ON THE STRUCTURE OF THE SPACE OF CUSP FORMS
FOR A SEMISIMPLE GROUP OVER A NUMBER FIELD

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Abstract. Let $G$ be a semisimple algebraic group defined over a number field $k$. We study unramified irreducible components of irreducible automorphic cuspidal representations in the space of cusp forms $A_{cusp}(G(k) \setminus G(k))$ using the action of an unramified Hecke algebra on compactly supported cuspidal Poincaré series.

1. Introduction

The existence of cusp forms is a fundamental problem in the modern theory of automorphic forms (\cite{1, 15, 14, 6, 7, 17, 15}). We study this problem in the following setup. Let $G$ be a semisimple algebraic group defined over a number field $k$. We write $\mathbb{A}$ for the ring of adèles of $k$. In \cite[Theorem 1-1]{11} we studied the existence of irreducible automorphic cuspidal representations having their ramified local components belonging to prescribed Bernstein classes (see below). In the present paper we study unramified local components of representations constructed in \cite[Theorem 1-1]{11}. To explain our results, we introduce some notation.

We write $V_f$ (resp., $V_\infty$) for the set of finite (resp., Archimedean) places. For $v \in V_\infty \cup V_f$, we write $k_v$ for the completion of $k$ at $v$; if $v \in V_f$, then we let $\mathcal{O}_v$ be the ring of integers of $k_v$. Let $G_\infty = \prod_{v \in V_\infty} G(k_v)$. This is a semisimple Lie group with finite center. It follows from \cite[3.9.1]{16} that a group $G$ is unramified over $k_v$ for almost all $v \in V_f$. In this case $G$ is defined over $\mathcal{O}_v$ and $K_v \overset{\text{def}}{=} G(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup of $G(k_v)$.

If $T$ is a finite set of places in $V_f$ such that $G$ is unramified over $k_v$, then we write $\mathcal{H}_T$ for the algebra under convolution consisting of all $\prod_{v \in T} K_v$–bi–invariant smooth compactly supported complex functions on $\prod_{v \in T} G(k_v)$. It is well–known that the Satake isomorphism can be used to identify this algebra with the algebra of regular functions on a certain complex affine algebraic variety (see \cite{4}; see also \cite{13}, Section 1). We denote this variety by $\text{Spec}_{\text{max}} \mathcal{H}_T$.

Let $v \in V_f$. Then there exists a well–known notion of Bernstein classes for $G(k_v)$ (see for example \cite{11}, Section 5). A Bernstein class $\mathfrak{M}_v$ is determined by the pair $(M_v, \rho_v)$, where $M_v$ is a Levi subgroup of $G(k_v)$ and $\rho_v$ is an (irreducible) supercuspidal representation of $M_v$. We say that $\pi_v \in \text{Irr} G(k_v)$ belongs to $\mathfrak{M}_v$ if $\pi_v$...
is a subquotient of $\text{Ind}_{P_v}^{G_v}(\chi_v P_v)$, for some unramified character $\chi_v$ of $M_v$. Here $P_v$ is an arbitrary parabolic subgroup of $G_v$ containing $M_v$ as a Levi subgroup.

The main result of this paper is the following theorem:

**Theorem 1-1.** Let $S$ be a finite set of places containing $V_\infty$ such that $G$ is unramified over $k_v$ for all $v \in V_f - S$. For every $v \in S - V_\infty$, let $M_v$ be a Bernstein class of $G(k_v)$. Let $\nu_0 \in V_f - S$ be a place. We write $\mathcal{E}_{\nu_0, S}$ for the set of equivalence classes of irreducible automorphic cuspidal representations $\pi = \pi_\infty \otimes \bigotimes_{v \in V_f} \pi_v$ such that $\pi_v$ belongs to the class $\mathcal{M}_v$ for $v \in S - V_\infty$, and $\pi_v$ is unramified for $V_f - S - \{\nu_0\}$. Then, for a non-empty finite subset $T$ of $V_f - S - \{\nu_0\}$, the set consisting of all Satake parameters of unramified representations $\bigotimes_{v \in T} \pi_v$, when $\pi$ ranges over $\mathcal{E}_{\nu_0, S}$, is Zariski dense in $\text{Spec}_{\max} \mathcal{H}_T$. Moreover, we can choose infinitely many $\pi \in \mathcal{E}_{\nu_0, S}$ such that the representations $\bigotimes_{v \in T} \pi_v$ are mutually non-equivalent and every $\bigotimes_{v \in T} \pi_v$ is isomorphic to the corresponding full unramified principal series (determined by its Satake parameter $A$).

The proof of Theorem 1-1 is given in Section 3. It is based on a spectral decomposition of Poincaré series for compactly supported cuspidal functions in $L^2(G(k) \setminus G(\mathbb{A}))$, a method that was already successfully applied to study cusp forms in $[11]$ and $[12]$, combined with a method of reducing the problem to $S$-arithmetic cuspidal forms $[13]$, Theorem 0-3 and Theorem 0-4. The idea for the studying of $\mathcal{H}_T$-modules of Poincaré series to obtain results such as Theorem 1-1 is taken from $[13]$, but the main problem here is to find the correct form of a cuspidal compactly supported Poincaré series in order to prove Theorem 1-1 (see Lemma 3-3). Lemma 3-3 is proved in Section 4.

2. Preliminary results

In this section we fix the notation used in this paper. We let $G$ be a semisimple algebraic group defined over a number field $k$. We write $V_f$ (resp., $V_\infty$) for the set of finite (resp., Archimedean) places. For $v \in V \overset{\text{def}}{=} V_\infty \cup V_f$, we write $k_v$ for the completion of $k$ at $v$. If $v \in V_f$, then we let $\mathcal{O}_v$ denote the ring of integers of $k_v$. Let $\mathbb{A}$ be the ring of adèles of $k$. For almost all places of $k$, $G$ is defined over $\mathcal{O}_v$ ($[16]$, 3.9.1). We write $K_v \overset{\text{def}}{=} G(\mathcal{O}_v)$. The group of adèle points $G(\mathbb{A}) = \prod_v G(k_v)$ is a restricted product over all places of $k$ of the groups $G(k_v)$: $g = (g_v)_{v \in V} \in G(\mathbb{A})$ if and only if $g_v \in K_v$ for almost all $v$. The group $G(\mathbb{A})$ is a locally compact group and $G(k)$ is embedded diagonally as a discrete subgroup of $G(\mathbb{A})$.

For a finite subset $S \subset V$, we let

$$G_S = \prod_{v \in S} G(k_v).$$

In addition, if $S$ contains all Archimedean places $V_\infty$, then we let $G^S = \prod_{v \notin S} G(k_v)$. Then

$$G(\mathbb{A}) = G_S \times G^S. \tag{2-1}$$

We let $G_\infty = GV_\infty$ and $G(\mathbb{A}_f) = GV_\infty$.

The group $G_\infty$ is a semisimple Lie group. It might not be connected, but it has a finite center. The group $G(\mathbb{A}_f)$ is a totally disconnected group. Let $K_\infty \subset G_\infty$ be a maximal compact subgroup. Let $g_\infty = \text{Lie}(G_\infty)$ be the (real) Lie algebra of
Let $\mathcal{U}(g_{\infty})$ be the universal enveloping algebra of the complexified Lie algebra $g_{\infty, \mathbb{C}} = g_{\infty} \otimes_{\mathbb{R}} \mathbb{C}$. Let $Z(g_{\infty})$ be the center of $\mathcal{U}(g_{\infty})$.

We say that a continuous function $f : G(\mathbb{A}) \to \mathbb{C}$ is smooth if $f(g, g_f) \in C^\infty(G_{\infty})$ for all $g_f \in G(\mathbb{A}_f)$ and if there exists an open compact subgroup $L \subset G(\mathbb{A}_f)$ such that $f(g_{\infty}, g_f \cdot l) = f(g_{\infty}, g_f)$ for all $(g_{\infty}, g_f) \in G_{\infty} \times G(\mathbb{A}_f)$ and $l \in L$. Here we consider $f$ as a function of two variables, $f(g) = f(g_{\infty}, g_f)$, where $g = (g_{\infty}, g_f)$.

We write $C^\infty(G(\mathbb{A}))$ for the vector space of all smooth functions on $G(\mathbb{A})$. It is easy to show that $C^\infty_c(G(\mathbb{A}))$ is a span of the functions $f_\infty \otimes \bigotimes_{v \in V_f} f_v$, where $f_\infty \in C^\infty_c(G_{\infty})$, $f_v \in C^\infty_c(G_{k_v})$ ($v \in V_f$), and $f_v = char_{K_v}$ for almost all $v$.

By definition, we let $C^\infty(G(k) \setminus G(\mathbb{A})) \subset C^\infty_c(G(\mathbb{A}))$ be the subspace consisting of all functions $f \in C^\infty(G(\mathbb{A}))$ such that $f(\gamma \cdot g) = f(g)$ for all $\gamma \in G(k)$ and $g \in G(\mathbb{A})$. The space $C^\infty_c(G(k) \setminus G(\mathbb{A}))$ is a subset of $C^\infty(G(k) \setminus G(\mathbb{A}))$ consisting of functions compactly supported modulo $G(k)$.

An automorphic form is a function in $C^\infty_c(G(k) \setminus G(\mathbb{A}))$ which is $K_{\infty}$-finite on the right, $Z(g_{\infty})$–finite and which satisfies a certain growth condition (see [3], 4.2). The space of all automorphic forms we denote by $\mathcal{A}(G(k) \setminus G(\mathbb{A}))$. The subspace of cuspidal automorphic forms we denote by $\mathcal{A}_{\text{cusp}}(G(k) \setminus G(\mathbb{A}))$. By definition, $f \in \mathcal{A}(G(k) \setminus G(\mathbb{A}))$ is a cuspidal automorphic form if

\[(2-2) \quad \int_{U_P(k) \setminus U_P(\mathbb{A})} f(ng)dn = 0 \text{ (for all } g \in G(\mathbb{A})),\]

for all proper $k$–parabolic subgroups $P$ of $G$. In this paper we write $U_P$ for the unipotent radical of a $k$–parabolic subgroup $P$ of $G$. In general, we say that a locally integrable function $f : G(k) \setminus G(\mathbb{A}) \to \mathbb{C}$ is a cuspidal function if it satisfies \[(2-2)\] for almost all $g \in G(\mathbb{A})$. The space of cuspidal automorphic forms is denoted by $\mathcal{A}_{\text{cusp}}(G(k) \setminus G(\mathbb{A}))$.

The topological space $G(k) \setminus G(\mathbb{A})$ has a finite volume $G(\mathbb{A})$–invariant measure:

\[(2-3) \quad \int_{G(k) \setminus G(\mathbb{A})} P(f)(g)dg = \int_{G(\mathbb{A})} f(g)dg \quad (f \in C^\infty_c(G(\mathbb{A}))),\]

where the adèlic compactly supported Poincaré series $P(f)$ is defined as follows:

\[(2-4) \quad P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma \cdot g) \in C^\infty_c(G(k) \setminus G(\mathbb{A})).\]

The measure introduced in \[(2-3)\] enables us to introduce the Hilbert space $L^2(G(k) \setminus G(\mathbb{A}))$ and its closed subspaces $L^2_{\text{cusp}}(G(k) \setminus G(\mathbb{A}))$ consisting of all cuspidal functions in $L^2(G(k) \setminus G(\mathbb{A}))$. Both of them are unitary representations of $G(\mathbb{A})$. Moreover, the space $L^2_{\text{cusp}}(G(k) \setminus G(\mathbb{A}))$ can be decomposed into a direct sum of irreducible unitary representations of $G(\mathbb{A})$, each occurring with a finite multiplicity (see [3]).

Let $S$ be a finite set of places of $V$ containing $V_{\infty}$. Then we write $G(\mathbb{A})$ using the decomposition \[(2-1)\]. Let $L \subset G^S$ be an open compact subgroup. Let

\[(2-5) \quad \Gamma = \Gamma_L = G(k) \cap L \subset G^S,\]

where we identify $G(k)$ with its image under the diagonal embedding into $G^S$. Identifying $\Gamma$ with its image under another diagonal embedding $G(k) \to G_S$, it can be considered as a discrete subgroup of $G_S$. It is called an $S$–arithmetic subgroup.
of $G(k)$. It is well known that we can fix a finite volume $G_S$–invariant measure on $\Gamma \setminus G_S$ by

(2-6) \[ \int_{\Gamma \setminus G_S} P_L(f_S)(g_S)dg_S = \int_{G_S} f_S(g_S)dg_S \]

for $f_S \in C_c^\infty(G_S)$, where the compactly supported Poincaré series (for $\Gamma$) is defined as follows:

(2-7) \[ P_L(f_S)(g_S) \overset{\text{def}}{=} \sum_{\gamma \in \Gamma} f_S(\gamma \cdot g_S). \]

The function $P_L(f_S)$ belongs to the space $C_c^\infty(\Gamma \setminus G_S)$.

Similarly as before, we define the notion of $\Gamma$–cupidality by letting $U_{P,S} = \prod_{v \in S} U_P(k_v)$, and integrating over $\Gamma \cap U_{P,S} \setminus U_{P,S}$, for any proper $k$–parabolic subgroup $P$ of $G$. We let $C_{c,\text{cusp}}(\Gamma \setminus G_S)$ be the subspace of $\Gamma$–cupidal functions in $C_c^\infty(\Gamma \setminus G_S)$. In this paper we use the following analogue of ([11], Lemma 3-3):

**Lemma 2-8.** Let $L \subset G^S$ be an open compact subgroup and let $\psi : U_P(k) \setminus U_P(A) \to \mathbb{C}$ be a continuous function which is right–invariant under the open compact subgroup $L \cap U_P^S$. Then, we have the following formula:

\[ \int_{U_P(k) \setminus U_P(A)} \psi(u)du = \text{vol}_{U_P^S}(L \cap U_P^S) \cdot \int_{\Gamma_L \cap U_{P,S} \setminus U_{P,S}} \psi(u_S)du_S. \]

The next lemma will be used several times in this paper.

**Lemma 2-9.** Let $L \subset G^S$ be an open compact subgroup and let $\varphi_S \in C_c^\infty(G_S)$. Then, $P(\varphi_S \otimes \text{char}_L)$ is cuspidal if and only if $P_{\Gamma_L}(\varphi_S)$ is $\Gamma_L$–cuspidal.

**Proof.** We can write $G^S = G(k) \cdot C \cdot L$, for some finite set $C$ of minimal cardinality ([2]). (Here we identify $G(k)$ with its image in $G^S$ under the diagonal embedding.) We may assume that $1 \in C$.

Let $P$ be a proper $k$–parabolic subgroup of $G$. Let $g^S = \gamma cl \in G^S = G(k) \cdot C \cdot L$. Then, we have the following:

\[
\begin{align*}
&\int_{U_P(k) \setminus U_P(A)} P(\varphi_S \otimes \text{char}_L)(u(g_S, g^S))du \\
&= \int_{U_P(k) \setminus U_P(A)} P(\varphi_S \otimes \text{char}_L)(u(g_S, \gamma cl))du \\
&= \int_{U_P(k) \setminus U_P(A)} P(\varphi_S \otimes \text{char}_L)(u(g_S, \gamma c))du \\
&= \int_{\gamma^{-1}U_P(k) \gamma \setminus \gamma^{-1}U_P(A) \gamma} P(\varphi_S \otimes \text{char}_L)(\gamma u \gamma^{-1}(g_S, \gamma c))du \\
&= \int_{\gamma^{-1}U_P(k) \gamma \setminus \gamma^{-1}U_P(A) \gamma} P(\varphi_S \otimes \text{char}_L)(u(\gamma^{-1}g_S, c))du.
\end{align*}
\]

Thus, $P(\varphi_S \otimes \text{char}_L)$ is cuspidal on $G(A)$ if and only if

(2-10) \[ \int_{U_P(k) \setminus U_P(A)} P(\varphi_S \otimes \text{char}_L)(u(g_S, c))du = 0 \quad (g_S \in G_S, \ c \in C), \]

for every proper $k$–parabolic subgroup $P$ of $G$. Since the function $u \in U_P(A) \mapsto P(\varphi_S \otimes \text{char}_L)(u(g_S, c))$ is right–invariant under $U_P^S \cap cL c^{-1}$, Lemma 2-8 shows
that the integral on the left–hand side of (2-10) can be computed as follows:
\[
\begin{align*}
&\int_{U_p(k)\setminus U_p(A)} P(\varphi_S \otimes \chi)(u(g_S, e)) du \\
&= \int_{\Gamma_{L} \cap U_{P,S} \setminus U_{P,S}} P(\varphi_S \otimes \chi)(u g_S, e) du \\
&= \int_{\Gamma_{L} \cap U_{P,S} \setminus U_{P,S}} \sum_{\gamma \in G(k)} \varphi_S(\gamma u g_S) \chi(\gamma c) du.
\end{align*}
\]

In order to evaluate the last integral in (2-11), we note that the inner sum is non–zero only if \( c \in G(k) \cdot L \) and \( 1 \in C \), we obtain \( c = 1 \). Thus, if \( c \neq 1 \), then the integral in (2-11) is zero, while for \( c = 1 \) it becomes
\[
\int_{\Gamma_{L} \cap U_{P,S} \setminus U_{P,S}} \sum_{\gamma \in G(k)} \varphi_S(\gamma u g_S) \chi(\gamma c) du = \int_{\Gamma_{L} \cap U_{P,S} \setminus U_{P,S}} P_{\chi}(\varphi_S)(u g_S) du.
\]

This proves the lemma. \( \square \)

3. THE PROOF OF THEOREM [13]

Using the notation introduced in the statement of Theorem [11], we put \( S' = S \cup \{v_0\} \). Let \( K'^S \subset G'^S \) be the open compact subgroup defined by
\[
K'^S = \prod_{v \in S'} K_v.
\]

We also put \( G_T = \prod_{v \in T} G(k_v) \) and \( K_T = \prod_{v \in T} K_v \). We consider the commutative algebra \( \mathcal{H}_T \) described in the introduction.

It is well–known (see [4]) that the set of equivalence classes of irreducible unramified representations (i.e., having a non–zero \( K_T \)–fixed vector) and \( \text{Spec}_{\max} \mathcal{H}_T \) are in a one–to–one correspondence defined in the following way: for an unramified irreducible representation \( \pi_T \) of \( G_T \), the algebra \( \mathcal{H}_T \) acts on the one–dimensional space \( \pi_T^{K_T} \) by a character \( \omega_{\pi_T} \in \text{Spec}_{\max} \mathcal{H}_T \).

Let \( p_{v_0} \) be the maximal ideal in the ring \( \mathcal{O}_{v_0} \), and let \( U_{v_0} \) be the kernel of the reduction homomorphism \( G(\mathcal{O}_{v_0}) \to G(\mathcal{O}_{v_0}/p_{v_0}) \). Let \( \mathcal{F} \) be the family of open compact subgroups \( L^S \) of \( G^S \) of the following form:
\[
L^S = K'^S \times L_{v_0}, \quad L_{v_0} \subset U_{v_0} \text{ is a variable group.}
\]

If we let \( \Gamma_{L^S} = G(k) \cap L \), where \( G(k) \) is identified with a subgroup of \( G^S \), then the image of \( \Gamma_{L^S} \) under the diagonal embedding of \( G(k) \) into \( G^S \) is a discrete subgroup. We continue to call this discrete subgroup \( \Gamma_{L^S} \). The following lemma is the key fact for the proof of Theorem [13]. It is proved in Section [13] and the proof uses many fine parts of [13].

**Lemma 3-3.** Assume that \( \varphi_v \in C_c^\infty(G(k_v)) \) (\( v \in S \setminus V_\infty \)) are non–zero functions. Then, there exists \( \varphi_\infty \in C_c^\infty(G_\infty) \) such that for every open compact subgroup \( L^S \in \mathcal{F} \) the Poincaré series \( P(\varphi_X \otimes \chi_{L^S}) \) is a cuspidal function on \( G(k) \setminus G(\mathbb{A}) \) and non–zero when restricted to \( G_\infty \).
Clearly, cuspidal functions $P(\varphi_S \otimes \text{char}_{L^S})$ introduced in Lemma 3.3 are all $K_T$–invariant. So, we may study the structure of the $H_T$–module generated by $P(\varphi_S \otimes \text{char}_{L^S})$:

\[(3-4) \quad J(\varphi_S, L^S)^T \overset{\text{def}}{=} H_T.P(\varphi_S \otimes \text{char}_{L^S}) \subset C_{c,\text{cusp}}(G(k) \setminus G(\mathbb{A}))^{K_T}, \quad L^S \in \mathcal{L}.\]

We denote its annihilator as follows:

\[(3-5) \quad I(\varphi_S, L^S)^T \overset{\text{def}}{=} \text{Ann} J(\varphi_S, L^S)^T.\]

Since the module $J(\varphi_S, L^S)^T$ is generated by the image of the identity and the algebra $H_T$ is commutative, we have the following isomorphism of $H_T$–modules:

\[
J(\varphi_S, L^S)^T \cong H_T/I(\varphi_S, L^S)^T.
\]

We let $U_{L^S}$ denote the closed $G(\mathbb{A})$–invariant subspace generated by $P(\varphi_S \otimes \text{char}_{L^S})$ in $L^2_{\text{cusp}}(G(k) \setminus G(\mathbb{A}))$. We can decompose $U_{L^S}$ into the Hilbert direct sum of closed irreducible $G(\mathbb{A})$–invariant subspaces $\mathcal{H}$.

\[(3-6) \quad U_{L^S} = \bigoplus_j \mathcal{H}_{L^S}^j.\]

We write

\[(3-7) \quad \pi_{L^S}^{(j)} = \pi_{\infty, L^S}^{(j)} \otimes \bigotimes_{v \in V_f} \pi_{v, L^S}^{(j)}\]

for the $(\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f)$–module of $\mathcal{H}_{L^S}^j$. Now, we are ready to prove the following lemma:

**Lemma 3.8.** Under the above assumptions, we have the following:

(i) $\bigotimes_{v \in T} \pi_{v, L^S}^{(j)}$ contains a non-zero $K_T$–invariant vector; i.e., $\bigotimes_{v \in T} \pi_{v, L^S}^{(j)}$ is unramified.

(ii) The Zariski closure in $\text{Spec}_{\text{max}} H_T$ of the set of all

\[
\omega_{T,L^S,j} \overset{\text{def}}{=} \text{the character of } H_T \text{ associated to } \bigotimes_{v \in T} \pi_{v, L^S}^{(j)}
\]

is exactly the algebraic set $\mathcal{V}(I(\varphi_S, L^S)^T)$ associated to the ideal $I(\varphi_S, L^S)^T$.

In fact, $I(\varphi_S, L^S)^T$ is the intersection of the kernels of all characters $\omega_{T,L^S,j}$. In particular, $I(\varphi_S, L^S)^T$ is a radical ideal. (This notion is recalled in the course of the proof.)

(iii) Let us select a decreasing sequence of open compact subgroups of $G(k_{v_0})$: $U_{v_0} \supset L_{v_0,1} \supset L_{v_0,2} \supset \cdots$ such that $\bigcap_{n \geq 1} L_{v_0,n} = \{1\}$. Let $L_n^S = K^S \times L_{v_0,n}$. Then the union $\bigcup_{n \geq 1} \mathcal{V}(I(\varphi_S, L_n^S)^T)$ is Zariski dense in $\text{Spec}_{\text{max}} H_T$.

**Proof.** We prove (i). First, we can decompose according to (3-6):

\[(3-9) \quad P(\varphi_S \otimes \text{char}_{L^S}) = \sum_j \psi_j, \quad \psi_j \in \mathcal{H}_{L^S}^j.
\]

Since $U_{L^S}$ is generated by $P(\varphi_S \otimes \text{char}_{L^S})$, $\psi_j \neq 0$ and $\psi_j$ is $K^T$–invariant for all $j$. This implies (i). Now, we prove (ii). Let $f \in H_T$. Then we have the following:

\[(3-10) \quad f.P(\varphi_S \otimes \text{char}_{L^S}) = \sum_j f.\psi_j = \sum_j \omega_{T,L^S,j}(f)\psi_j,
\]
since $\psi_j$ is $K^T$-invariant in $\mathfrak{H}^1_{L, S}$ for all $j$. This implies that $f \in I(\varphi_S, L^S)^T$ if and only if $\omega_{T, L^S}(f) = 0$ for all $j$. Hence, we see that the ideal $I(\varphi_S, L^S)^T$ is a radical ideal (i.e., $f^m \in I(\varphi_S, L^S)^T$, for $f \in I_T$ and $m > 0$, implies that $f \in I(\varphi_S, L^S)^T$). The remainder of (i) follows from the usual “radical ideal–algebraic set” correspondence in algebraic geometry. This proves (ii). Finally, we prove (iii). The proof of (iii) can be done on the (simplified) lines of the proof of ([13], Theorem 2-8 (ii)). We give some hints and leave the details to the reader.

We write every group $G$ in the form $G = G^{\operatorname{max}} \times T$, where $G^{\operatorname{max}}$ is an open compact subgroup of $G_{S, UT}$ (see (2-2)). We define a discrete subgroup $\Gamma_{L^S}$ of $G_{S, UT}$ in the standard way (see (2-20)). The function $P_{G_{\mathfrak{L}, S}}(\varphi_S \otimes \operatorname{char}_{K_T})$ is $\Gamma_{L^S}$-cuspidal applying Lemmas 2-9 and 3-3. Direct computation shows that for $g_\infty \in G_\infty \subset G_{S, UT}$, we have the following:

$$P_{G_{\mathfrak{L}, S}}(\varphi_S \otimes \operatorname{char}_{K_T})(g_\infty) = P(\varphi_S \otimes \operatorname{char}_{L^S})(g_\infty, 1).$$

Thus, by Lemma 2-8 (3), $P_{G_{\mathfrak{L}, S}}(\varphi_S \otimes \operatorname{char}_{K_T})$ is non–trivial. Then, the $\mathcal{H}_T$-submodule of $C_{c, \operatorname{cusp}}(\Gamma_{L^S} \setminus G_{S, UT})^{\mathbf{K}_T}$ generated by $P_{G_{\mathfrak{L}, S}}(\varphi_S \otimes \operatorname{char}_{K_T})$ is non–trivial. We denote this module by $J(\varphi_S, L^S)^T$. Let $I(\varphi_S, L^S)^T$ be its annihilator. Then, as in ([13], Lemma 2-20), we have the epimorphism of $\mathcal{H}_T$-modules,

$$J(\varphi_S, L^S)^T \rightarrow J(\varphi_S, L^S_1)^T \rightarrow 0,$$

and, consequently, we have

$$I(\varphi_S, L^S)^T \subset I(\varphi_S, L^S_1)^T. \tag{3-11}$$

We write $L^S_n = K^T \times (L_n)^S$ for $n \geq 1$. Then $(L_1)^S \supset (L_2)^S \supset \cdots$, and we have an epimorphic $\mathcal{H}_T$-module map,

$$J(\varphi_S, (L_{n+1})^S_1)^T \rightarrow J(\varphi_S, (L_n)^S_1)^T \rightarrow 0 \text{ for all } n \geq 1,$$

defined by

$$P_{(L_n)^S_1}(\varphi_S, (L_{n+1})^S_1)(f)(g_{S, UT}) = \sum_{\gamma \in \Gamma_{(L_{n+1})^S_1 \setminus \Gamma_{(L_n)^S_1}}} f(\gamma \cdot g_{S, UT}).$$

This implies that $I(\varphi_S, (L_{n+1})^S_1)^T \subset I(\varphi_S, (L_n)^S_1)^T$ for all $n \geq 1$ ([13], Lemma 2-24). We show the following:

$$\bigcap_{n \geq 1} I(\varphi_S, (L_n)^S_1)^T = \{0\}. \tag{3-12}$$

First, identifying $G(k)$ with its image under the diagonal embedding in $G_{S, UT}$, we have $\Gamma_{(L_n)^S_1} = G(k) \cap (L_n)^S_1$ (see (3-2)). Furthermore, this observation and the assumption $\bigcap_{n \geq 1} I(\varphi_S, (L_n)^S_1)^T$ show us that $\bigcap_{n \geq 1} I(\varphi_S, (L_n)^S_1)^T = \{1\}$ by considering the $v_0$-th component. Now, as in the proof ([13], Lemma 2-27), we show the claim in (3-12).

Now, (3-11) and (3-12) show that $\bigcap_{n \geq 1} I(\varphi_S, L_n^S)^T = \{0\}$. (See [13], Lemma 2-28). Since, by (ii), the ideals $I(\varphi_S, L_n^S)^T$ are radical, the following simple argument shows that the union $\bigcup_{n \geq 1} V(I(\varphi_S, L_n^S)^T)$ is Zariski dense in $\text{Spec}_{\mathbf{K}_T} \mathcal{H}_T$. If $f \in \mathcal{H}_T$ vanishes on that union, then by Nullstellensatz for each $n \geq 1$ there exists $m_n \in \mathbb{Z}_{>0}$ such that $f^{m_n} \in I(\varphi_S, L_n^S)^T$. Since $I(\varphi_S, L_n^S)^T$ is radical, we obtain $f \in I(\varphi_S, L_n^S)^T$ for all $n \geq 1$. Hence, $f \in \bigcap_{n \geq 1} I(\varphi_S, L_n^S)^T = \{0\}$. Thus, $f = 0$. This proves the claim. The proof of (iii) is complete. \qed
Corollary 3-13. Under the assumptions of Lemma 3-8 (iii), the union
\[ \bigcup_{n \geq 1} \{ \omega_{T,L_n} \}; \] all \( j \) appearing in (3-9) for \( L^S = L_n^S \)
is Zariski dense in \( \text{Spec}_{\text{max}} \mathcal{H}_T \). Moreover, it contains a subset which is also Zariski
dense in \( \text{Spec}_{\text{max}} \mathcal{H}_T \) such that every unramified representation \( \bigotimes_{v \in T} \pi_{v,L}^{(j)} \) in the
corresponding set of unramified representations is isomorphic to the corresponding
full unramified principal series of \( G_T \).

Proof. The first claim follows directly from Lemma 3-8 (iii). The other claim has
the same proof as (13, Theorem 3-1). \( \square \)

Now, we begin the proof of Theorem 1-1. Let \( v \in S - V_\infty \). Then \( G_{v}(k_v) \) acts
by right translations on \( C^c_v(G(k_v)) \). We showed in (11, Lemma 5-2 (i)) that
\( C^c_\infty(G(k_v))(2 \mathcal{R}_v) \neq 0 \). We select \( \varphi_v \in C^c_\infty(G(k_v))(2 \mathcal{R}_v) \), \( \varphi_v(1) \neq 0 \). Next, for a
\( K_\infty \)-type \( \delta \), we write \( d(\delta) \) and \( \xi_\delta \) for the degree and character of \( \delta \), respectively.
Since \( P(\varphi \otimes \text{char}_{L^S}) \) is non–zero, there exists \( \delta \) such that its \( \delta \)-isotypic component

\[ P(\varphi_\infty) \otimes \bigotimes_{v \in S - V_\infty} \varphi_v \otimes \text{char}_{L^S}(g) \]

is a non–zero function, where \( E_\delta(\varphi_\infty) = \int_{K_\infty} d(\delta) \xi_\delta(k) \varphi_\infty(g_\infty k) \) transforms as \( \delta \) on the right. Now, in the settings of the paragraph before Lemma 3-8 and using (3-9), the spectral decomposition of \( P(E_\delta(\varphi_\infty) \otimes \bigotimes_{v \in S - V_\infty} \varphi_v \otimes \text{char}_{L^S}) \) is given by

\[ P(E_\delta(\varphi_\infty) \otimes \bigotimes_{v \in S - V_\infty} \varphi_v \otimes \text{char}_{L^S}) = \sum_j E_\delta(\psi_j). \]

Using this, we see that \( \pi_{v,L^S}^{(j)} \) belongs to the class of \( \mathcal{M}_v \) for all \( v \in S - V_\infty \),
applying the argument of the proof of (11, Theorem 7-2 (ii)). It is clear that \( \pi_{v,L^S}^{(j)} \)
is unramified for \( v \in V_f - S - \{v_0\} \). Now, Corollary 3-13 completes the proof of
Theorem 1-1.

4. The Proof of Lemma 3-3

The proof of Lemma 3-3 reveals a new way of constructing cuspidal compactly supported
Poincaré series. We continue with the notation introduced in Section 3
before the statement of Lemma 3-3.

First, using a convenient right translation, we may assume that \( \varphi_v(1) \neq 0 \) for all \( v \in S - V_\infty \). We let \( \varphi_{v_0} = \text{char}_{U_{v_0}}. \)

Put \( \Gamma_{S'} = G(k) \cap K_{S'} \) (see (3-1)), where we identify \( G(k) \) with its image in \( G_{S'} \)
under the diagonal embedding. We can consider \( \Gamma_{S'} \) as a discrete subgroup of \( G_{S'} \),
identifying it with the image under the diagonal embedding of \( G(k) \) in \( G_{S'} \).

We select open compact subgroups \( L_v \subset G(k_v) \) for \( v \in S - V_\infty \) such that \( \varphi_v \) is
bi–invariant under \( L_v. \) We let

\[ U = K_{S'} \times \prod_{v \in S - V_\infty} L_v \times U_{v_0}. \]

Then \( \Gamma_U = G(k) \cap U \) can be considered as a discrete subgroup of \( G_{\infty}. \)
Now, we construct a very special \( f = f_\infty \otimes \bigotimes_{v \in V_f} f_v \in C_c^\infty(G(\mathbb{A})) \) such that \( P(f) \) is a cuspidal compactly supported Poincaré series. We let \( f_v = \text{char}_K_v \) for \( v \notin S \), and let \( f_v = \text{char}_{L_v} \) for \( v \in S - V_\infty \). We let \( f_{v_0}, f_{v_0}(1) \neq 0 \) be the matrix coefficient of a supercuspidal representation constructed in the following way. We select some supercuspidal irreducible representation \( \rho \) of a finite group \( G(O_{v_0}/p_{v_0}) \). Then the reduction homomorphism \( G(O_{v_0}) \to G(O_{v_0}/p_{v_0}) \) enables us to consider \( \rho \) as an irreducible representation of \( G(O_{v_0}) \). Now, (9, Proposition on page 20; see also (8), Proposition 6.6), implies that the compactly induced representation \( c - \text{Ind}_{G(O_{v_0})}^{G(O_{v_0})}(\rho) \) breaks into finitely many irreducible supercuspidal representations. Frobenius reciprocity implies that all of them contain \( \rho \) upon the restriction to \( G(O_{v_0}) \). Hence they have a vector invariant under \( U_{v_0} \). We select a matrix coefficient, \( f_{v_0}, f_{v_0}(1) \neq 0 \), of one of the irreducible subrepresentations of \( c - \text{Ind}_{G(O_{v_0})}^{G(O_{v_0})}(\rho) \) which is bi–invariant under \( U_{v_0} \).

Now, we apply (11, Theorem 4-2) to construct a convenient Archimedean component \( f_\infty \). To explain this, we review the Iwasawa decomposition for \( G_\infty \). Let \( P_\infty \) be a minimal parabolic subgroup of \( G_\infty \). Let \( N_\infty \) be its unipotent radical, let \( A_\infty \) be the maximal split torus, and \( M_\infty = Z_{K_\infty}(A_\infty) \). Then, we have the following diffeomorphism (the Iwasawa decomposition):

\[
N_\infty \times A_\infty \times K_\infty \overset{(n,a,k)\mapsto n.a.k}{\rightarrow} G_\infty = N_\infty A_\infty K_\infty.
\]

The Iwasawa decomposition implies that there exist unique \( C^\infty \)-functions \( a : G_\infty \to A_\infty \), \( n : G_\infty \to N_\infty \), and \( k : G_\infty \to K_\infty \) such that

\[
g_\infty = n(g_\infty) \cdot a(g_\infty) \cdot k(g_\infty), \quad g_\infty \in G_\infty.
\]

As in the proof of (11, Theorem 4-2), we can find neighborhoods of identities \( U \subset N_\infty \) and \( V \subset A_\infty \) such that

\[
(4-2) \quad \Gamma_{S'} \cap \left[ (UVK_\infty) \times \prod_{v \in S' - V_\infty} \text{supp} \left( f_v \right) \right] = \Gamma_{S'} \cap \left[ K_\infty \times \prod_{v \in S' - V_\infty} \text{supp} \left( f_v \right) \right].
\]

We pick "a variable small" neighborhood \( W \) of the identity in \( K_\infty \) and assume that

\[
(4-3) \quad \Gamma_{S'} \cap \left[ (WW^{-1}) \times \prod_{v \in S' - V_\infty} \text{supp} \left( f_v \right) \right] = \{1\}.
\]

The function \( f_\infty \) is selected in the following way:

\[
f_\infty(g_\infty) = \zeta(n(g_\infty))\eta(a(g_\infty))\xi(k(g_\infty)),
\]

where \( \zeta \in C_c^\infty(U) \), \( \eta \in C_c^\infty(V) \), and \( \xi \in C_c^\infty(W) \), \( \zeta(1), \eta(1), \xi(1) \neq 0 \). We have (see the proof of Theorem 4-2 in (11)) that

\[
(4-4) \quad P(f)(g_\infty, 1) = \sum_{\gamma \in \Gamma_{S'}} \left( \prod_{v \in S' - V_\infty} f_v(\gamma) \right) f_\infty(\gamma \cdot g_\infty).
\]

We require that

\[
\sum_{\gamma \in \Gamma_{S'}} \left( \prod_{v \in S' - V_\infty} f_v(\gamma) \right) \cdot f_\infty(\gamma \cdot k_\infty) \neq 0
\]
for some \( k_\infty \in K_\infty \) to insure that \( P(f)|_{G_\infty} \neq 0 \). By \([1-2]\), this is equivalent to (4-5)
\[
P(f)|_{G_\infty}(k_\infty) = \sum_{\gamma \in \Gamma_S' \cap K_\infty} \left( \prod_{v \in S' - V_\infty} f_v(\gamma) \right) \cdot \xi(\gamma \cdot k_\infty) \neq 0, \quad \text{for some } k_\infty \in K_\infty,
\]
where in the summation we identify \( \Gamma_S' \subset G(k) \) with a subgroup of \( G_\infty \). Although \( \Gamma_S' \) is not a discrete subgroup of \( G_\infty \), the sum is finite if we “throw out” all \( \gamma \)'s such that \( \prod_{v \in S' - V_\infty} f_v(\gamma) = 0 \). Applying \([4-3], \ (4-5)\) is satisfied for every \( \xi \in C_c^\infty(W) \) such that \( \xi(1) \neq 0 \).

The function \( P(f)|_{G_\infty} \) is a \( \Gamma_U \)-cuspidal compactly supported Poincaré series for \( G_\infty \) (see \([11], \ Proposition \ 3-2\)). Let \( \varphi_\infty \in C_c^\infty(G_\infty) \) be such that
\[
(4-6) \quad P_{\Gamma_U}(\varphi_\infty) = P(f)|_{G_\infty}.
\]
Hence
\[
(4-7) \quad P_{\Gamma_U}(\varphi_\infty) \text{ is } \Gamma_U \text{-cuspidal and non-zero}.
\]
Put \( \varphi_{S'} = \varphi_\infty \otimes \bigotimes_{\xi \in S' - V_\infty} \varphi_v \). Then, if we write \( G(A) = G_{S'} \times G_{S'}^\prime \) (see \([2-1]\)), then we have the following:
\[
P(\varphi_{S'} \otimes \text{char}_{K_{S'}})(g_\infty, 1) = \sum_{\gamma \in \Gamma_{S'}} \varphi_\infty(\gamma g_\infty) \prod_{v \in S' - V_\infty} \varphi_v(\gamma),
\]
for all \( g_\infty \in G_\infty \) considering \( G_\infty \subset G_{S'} \).

For a fixed \( g_\infty \in G_\infty \), this sum is actually finite since \( \Gamma_{S'} \) is a discrete subgroup of \( G_{S'} \). We transform the sum in a more convenient form:
\[
P(\varphi_{S'} \otimes \text{char}_{K_{S'}})(g_\infty, 1) = \sum_{\gamma \in \Gamma_U \setminus \Gamma_{S'}} P_{\Gamma_U}(\varphi_\infty)(\gamma g_\infty) \prod_{v \in S' - V_\infty} \varphi_v(\gamma),
\]
where we used our assumption that \( \varphi_v \) is bi-invariant under \( L_v \) for all \( v \in S - V_\infty \) and \( \varphi_{v_0} \) is bi-invariant under \( U_{v_0} \). We select the representatives in \( \Gamma_{S'} \) for cosets in \( \Gamma_U \setminus \Gamma_{S'} \) and let \( 1 = \gamma_1, \ldots, \gamma_l \in \Gamma_{S'} \) denote all of them satisfying
\[
\prod_{v \in S' - V_\infty} \varphi_v(\gamma_i) \neq 0.
\]
Then
\[
P(\varphi_{S'} \otimes \text{char}_{K_{S'}})(g_\infty, 1) = \sum_{i=1}^l P_{\Gamma_U}(\varphi_\infty)(\gamma_i g_\infty) \prod_{v \in S' - V_\infty} \varphi_v(\gamma_i), \quad g_\infty \in G_\infty.
\]
Thus, \([4-4] \) and \([4-6]\) imply that
\[
P(\varphi_{S'} \otimes \text{char}_{K_{S'}})(k_\infty, 1)
\]
\[
= \sum_{i=1}^l \sum_{\gamma \in \Gamma_{S'}} \left( \prod_{v \in S' - V_\infty} f_v(\gamma) \varphi_v(\gamma_i) \right) f_\infty(\gamma_i k_\infty), \quad k_\infty \in K_\infty.
\]
Using \([4-3]\), we see that \( f_\infty(\gamma_i k_\infty) = 0 \) unless \( \gamma_i \in UVK_\infty \). If this is so, then we have
\[
f_\infty(\gamma_i k_\infty) = \zeta(n(\gamma_i))\eta(a(\gamma_i))\xi(k(\gamma_i)k_\infty).
\]
We remark that \( \gamma \gamma_i \notin K_\infty \) implies that \( n(\gamma \gamma_i) \neq 1 \) or \( a(\gamma \gamma_i) \neq 1 \). Now, shrinking the supports of \( \zeta \) and \( \eta \), we can accomplish that \( \zeta(n(\gamma \gamma_i)) \eta(a(\gamma \gamma_i)) = 0 \) for all those \( \gamma \)'s which are not in \( K_\infty \). Now, the sum \( (4-8) \) is of the form

\[
P(\varphi_{S'} \otimes \text{char}_{K_{S'}})(k_\infty, 1)
\]

(4-9)

\[
= \sum_{\delta \in \Gamma_{S'} \cap K_\infty} \xi(\delta k_\infty) \sum_{i=1}^{l} \left( \prod_{v \in S' - V_\infty} f_v(\delta \gamma_i^{-1}) \varphi_v(\gamma_i) \right), \quad k_\infty \in K_\infty.
\]

The coefficient for the term \( \delta = 1 \) is given by

\[
\sum_{i=1}^{l} \left( \prod_{v \in S' - V_\infty} f_v(\delta \gamma_i^{-1}) \varphi_v(\gamma_i) \right) = \prod_{v \in S' - V_\infty} f_v(1) \varphi_v(1) \neq 0.
\]

(Indeed, since \( f_v(\gamma_i^{-1}) \neq 0 \) implies \( \gamma_i^{-1} \in L_v \), for \( v \neq v_0 \), and \( \varphi_v(\gamma_i) \neq 0 \) implies \( \gamma_i \in U_{v_0} \), we have \( \gamma_i \in \Gamma_U \). Hence \( i = 1 \) by the above selection of the representatives for \( \Gamma_U \setminus \Gamma_{S'} \).

Finally, we note that in \( (4-9) \) the sum runs over finitely many \( \delta \)'s independent of \( k_\infty \in K_\infty \). Thus, if we shrink \( W \) (see \( (3-2) \)), then we can accomplish that \( \delta W \cap W = \emptyset \) for \( \delta \neq 1 \). Now, for \( k_\infty \in W \), \( (4-9) \) implies that

\[
P(\varphi_{S'} \otimes \text{char}_{K_{S'}})(k_\infty, 1) = \left( \prod_{v \in S' - V_\infty} f_v(1) \varphi_v(1) \right) \xi(k_\infty).
\]

This shows that \( P(\varphi_{S'} \otimes \text{char}_{K_{S'}}) \) is non–zero when restricted to \( G_\infty \). Next, we show that it is cuspidal. The global Hecke algebra \( C_c^\infty(G(\mathbb{A})) \) acts by convolution on \( C_c^\infty(G(k) \setminus G(A)) \) and on the subspace of cuspidal functions as follows:

\[
F.\psi(g) = \int_{G(\mathbb{A})} \psi(gh)f(h)dh = \psi^* F'(g), \quad F \in C_c^\infty(G(\mathbb{A})), \psi \in C_c^\infty(G(k) \setminus G(A)),
\]

where \( F'(g) = F(g^{-1}) \). A similar action exists on \( C_c^\infty(G(\mathbb{A})) \), and the map \( \varphi \mapsto P(\varphi) \) is equivariant. Since

\[
\prod_{v \in S' - V_\infty} \text{vol}(L_v) \cdot \text{vol}(U_{v_0}) \cdot \left( \prod_{v \in S' - V_\infty} \varphi_v^\vee \otimes \bigotimes_{v \in S' - V_\infty} \text{char}_{K_{S'}} \right)
\]

maps \( \varphi_\infty \otimes \bigotimes_{v \in S' - V_\infty} \text{char}_{L_v} \otimes \text{char}_{U_{v_0}} \otimes \text{char}_{K_{S'}} \) onto

\[
\varphi_\infty \otimes \bigotimes_{v \in S' - V_\infty} \varphi_v \otimes \text{char}_{K_{S'}}.
\]

It is enough to show that \( P(\varphi_\infty \otimes \bigotimes_{v \in S' - V_\infty} \text{char}_{L_v} \otimes \text{char}_{U_{v_0}} \otimes \text{char}_{K_{S'}}) \) is cuspidal. But this follows from Lemma \( (2-9) \) and \( (4-7) \). This proves the lemma for \( L_{v_0} = U_{v_0} \).

In general, for \( L_{v_0} \subset U_{v_0} \), we define discrete subgroups \( \Gamma_{L_S} \) and \( \Gamma_{U_S} \) of \( G_S \) using open compact subgroups \( L_S = K_{S'} \times L_{v_0} \) and \( U_S = K_{S'} \times U_{v_0} \) of \( G_S \) (see \( (3-2) \)). Then, using the notation from the first part of the proof, we have the following:

\[
P(\varphi_{S'} \otimes \text{char}_{K_{S'}})(g_S, 1) = P_{U_S}(\varphi_{S})(g_S) = \sum_{\gamma \in \Gamma_{L_S} \setminus \Gamma_{U_S}} P_{L_S}(\varphi_S)(\gamma g_S), \quad g_S \in G_S.
\]

Hence \( P_{L_S}(\varphi_S) \) is non–zero when restricted to \( G_\infty \). Since, by a direct computation, we find

\[
P_{L_S}(\varphi_S)(g_S) = P(\varphi_S \otimes \text{char}_{L_S})(g_S, 1),
\]
where we use the decomposition (2-1), we see that $P(\varphi_S \otimes \text{char}_L^S)$ is non-zero when restricted to $G_\infty$. It remains to show the cuspidality of $P(\varphi_S \otimes \text{char}_L^S)$. As above, it is enough to show that $P(\varphi_\infty \otimes \prod_{v \in S - V_\infty} \text{char}_{L_v} \otimes \text{char}_{L_v^0} \otimes \text{char}_{K^S_v})$ is cuspidal. By Lemma (2-9), it is enough to show that $P(\Gamma^L(\varphi_\infty))$ is $\Gamma_L$-cuspidal, where $L = K^S \times \prod_{v \in S - V_\infty} L_v \times L_0$. But $\Gamma_L \subset \Gamma_U$, (4-7) and (12), Lemma 2-9) imply that.

References


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