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ON THE DISTRIBUTION AND INTERLACING OF THE ZEROS OF STIELTJES POLYNOMIALS

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Abstract. Polynomial solutions to the generalized Lamé equation, the *Stieltjes polynomials*, and the associated *Van Vleck polynomials* have been studied since the 1830's, beginning with Lamé in his studies of the Laplace equation on an ellipsoid, and in an ever widening variety of applications since. In this paper we show how the zeros of Stieltjes polynomials are distributed and present two new interlacing theorems. We arrange the Stieltjes polynomials according to their Van Vleck zeros and show, firstly, that the zeros of successive Stieltjes polynomials of the same degree interlace, and secondly, that the zeros of certain Stieltjes polynomials of successive degrees interlace.

1. Introduction and main results

Let $\alpha_1 < \cdots < \alpha_p$ be any p distinct real numbers, and let $\rho_1, \ldots, \rho_p$ be positive numbers. The *generalized Lamé equation*[1] is the second order ODE given by

$$S''(x) + \sum_{j=1}^{p} \frac{\rho_j}{x - \alpha_j} S'(x) = \frac{V(x)}{A(x)} S(x), \tag{1}$$

where $A(x) = \prod_{j=1}^{p}(x - \alpha_j)$ and $V(x)$ is a polynomial of degree $p - 2$. A result of Stieltjes [22], known as the Heine-Stieltjes Theorem [23], says that there exist exactly $\binom{k+p-2}{k}$ polynomials $V(x)$ for which (1) has a polynomial solution S of degree k. These polynomial solutions are often called *Stieltjes* or *Heine-Stieltjes polynomials*, and the corresponding polynomials $V(x)$ are known as *Van Vleck polynomials*.

In this paper we will consider the case when $p = 3$, in which case (1) is a Heun equation and the Heine-Stieltjes Theorem says that there are $k + 1$ values of ν for which (1) has a polynomial solution of degree k with $V(x) = \mu(x - \nu)$. We refer to these values of ν as the Van Vleck zeros of order k.

The equation (1) was studied by Lamé in the 1830's in the special case, $\rho_j = 1/2$, in connection with the separation of variables in the Laplace equation using elliptical coordinates [25, Ch. 23]. The equation has since found a strikingly wide variety

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[1]Usually the term *generalized Lamé equation* refers to such equations in which the α_j's are allowed to be complex. But here we only consider the real case.

of other applications, from electrostatics [10, 13, 8, 17] to completely quantum integrable systems such as the quantum C. Neumann oscillators, the asymmetric top and the geodesic flow on an ellipsoid [1, 9, 5, 11]. In particular, the zeros of the Stieltjes polynomials may be nicely interpreted as the equilibrium positions of k unit charges in a logarithmic potential in which at each position α_j in the complex plane is fixed a charge of magnitude ρ_j.

Much is known about the properties of Stieltjes and Van Vleck polynomials in the case when the degree of the Stieltjes polynomial is fixed. A few important facts are that the zeros of any Stieltjes polynomial are simple, lie inside the interval (α_1, α_3), and none of them can equal α_2 or its corresponding Van Vleck zero. Similarly, for fixed k, the Van Vleck zeros are distinct and also lie within (α_1, α_3). The proofs of these results can be found in [24, 23, 18, 19].

Recently we showed in [6] that the Van Vleck zeros of successive orders interlace. That is, if the Van Vleck zeros of order k are written in increasing order as $\nu_1^{(k)} < \nu_2^{(k)} < \cdots < \nu_{k+1}^{(k)}$, then

$$\alpha_1 < \nu_i^{(k+1)} < \nu_i^{(k)} < \nu_{i+1}^{(k+1)} < \alpha_3, \quad i = 1, \ldots, k+1. \tag{2}$$

Much of the research in the past several years has focused on the asymptotic properties of the zeros of Stieltjes and Van Vleck polynomials as the degree of the corresponding Stieltjes polynomials tends toward infinity [4, 3, 16, 14, 15, 20, 21]. Interlacing theorems such as the one above are interesting, not only for what they tell us about the zeros for a finite degree, but also because they help us to understand such asymptotic limits. They are "classical" results in the sense of being statements about finite degree polynomials, and they are also a bridge to connect other classical results with asymptotic limits. Our results below are further steps in this direction.

In this paper we present a theorem on the distribution of the Stieltjes zeros and two additional interlacing theorems. First we note that the leading coefficient μ of $V(x)$ is determined by the degree k of the Stieltjes polynomials by substitution and identification of the powers as

$$\mu = \mu_k = k\left(k - 1 + \rho_1 + \rho_2 + \rho_3\right).$$

For each positive integer k we label the $k+1$ Stieltjes polynomials of degree k according to their Van Vleck zeros as $S_j^{(k)}(x)$. That is, $S_j^{(k)}(x)$ is the polynomial of degree k that satisfies

$$\left[\frac{d^2}{dx^2} + \sum_{j=1}^{3} \frac{\rho_j}{x - \alpha_j}\frac{d}{dx} - \frac{\mu_k\left(x - \nu_j^{(k)}\right)}{A(x)}\right] S_j^{(k)}(x) = 0. \tag{3}$$

Our first result is a strengthening of a classical result of Stieltjes. Let I be the closed interval bounded by α_2 and the Van Vleck zero $\nu_j^{(k)}$. Then we have the following.

Theorem 1. *There are exactly $j-1$ zeros of $S_j^{(k)}$ in the interval $(\alpha_1, \alpha_2) \setminus I$ and $k-j+1$ zeros of $S_j^{(k)}$ in the interval $(\alpha_2, \alpha_3) \setminus I$.*

The following two theorems establish that the zeros of successive Stieltjes polynomials of the same degree interlace and that the zeros of certain Stieltjes polynomials of successive degrees interlace. The interlacing properties are illustrated in Figure 1.

Theorem 2. *The zeros of $S_j^{(k)}$ and $S_{j+1}^{(k)}$ interlace; between any two consecutive zeros of $S_j^{(k)}$ there is exactly one zero of $S_{j+1}^{(k)}$. Moreover, the smallest zero of $S_{j+1}^{(k)}$ is less than the smallest zero of $S_j^{(k)}$. That is, let $x_{i,j}^{(k)}$, $i = 1, \ldots, k$, be the zeros of the Stieltjes polynomial $S_j^{(k)}$, arranged in increasing order. Then*

$$\alpha_1 < x_{1,j+1}^{(k)} < x_{1,j}^{(k)} < x_{2,j+1}^{(k)} < x_{2,j}^{(k)} < \cdots < x_{k,j+1}^{(k)} < x_{k,j}^{(k)} < \alpha_3. \tag{4}$$

Theorem 3. *The zeros of $S_j^{(k)}$ and $S_l^{(k+1)}$ interlace if and only if $l = j$ or $l = j+1$. That is,*

$$\begin{aligned} &\alpha_1 < x_{i,j}^{(k+1)} < x_{i,j}^{(k)} < x_{i+1,j}^{(k+1)} < \alpha_3, \quad \textit{and} \\ &\alpha_1 < x_{i,j+1}^{(k+1)} < x_{i,j}^{(k)} < x_{i+1,j+1}^{(k+1)} < \alpha_3, \quad i = 1, \ldots, k. \end{aligned} \tag{5}$$

If l is neither j nor $j+1$, then the zeros of $S_j^{(k)}$ and $S_l^{(k+1)}$ do not interlace.

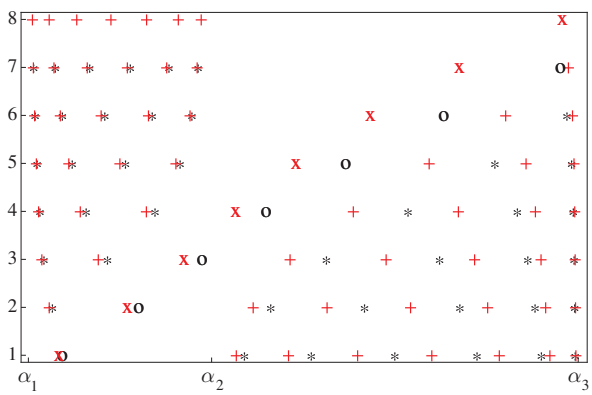

FIGURE 1. Zeros of Stieltjes and Van Vleck polynomials. On row j are shown $*$: zeros of $S_j^{(k)}$; $+$: zeros of $S_j^{(k+1)}$; O: Van Vleck zero $\nu_j^{(k)}$; X: Van Vleck zero $\nu_j^{(k+1)}$, for $k = 6$. Values of the parameters are $(\alpha_1, \alpha_2, \alpha_3) = (-1, 0, 2)$, $(\rho_1, \rho_2, \rho_3) = (1, 2, 1/3)$.

In the following section we prove these results. Following this, we conclude with a section of remarks and comments on open problems. We show how these results can be combined with known asymptotic properties of Stieltjes and Van Vleck polynomials. In particular, we construct sequences of Van Vleck zeros to converge to any number in $[\alpha_1, \alpha_3]$. We also comment on the possible existence of orthogonal sequences of Stieltjes polynomials. This is a natural question to ask, for several reasons. For one thing, we see from Theorem 3 that any sequence of Stieltjes polynomials $(S_{j_k}^{(k)})$ with $j_{k+1} = j_k$ or $j_k + 1$ shares at least one thing in common with orthogonal polynomials, namely the well known fact that the zeros of orthogonal polynomials of successive degrees interlace. We will also show that certain of the sequences $(S_{j_k}^{(k)})$ have the same asymptotic zero distribution as sequences of orthogonal polynomials.

The relationship between orthogonal polynomials and second order differential equations has been for many years a rich source of results and problems. Bochner

[2] classified all orthogonal polynomials satisfying an eigenvalue problem of the type

$$LP_n = \lambda_n P_n, \tag{6}$$

where L is a second order differential operator. (See [12] for a recent generalization of these results.) In the Bochner problem, however, the operator L does not depend on the degree of the polynomial solution, and there is exactly one polynomial solution for each degree. This is not the case for the equation (3) considered in this paper, which is instead an eigenvalue problem of the type

$$L_n P_n = \lambda_n P_n, \tag{7}$$

where now L_n depends on the degree of the polynomial solutions, and there is more than one solution for each n. Determining the existence of orthogonal sequences for (7) is thus much more difficult than for (6).

2. Proofs of main results

For the proofs of the theorems we will need the following lemma.

Lemma 1. *There is a zero of $S_{j+1}^{(k)}$ between every two zeros of $S_j^{(k)}$ in the interval (α_1, α_2), and between every zero of $S_j^{(k)}$ in (α_1, α_2) and either of the singular points α_1, α_2. Likewise, there is a zero of $S_j^{(k)}$ between every two zeros of $S_{j+1}^{(k)}$ in the interval (α_2, α_3), and between every zero of $S_{j+1}^{(k)}$ in (α_2, α_3) and either of the singular points α_2, α_3.*

Proof. For this proof, since we are dealing with Stieltjes polynomials of fixed degree k, we will omit the superscripts and simply write $S_j = S_j^{(k)}$ and $S_{j+1} = S_{j+1}^{(k)}$. Now, the Stieltjes polynomials S_j and S_{j+1} satisfy

$$\left[\frac{d^2}{dx^2} + \sum_{j=1}^{3} \frac{\rho_j}{x-\alpha_j}\frac{d}{dx} - \frac{\mu_k\left(x-\nu_j^{(k)}\right)}{A(x)}\right] S_j(x) = 0, \tag{8}$$

$$\left[\frac{d^2}{dx^2} + \sum_{j=1}^{3} \frac{\rho_j}{x-\alpha_j}\frac{d}{dx} - \frac{\mu_k\left(x-\nu_{j+1}^{(k)}\right)}{A(x)}\right] S_{j+1}(x) = 0. \tag{9}$$

Define the integrating factor

$$J(x) = \prod_{j=1}^{3} |x-\alpha_j|^{\rho_j}.$$

Then

$$J'(x) = J(x)\sum_{j=1}^{3}\frac{\rho_j}{x-\alpha_j}.$$

Upon multiplying (8) by S_{j+1} and (9) by S_j, taking the difference of the result, and then multiplying by J, we obtain

$$\frac{d}{dx}\left[J\left(S_{j+1}'S_j - S_{j+1}S_j'\right)\right] = QS_jS_{j+1}, \tag{10}$$

where

$$Q(x) = (\nu_j - \nu_{j+1})\frac{J(x)}{A(x)}.$$

Note that $Q < 0$ in (α_1, α_2) and $Q > 0$ in (α_2, α_3). Now, consider two consecutive zeros of S_j, x_1 and x_2, in the interval (α_1, α_2). Then S_j' must alternate signs at x_1 and x_2. Thus, the expression

$$J\left(S_{j+1}'S_j - S_{j+1}S_j'\right)\Big|_{x=x_1}^{x=x_2} = -\,J\left(S_{j+1}S_j'\right)\Big|_{x=x_1}^{x=x_2}$$

has the same sign as S_jS_{j+1} in (x_1, x_2). But (10) implies that this expression is negative if $S_jS_{j+1} > 0$ in (x_1, x_2) and positive if $S_jS_{j+1} < 0$ in (x_1, x_2). Thus it must be that S_{j+1} changes sign in (x_1, x_2), and we have shown that between any two consecutive zeros of S_j in (α_1, α_2), there is a zero of S_{j+1}.

Now let x_1 be the smallest zero of S_j in (α_1, α_2). Then, since $J(\alpha_1) = 0$,

$$J\left(S_{j+1}'S_j - S_{j+1}S_j'\right)\Big|_{x=\alpha_1}^{x=x_1} = -J(x_1)S_{j+1}(x_1)S_j'(x_1)$$

also has the same sign as S_jS_{j+1} in (α_1, x_1) if S_{j+1} does not change sign in this interval, again contradicting (10). A similar argument can be applied to the largest zero of S_j in (α_1, α_2) and α_2, which establishes that between every zero of S_j in (α_1, α_2) and either of the singular points α_1, α_2, there is a zero of S_{j+1}.

This argument may be exactly repeated in the interval (α_2, α_3), noting that $Q > 0$ in this interval. It follows that between any two consecutive zeros of S_{j+1} in (α_2, α_3), or between α_2 and the smallest zero of S_{j+1} in (α_2, α_3), or between the largest zero of S_{j+1} in (α_2, α_3) and α_3 there is a zero of S_j. □

Proof of Theorem 1. First we show that there are no zeros of S_j between α_2 and its corresponding Van Vleck zero ν. According to Shah [18, cf. Theorem 2], between any zero of S_j and ν there is either a zero of $S_j'(x)$ or α_2. Suppose $\nu > \alpha_2$ and that there is a zero x_1 of $S_j(x)$ between α_2 and ν. Since there is a zero of $S_j'(x)$ between x_1 and ν, there must be a zero x_2 of $S_j(x)$ greater than ν. We may assume that x_1 and x_2 are consecutive. But since the zeros of $S_j(x)$ are simple, there cannot be a zero of $S_j'(x)$ in both intervals (x_1, ν) and (ν, x_2), which is a contradiction. The case when $\nu < \alpha_2$ is similar.

The theorem will thus be proved once we have shown that there are $j - 1$ zeros of S_j in the interval (α_1, α_2). According to a classical result of Stieltjes [22, 25], every possible distribution of the zeros of the Stieltjes polynomials in the intervals (α_1, α_2) and (α_2, α_3) occurs. That is, for each integer k and any integer $0 \le m \le k$, there is a Stieltjes polynomial of degree k with m zeros in (α_1, α_2) and $k - m$ zeros in (α_2, α_3). Thus it suffices to show that there are at least as many zeros of $S_{j+1}^{(k)}$ as there are of $S_j^{(k)}$ in (α_1, α_2). But this is an immediate consequence of Lemma 1. □

Proof of Theorem 2. Combining Lemma 1 with Theorem 1, we see that between the $j-1$ zeros of S_j in (α_1, α_2) there are $j-2$ zeros of S_{j+1}. The other two zeros of S_{j+1} in (α_1, α_2) lie between α_1 and the smallest zero of S_j and between the largest zero of S_j in (α_1, α_2) and α_2. Similarly, between the $k-j$ zeros of S_{j+1} in (α_2, α_3) there are $k-j-1$ zeros of S_j, and the other two zeros of S_j in (α_2, α_3) lie between α_2 and the smallest zero of S_{j+1} in (α_2, α_3) and between the largest zero of S_{j+1} in (α_2, α_3) and α_3. We have thus accounted for all of the zeros of S_j and S_{j+1}, and the interlacing is proved. □

We have actually proved a stronger statement than Theorem 2. We note this in the following corollary.

Corollary 1. *Let $l > j$. Then between any two zeros of $S_j^{(k)}$ in (α_1, α_2) there is a zero of $S_l^{(k)}$. Between any two zeros of $S_l^{(k)}$ in (α_2, α_3) there is a zero of $S_j^{(k)}$.*

Proof of Theorem 3. First we prove the "only if" part. In order for the zeros of $S_i^{(k+1)}$ and $S_j^{(k)}$ to interlace, it must be the case that the smallest zero of $S_i^{(k+1)}$ is smaller than the smallest zero of $S_j^{(k)}$, which is impossible if $i < j$, since then there would be fewer zeros of $S_i^{(k+1)}$ than of $S_j^{(k)}$ in the interval (α_1, α_2). Similarly, if $i > j+1$, then there are fewer zeros of $S_i^{(k+1)}$ than of $S_j^{(k)}$ in the interval (α_2, α_3).

For the "if" part, as in the proof of Lemma 1, we derive the following expression:

$$\frac{d}{dx}\left[J\left(\frac{dS_i^{(k+1)}}{dx}S_j^{(k)} - S_i^{(k+1)}\frac{dS_j^{(k)}}{dx}\right)\right] = QS_j^{(k)}S_i^{(k+1)}, \tag{11}$$

where in this case,

$$\begin{aligned} Q(x) &= (\mu_{k+1} - \mu_k)\,\frac{J(x)}{A(x)}\,(x - \hat{\nu}_i)\,, \quad \text{and} \\ \hat{\nu}_i &= \frac{\mu_{k+1}\nu_i^{(k+1)} - \mu_k\nu_j^{(k)}}{\mu_{k+1} - \mu_k}. \end{aligned}$$

Note that (2) implies that

$$\hat{\nu}_j < \nu_j^{(k+1)} < \alpha_3 \quad \text{and} \quad \alpha_1 < \nu_{j+1}^{(k+1)} < \hat{\nu}_{j+1}.$$

Suppose, for the moment, that $\alpha_1 < \hat{\nu}_j < \alpha_2$. Then, since $Q > 0$ in $(\hat{\nu}_j, \alpha_2)$, between every two consecutive zeros of $S_j^{(k+1)}$ in $(\hat{\nu}_j, \alpha_2)$ and between the largest zero of $S_j^{(k+1)}$ in $(\hat{\nu}_j, \alpha_2)$ and α_2 there is a zero of $S_j^{(k)}$. Since $Q < 0$ in $(\alpha_1, \hat{\nu}_j)$, there is a zero of $S_j^{(k+1)}$ between every two zeros of $S_j^{(k)}$ in $(\alpha_1, \hat{\nu}_j)$ and between α_1 and the smallest zero of $S_j^{(k)}$ in $(\alpha_1, \hat{\nu}_j)$. This accounts for all of the $j-1$ zeros of $S_j^{(k)}$ and of $S_j^{(k+1)}$ in (α_1, α_2). Similarly, since $Q < 0$ in (α_2, α_3), between every two zeros of $S_j^{(k)}$ in (α_2, α_3) and between every zero of $S_j^{(k)}$ in (α_2, α_3) and either of the singular points α_2, α_3 there is a zero of $S_j^{(k+1)}$. This accounts for the k zeros of $S_j^{(k)}$ and the $k+1$ zeros of $S_j^{(k+1)}$, and the interlacing is proved in this case.

A nearly identical argument holds in the case when $\alpha_2 < \hat{\nu}_{j+1} < \alpha_3$ to show that the zeros of $S_{j+1}^{(k+1)}$ and $S_j^{(k)}$ interlace. The proof is thus completed by the following lemma. □

Lemma 2. *$\alpha_1 < \hat{\nu}_j < \alpha_2$ and $\alpha_2 < \hat{\nu}_{j+1} < \alpha_3$.*

Proof. Suppose that $\hat{\nu}_j \le \alpha_1$. Then it would be the case that $Q > 0$ in (α_1, α_2) and by arguments similar to the above, between every two zeros of $S_j^{(k+1)}$ in (α_1, α_2) and between every zero of $S_j^{(k+1)}$ in (α_1, α_2) and either of the singular points α_1, α_2 there is a zero of $S_j^{(k)}$. But this would imply the existence of at least j zeros of $S_j^{(k)}$ in (α_1, α_2), which contradicts Theorem 1. Likewise, if $\hat{\nu}_j \ge \alpha_2$, a similar argument would imply the existence of at least j zeros of $S_j^{(k+1)}$ in (α_1, α_2), which is also a contradiction. This argument may be repeated to prove the second statement. □

3. Concluding remarks

In [4] and [1] it is shown that the asymptotic density of the Van Vleck zeros is supported on (α_1, α_3). Thus, given any point $\nu \in (\alpha_1, \alpha_3)$ there is a sequence of Van Vleck zeros that converges to ν as $k \to \infty$. Martínez-Finkelshtein and Saff [16] also consider the asymptotics of Stieltjes and Van Vleck zeros. Theorems 1 and 2 of [16] give the asymptotic distribution of the zeros of a sequence of Stieltjes and Van Vleck polynomials under the assumption that the proportion of Stieltjes zeros in the interval (α_1, α_2) tends to $\theta \in [0, 1]$. Using the ordering in Theorem 1 of the present paper, we may modestly improve these results by explicitly constructing a sequence of Van Vleck zeros to converge to any number in (α_1, α_3).

Suppose the sequence $\left(S_{j_k}^{(k)}\right)$ is chosen such that the fraction of the zeros of $S_{j_k}^{(k)}$ in (α_1, α_2) tends to θ. Some simple but tedious calculations show that the limit $\lim_{k\to\infty} \nu_{j_k}^{(k)} = \nu$ of the corresponding Van Vleck zeros is determined by

$$\frac{1}{\pi}\int_{\alpha_1}^{\min(\alpha_2,\nu)} \sqrt{\left|\frac{\nu - x}{A(x)}\right|}dx = \theta. \tag{12}$$

Moreover, let I by the interval bounded by α_2 and ν. Then the asymptotic distribution of the Stieltjes polynomials $S_{j_k}^{(k)}$ is given by

$$\rho_S(x) = \begin{cases} \dfrac{1}{\pi}\sqrt{\dfrac{\nu - x}{A(x)}} & \text{if} \quad x \in (\alpha_1, \alpha_3) \setminus I, \\ 0 & \text{if} \quad x \in I. \end{cases} \tag{13}$$

We note, in particular, that if $\nu = \alpha_1, \alpha_2$ or α_3, then ρ_S is the so-called "arcsine distribution" supported on the intervals (α_2, α_3), (α_1, α_3) and (α_1, α_2), respectively.

Equations (12) and (13) follow from the results of [16] and demonstrate that for any $\nu \in [\alpha_1, \alpha_3]$ there exists a sequence of Van Vleck zeros that converges to ν. With Theorem 1 of the present paper we can provide a simple way to explicitly construct such a sequence. Since, by Theorem 1, $j_k - 1$ zeros of $S_{j_k}^{(k)}$ lie in (α_1, α_2), if $j_k/k \to \theta$, then the fraction of zeros of $S_{j_k}^{(k)}$ in (α_1, α_2) tends to θ. Combining the ordering of Theorem 1 with the asymptotic properties of [16], we have the following.

Proposition 1. *(i) Let (j_k) be a sequence of positive integers such that $j_k/k \to \theta$. Then we have the limit of sequences of Van Vleck zeros:*

$$\lim_{k\to\infty} \nu_{j_k}^{(k)} = \nu, \tag{14}$$

where ν is determined by (12).

(ii) Given any $\nu \in [\alpha_1, \alpha_3]$, if θ is defined by (12) *and $j_k/k \to \theta$, then the sequence $\left(\nu_{j_k}^{(k)}\right)$ converges to ν.*

We may also calculate conditions under which the limiting distribution of Stieltjes zeros is supported throughout (α_1, α_3). A necessary condition given by Theorem 1 is that $\nu = \alpha_2$, and (13) tells us that this is also a sufficient condition. In order for the limiting distribution to be supported on (α_1, α_3), the fraction of the zeros of $S_{j_k}^{(k)}$ in (α_1, α_2) must tend to θ_c, where θ_c is calculated by setting $\nu = \alpha_2$ in (12). A simple calculation shows that this is (see also [16, Prop. 1])

$$\theta_c = \frac{2}{\pi}\sin^{-1}\sqrt{\frac{\alpha_2 - \alpha_1}{\alpha_3 - \alpha_1}}. \tag{15}$$

Perhaps the most intriguing open question in the $p = 3$ case is whether there exist sequences of orthogonal Stieltjes polynomials, for, by Theorem 3, if the sequence (j_k) satisfies $j_{k+1} = j_k$ or $j_k + 1$ (for example $j_k = \lceil k\theta + 1 \rceil$), then the sequence of Stieltjes polynomials $\left(S_{j_k}^{(k)}\right)$ is an infinite sequence of polynomials with interlacing zeros and asymptotic zero distribution given by (13). Moreover, when ν in (13) is one of $\alpha_1, \alpha_2, \alpha_3$ or, equivalently, if $\theta = 0, 1$ or θ_c, the sequence $\left(S_{\lceil k\theta+1 \rceil}^{(k)}\right)$ is an infinite sequence of polynomials with interlacing zeros and asymptotic zero distribution identical to the asymptotic zero distribution of a sequence of orthogonal polynomials.

However, orthogonality is a rather strict condition, and we conjecture that given any measure ω, when $p > 2$ there is no sequence of Stieltjes polynomials that is orthogonal with respect to ω. If it turns out that this conjecture is correct, it would be interesting to see what role is played by these special sequences of Stieltjes polynomials that share so many properties with orthogonal polynomials, which we might call "nearly orthogonal systems".

Remarks. Some of the results of the current paper have recently been generalized by P. Brändén [7] to the case when $p > 3$. However, the proofs of these results require a large amount of additional information and numerous techniques.

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