorem [2] Theorem 2.8] there exist mutually orthogonal minimal projections $p_1, \ldots, p_n$ and distinct non-zero scalars $\lambda_1, \ldots, \lambda_n$ such that $ay = \sum_{j=1}^n \lambda_j p_j$. Since for each $j$,  
\[ p_j = \frac{1}{2\pi i} \int_{\Gamma_j} (\lambda - ay)^{-1} d\lambda, \]
where $\Gamma_j$ is a small circle surrounding $\lambda_j$ from the remaining spectrum of $ay$, we see that $y^{-1}$ commutes with the integrand and thus with $p_j$. From the minimality of $p_j$ there exists a corresponding $\beta_j \in \mathbb{C}$ such that  
\[ a = \sum_{j=1}^n \lambda_j p_j y^{-1} = \sum_{j=1}^n \lambda_j p_j y^{-1} p_j = \sum_{j=1}^n \lambda_j \beta_j p_j. \]
If some $\beta_j = 0$, then $p_j y^{-1} = y^{-1} p_j$ implies that $y^{-1} p_j = p_j y^{-1} p_j = \beta_j p_j = 0$ and consequently $p_j = 0$, which is a contradiction. So, $\beta_j \neq 0$ and the thesis follows with $\alpha_j = \lambda_j / \beta_j$. \hfill \Box

Observe that if $y = 1$, then Lemma 0.1 is precisely Aupetit and Mouton’s diagonalization theorem, from which one may further conclude that the $\alpha_j$ are distinct.

For Lemma 0.2 recall that if $A$ is a semisimple Banach algebra and $p$ is a projection in $A$, then $pAp$ is a semisimple Banach algebra with identity element $p$. Moreover, if $z \in pAp$, then $\sigma_A(z) = \sigma'_{pAp}(z)$.

**Lemma 0.2.** If $A$ is a semisimple Banach algebra and $a \in \text{soc}(A)$ is a linear combination of mutually orthogonal projections, say $\{p_1, \ldots, p_n\}$, then there exists a finite dimensional, unital and semisimple subalgebra $B$ of $A$ such that $a \in B$ and $\text{rank}_A(a) = \text{rank}_B(a)$. In particular, we can take $B = pAp$ where $p = \sum_{j=1}^n p_j$ is the identity in $B$ and, moreover, $a$ is then invertible with respect to $p$ in $B$.

**Proof.** If $a \in \text{soc}(A)$ and $a = \lambda_1 p_1 + \cdots + \lambda_n p_n$, then the orthogonality of the $p_j$ implies that each $p_j \in \text{soc}(A)$. Also, it is obvious that each $p_j \in pAp$. From [2] Theorem 2.16] we have that $\text{rank}_A(a) = \sum_{j=1}^n \text{rank}_A(p_j)$ and similarly, applying [2] Theorem 2.16] in the algebra $pAp$, $\text{rank}_{pAp}(a) = \sum_{j=1}^n \text{rank}_{pAp}(p_j)$. But, from the remarks preceding Lemma 0.2 (together with Jacobson’s lemma), we have, for each $j$,  
\[ \text{rank}_{pAp}(p_j) = \sup_{x \in A} \#\sigma'_{pAp}(p_j pxp) = \sup_{x \in A} \#\sigma'_{A}(p_j pxp) \]
\[ = \sup_{x \in A} \#\sigma'_{A}(p_j x) = \text{rank}_A(p_j), \]
which proves the first part. The inverse of $a$ in $pAp$ is of course $\sum_{j=1}^n \frac{1}{\lambda_j} p_j$. \hfill \Box

In the remaining part of this paper, $Z(A)$ denotes the center of $A$. 

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Theorem 0.3. If $A$ is a semisimple Banach algebra, then $\text{rank}_A(ab) = \text{rank}_A(ba)$ for all $a, b \in A$ if and only if both of the following conditions are met:

(i) $\text{soc}(A) \subset Z(A)$;
(ii) $ab \in \text{soc}(A) \Rightarrow ba \in \text{soc}(A)$.

Proof. $\Rightarrow$ Since the finite rank elements coincide with $\text{soc}(A)$, it follows trivially that (ii) holds. To prove (i) it suffices to show, from [3, Corollary 2.2], that 0 is the only nilpotent element of $\text{soc}(A)$. Suppose there is a nilpotent finite rank element $a$ with $a^n = 0 (n \geq 2)$ and $a^k \neq 0$ for $k < n$. With our hypothesis we have, for each $b \in A$, that $\text{rank}_A(aba^{n-1}) = \text{rank}_A(a^n b) = 0$. So

$$\text{rank}_A(a^{n-2}(aba^{n-1})) \leq \text{rank}_A(aba^{n-1}) \Rightarrow \text{rank}_A(a^{n-1}ba^{n-1}) = 0,$$

from which it follows that $(a^{n-1}b)^2 = 0$ and hence that $\sigma_A(a^{n-1}b) = \{0\}$. Since $A$ is semisimple we then have $a^{n-1} = 0$, contradicting the nilpotency index of $a$.

$\Leftarrow$ Let $a, b \in A$ be arbitrary. From the assumption (ii) we may suppose $\text{rank}(ab) = n$. Then, again from (ii), $ba \in \text{soc}(A) \subset Z(A)$. Since $ab \in Z(A)$, Lemma 0.1 gives

$$ab = \alpha_1 p_1 + \cdots + \alpha_n p_n,$$

where the $p_i$ are distinct non-zero and mutually orthogonal minimal projections. Writing $p = \sum_{i=1}^n p_i$ it follows that $ab$ is invertible in the semisimple finite dimensional algebra $pAp$. If we observe that $(ab)^2 = (ba)^2$, then $(ba)^2$ belongs to and is invertible in $pAp$. Invertible elements all having equal rank, we get, using Lemma 0.2,

$$\text{rank}_A(ab) = \text{rank}_{pAp}(ab) = \text{rank}_{pAp}((ba)^2) = \text{rank}_A((ba)^2) \leq \text{rank}_A(ba).$$

So we have shown, for all $a, b \in A$, that $\text{rank}_A(ab) \leq \text{rank}_A(ba)$ and hence the theorem is proved.

The following simple example shows that (ii) is not superfluous in Theorem 0.3:

Let $C(l^2)$ be the ideal of compact operators on $l^2$ and denote the Calkin algebra by $C = B(l^2)/C(l^2)$. Then $C$ is a $B^*$-algebra and hence semisimple. Denote by $\Omega : B(l^2) \to C$, $\Omega(T) = \bar{T} = T + C(l^2)$ the canonical quotient homomorphism. If $\text{soc}(C) \neq \{0\}$, then $\Omega^{-1}(\text{soc}(C)) \neq B(l^2)$ would be a two-sided ideal of $B(l^2)$ properly containing $C(l^2)$, which violates the fact that $C(l^2)$ is the unique non-trivial two-sided ideal of $B(l^2)$. So $\text{soc}(C) = \{0\}$. Let $T \in B(l^2)$ be defined by the standard unilateral shift followed by the projection on $l^2$ which annihilates odd coordinates; let $S \in l^2$ be the projection which annihilates even coordinates. So if we take the semisimple algebra $A = C \times \mathbb{C}^n$, then $\text{soc}(A) \subset Z(A)$. However with $a = (S, 1)$ and $b = (T, 1)$ we get $\text{rank}_A(ab) = n$ but $\text{rank}_A(ba) = \infty$.

References


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