ON THE LENGTH OF CRITICAL ORBITS OF STABLE QUADRATIC POLYNOMIALS

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Abstract. We use the Weil bound of multiplicative character sums, together with some recent results of N. Boston and R. Jones, to show that the critical orbit of quadratic polynomials over a finite field of \( q \) elements is of length \( O(q^{3/4}) \), improving upon the trivial bound \( q \).

1. Introduction

Let \( \mathbb{F}_q \) be a finite field of \( q \) elements. For a polynomial \( f \in \mathbb{F}_q[X] \) we define the sequence of iterations:

\[
 f^{(0)}(X) = X, \quad f^{(n)}(X) = f\left(f^{(n-1)}(X)\right), \quad n = 1, 2, \ldots.
\]

Following [1, 2, 8, 9], we say that \( f \) is stable if all polynomials \( f^{(n)} \) are irreducible over \( \mathbb{F}_q \).

We now assume that \( q \) is odd.

As in [9], for a quadratic polynomial \( f(X) = aX^2 + bX + c \in \mathbb{F}_q[X], a \neq 0 \), we define \( \gamma = -b/(2a) \) as the unique critical point of \( f \) (that is, the zero of the derivative \( f' \)) and consider the set

\[
 \text{Orb}(f) = \{f^{(n)}(\gamma) : n = 2, 3, \ldots\},
\]

which is called the critical orbit of \( f \). Clearly there is some \( t \) such that \( f^{(t)}(\gamma) = f^{(s)}(\gamma) \) for some positive integer \( s < t \). Then \( f^{(n+t)}(\gamma) = f^{(n+s)}(\gamma) \) for any \( n \geq 0 \). Accordingly, for the smallest value of \( t_f \) with the above condition, we have

\[
 \text{Orb}(f) = \{f^{(n)}(\gamma) : n = 2, \ldots, t_f\}
\]

and \( \#\text{Orb}(f) = t_f - 1 \) or \( \#\text{Orb}(f) = t_f - 2 \) (depending on whether \( s = 1 \) or \( s \geq 2 \) in the above). It is shown in [7, 8, 9] that critical orbits play a very important role in the dynamics of polynomial iterations.

Trivially we have \( t_f \leq q + 1 \). In fact, by the Birthday Paradox one expects that \( t_f \) is of order \( q^{1/2} \) (for a sufficiently large \( q \)). Indeed, it is natural to expect that the map \( x \mapsto f(x) \) behaves like a random map on \( \mathbb{F}_q \), for which the trajectory length is of this order; see [3] for a detailed treatment of cycle structure of random maps on finite sets. For example, the Pollard integer factorisation algorithm (where a quadratic
polynomial \( f(X) = X^2 + c \) is iterated in a residue ring; see \([3\) Section 5.2.1]) is based on this assumption.

Here we obtain a nontrivial upper bound on the orbit length of stable quadratic polynomials:

**Theorem 1.** For any odd \( q \) and any stable quadratic polynomial \( f \in \mathbb{F}_q[X] \) we have

\[
t_f = O \left( q^{3/4} \right).
\]

By \([3\) Proposition 3], a quadratic polynomial \( f \in \mathbb{F}_q[X] \) is stable if the *adjusted orbit*

\[
\text{Orb}(f) = \{-f(\gamma)\} \cup \text{Orb}(f)
\]

contains no squares. We also recall that \( \alpha \in \mathbb{F}_q \) is a square if either \( \alpha = 0 \) or \( \alpha^{(q-1)/2} = 1 \), which can be tested (via repeated squaring) in \( O(\log q) \) field operations. Combining these with the bound of Theorem 1 we immediately obtain:

**Corollary 2.** For any odd \( q \), a quadratic polynomial \( f \in \mathbb{F}_q[X] \) can be tested for stability in time \( q^{3/4+\omega(1)} \).

Our proof is based on the Weil bound for multiplicative character sums with polynomials; see \([6\) Theorem 11.23].

Finally, we remark that estimating the size of the set of stable quadratic polynomials \( aX^2 + bX + c \in \mathbb{F}_q[X] \) is a very interesting question to which we hope our technique can apply as well.

### 2. Proof of Theorem 1

Let \( \chi \) be the quadratic character of \( \mathbb{F}_q \).

By \([3\) Proposition 3], if a quadratic polynomial \( f \in \mathbb{F}_q[X] \) is stable, then \( \text{Orb}(f) \) contains no squares, that is, \( \chi \left( f^{(n)}(\gamma) \right) = -1, n = 2, 3, \ldots \)

We now fix an integer parameter \( K \) and note that for any \( n \geq 1 \), we have simultaneously

\[
\chi \left( f^{(k+n)}(\gamma) \right) = -1, \quad k = 1, \ldots, K,
\]

which we rewrite as

\[
(1) \quad \chi \left( f^{(k)} \left( f^{(n)}(\gamma) \right) \right) = -1, \quad k = 1, \ldots, K.
\]

Since by the definition of \( t_f \) the values \( f^{(n)}(\gamma), n = 1, \ldots, t_f - 1 \), are pairwise distinct elements of \( \mathbb{F}_q \), we derive from (1) that

\[
(2) \quad t_f - 1 \leq \#T_q(K),
\]

where

\[
T_q(K) = \left\{ x \in \mathbb{F}_q : \chi \left( f^{(k)}(x) \right) = -1, k = 1, \ldots, K \right\}.
\]

We have

\[
(3) \quad \#T_q(K) = \frac{1}{2K} \sum_{x \in \mathbb{F}_q} \prod_{k=1}^{K} \left( 1 - \chi \left( f^{(k)}(x) \right) \right)
\]

since for every \( x \in T_q(K) \) the product on the right hand side of (3) is \( 2^K \); otherwise it is 0 when \( \chi(f^{(k)}(x)) = 1 \) for at least one \( k = 1, \ldots, K \) (note that since by our assumption \( f^{(k)}(X) \) is irreducible over \( \mathbb{F}_q \) we have \( f^{(k)}(x) \neq 0 \) for \( x \in \mathbb{F}_q \)).
Just expanding the product in (3), we obtain $2^k - 1$ character sums of the shape

$$\sum_{x \in \mathbb{F}_q} \chi \left( \prod_{j=1}^{\nu} f^{(k_j)}(x) \right), \quad 1 \leq k_1 < \ldots < k_\nu \leq K,$$

with $\nu \geq 1$ and one trivial sum that equals $q$ (corresponding to the terms 1 in the product in (3)).

Clearly $f^{(k)}(X)$ is a polynomial of degree $2^k$. Furthermore, by our assumption, each polynomial $f^{(k)}(X)$ is irreducible; therefore none of the polynomials

$$\prod_{j=1}^{\nu} f^{(k_j)}(X) \in \mathbb{F}_q[X], \quad 1 \leq k_1 < \ldots < k_\nu \leq K,$$

are a perfect square in the algebraic closure of $\mathbb{F}_q$. Therefore the Weil bound (see [6, Theorem 11.23]) applies to every sum (4) and implies that each of them is $O(2^Kq^{3/2})$. Therefore

$$\# T_q(K) = \frac{1}{2^K} q + O(2^Kq^{1/2}).$$

Choosing $K$ to satisfy

$$2^K \leq q^{1/4} < 2^{K+1}$$

and combining (2) and (5), we conclude the proof.

3. Comments

It is certainly interesting to obtain nontrivial estimates on the size $S_q$ of the set of triples $(a, b, c) \in \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q$ which correspond to stable quadratic polynomials $f(X) = aX^2 + bX + c$. Denoting by $F_k(a, b, c)$ the $k$th element of the critical orbit of $f$, we see that for any integer parameter $K$ we have

$$S_q \leq \# \mathcal{W}_q(K),$$

where

$$\mathcal{W}_q(K) = \{(a, b, c) \in \mathbb{F}_q^3 \times \mathbb{F}_q \times \mathbb{F}_q : \chi(F_k(a, b, c)) = -1, \ k = 1, \ldots, K\},$$

and as before $\chi$ denotes the quadratic character of $\mathbb{F}_q$. As in the proof of Theorem 1 we have

$$\# \mathcal{W}_q(K) \leq \frac{1}{2^K} \sum_{(a,b,c)\in \mathbb{F}_q^3 \times \mathbb{F}_q \times \mathbb{F}_q} \prod_{k=1}^{K} (1 - \chi(F_k(a, b, c)))$$

since for every triple $(a, b, c) \in \mathcal{W}_q(K)$ the product on the right hand side of (7) is $2^K$; otherwise it is either 0 (when $\chi(F_k(a, b, c)) = 1$ for at least one $k = 1, \ldots, K$) or 1 (when $F_1(a, b, c) = \ldots = F_K(a, b, c) = 0$).

Clearly $F_k(a, b, c)$ is a rational function in $a, b, c$ of degree at most $O(2^k)$. Thus expanding the product in (7), we obtain $2^K - 1$ character sums of the shape

$$\sum_{(a,b,c)\in \mathbb{F}_q^3 \times \mathbb{F}_q \times \mathbb{F}_q} \chi \left( \prod_{j=1}^{\nu} F_{k_j}(a, b, c) \right), \quad 1 \leq k_1 < \ldots < k_\nu \leq K,$$

with $\nu \geq 1$ and one trivial sum corresponding to 1 in (7). Assuming that one can prove that the Weil-type bound $O(2^Kq^{5/2})$ applies to all of them, we obtain from (6) that $S_q = O(q^3/2^K + 2^Kq^{5/2})$ and optimising the choice of $K$ we derive.
\[ S_q = O(q^{11/4}). \] In fact, for a nontrivial estimate of \( S_q \) it is enough to show that almost all sums admit a nontrivial estimate, which has actually been recently done in [5], where the bound \( S_q = O(q^{14/5}) \) is obtained.

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