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A COMPACT EMBEDDING OF A SOBOLEV SPACE IS EQUIVALENT TO AN EMBEDDING INTO A BETTER SPACE

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Abstract. We prove that the compact embedding of the Orlicz-Sobolev space is equivalent to the existence of a bounded embedding into a higher Orlicz space.

1. Introduction

If $\Omega \subset \mathbb{R}^n$ is a bounded domain and the Sobolev space $W^{1,p}(\Omega)$ is embedded into $L^q(\Omega)$, then for any $1 \leq s < q$, the embedding $W^{1,p}(\Omega) \Subset L^s(\Omega)$ is compact; see e.g. [5, Theorem 4]. This result generalizes to the setting of Orlicz-Sobolev spaces. Let A, Φ, Ψ be Young functions. If the embedding $W^{1,A}(\Omega) \subset L^{\Psi}(\Omega)$ is bounded and Ψ increases essentially faster than Φ, $\Psi \succ\succ \Phi$, then the embedding $W^{1,A}(\Omega) \Subset L^{\Phi}(\Omega)$ is compact; see [1]. The last statement contains the previous one since the function t^q grows essentially faster than t^s for $q > s$. The proof given in [5, Theorem 4] is based on the following consequence of the Rellich-Kondrachov theorem: Every bounded sequence in $W^{1,p}(\Omega)$ (or $W^{1,A}(\Omega)$) has a subsequence that is convergent a.e., and thus it is not surprising that the results can be generalized to the setting of abstract normed spaces W of measurable functions with the property that every bounded sequence has a subsequence convergent a.e.; see Theorem 3.4 for a precise statement. This is nothing really new. What is new is that the converse implication is also true: If for a Young function the embedding $W^{1,A}(\Omega) \Subset L^{\Phi}(\Omega)$ is compact, then there is a Young function Ψ that grows essentially faster than Φ such that the embedding $W^{1,A}(\Omega) \subset L^{\Psi}(\Omega)$ is bounded (in a special case a similar fact was observed in [6, Remark 4]). Hence a compact embedding is equivalent with an embedding into a better space; see Theorem 3.1. This result has several natural consequences. In particular it shows that the optimal embedding is never compact; see Corollary 3.2.

2. Notation and basic definitions

In this section we will recall basic definitions and facts from the theory of Orlicz spaces. For more details, see [1], [9].

We say that $\Phi : [0, \infty) \to [0, \infty)$ is a *Young function* if it is convex, continuous, strictly increasing, $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. If Φ and Ψ are two Young

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functions, we say that Ψ *grows essentially faster near infinity than* Φ if for every $k > 0$, $\Psi(t)/\Phi(kt) \to \infty$ as $t \to \infty$. We denote it by $\Psi \succ\succ \Phi$. Finally a Young function Φ is said to satisfy the Δ_2 *condition near infinity* if there are constants $K, t_0 > 0$ such that $\Phi(2t) \leq K\Phi(t)$ for all $t > t_0$.

Observe that if Φ satisfies the Δ_2 condition near infinity, then $\Psi \succ\succ \Phi$ if and only if $\Psi(t)/\Phi(t) \to \infty$ as $t \to \infty$.

Let Φ be a Young function and (X, μ) be a measure space. For simplicity we will *always* assume that $\mu(X) < \infty$. The *Orlicz space* $L^\Phi(X)$ consists of all measurable functions u on X such that

$$\int_X \Phi(\lambda|u(x)|)\, d\mu < \infty \quad \text{for some } \lambda > 0.$$

It follows from the convexity of Φ that $L^\Phi(X)$ is a linear space, and one can prove that this space, equipped with the *Luxemburg norm*

$$\|u\|_\Phi = \inf\left\{k > 0 : \int_X \Phi\left(\frac{|u(x)|}{k}\right) d\mu \leq 1\right\},$$

is a Banach space. Note that

$$\int_X \Phi\left(\frac{|u(x)|}{\|u\|_\Phi}\right) d\mu \leq 1\,.$$

If Φ satisfies the Δ_2 condition near infinity, then

$$L^\Phi(X) = \left\{u : \int_X \Phi(|u(x)|)\, d\mu < \infty\right\},$$

but this claim is not true without the Δ_2 condition.

Convexity of Φ implies that for $0 < \varepsilon \leq 1$, $\Phi(x) \leq \varepsilon\Phi(x/\varepsilon)$, and hence it is easy to see that convergence $u_n \to u$ in L^Φ implies that

$$\int_X \Phi(|u_n - u|)\, d\mu \to 0\,. \tag{2.1}$$

Convergence (2.1) is called *convergence in mean*, and we note here that convergence in mean implies convergence in the Luxemburg norm only if Φ satisfies the Δ_2 condition near infinity.

Given an open set $\Omega \subset \mathbb{R}^n$ and a Young function A we can define in a natural way the *Orlicz-Sobolev* space $W^{1,A}(\Omega)$. If $A(t) = t^p$, then $W^{1,A}(\Omega) = W^{1,p}(\Omega)$. Convexity of A implies that $A(t) \geq at$ for $t \geq t_0$ and hence $W^{1,A}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega)$. Thus it follows from the Rellich-Kondrachov theorem and the standard diagonal argument that every bounded sequence in $W^{1,A}(\Omega)$ has a subsequence that is convergent a.e.

We say that a family of functions $\mathcal{F} \subset L^1(X)$ is *equi-integrable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sup_{f \in \mathcal{F}} \int_E |f| d\mu < \varepsilon \quad \text{whenever } \mu(E) < \delta.$$

Note that equi-integrability does not imply in general that the family $\mathcal{F}$ is bounded in $L^1(X)$ even if $\mu(X) < \infty$ (which is our standing assumption), because the measure may have atoms.

We will need the following result of de la Vallée Poussin, which we state as a lemma. For a proof, see [4], [9].

Lemma 2.1 (de la Vallée Poussin)**.** *Let (X, μ) be a measure space with $\mu(X) < \infty$ and let $\mathcal{F} \subset L^1(\mu)$ be bounded. Then $\mathcal{F}$ is equi-integrable if and only if there is a Young function Φ, $\lim_{t\to\infty} \Phi(t)/t = \infty$ such that*

$$\sup_{f\in\mathcal{F}} \int_X \Phi(|f|)d\mu \leq 1\,. \tag{2.2}$$

In most of the statements found in the literature the condition is that the integral (2.2) is finite. Dividing Φ by an appropriate constant we may further require that the integral is less than or equal to 1.

3. Main theorems

The following theorem is the main result of the paper.

Theorem 3.1. *Let $W^{1,A}(\Omega)$ be an Orlicz-Sobolev space on $\Omega \subset \mathbb{R}^n$, $|\Omega| < \infty$. Then the following conditions are equivalent:*

(a) *$W^{1,A}(\Omega)$ is compactly embedded into $L^\Phi(\Omega)$, denoted $W^{1,A}(\Omega) \Subset L^\Phi(\Omega)$.*
(b) *There is a Young function $\Psi \succ\succ \Phi$ such that $W^{1,A}(\Omega)$ is continuously embedded into $L^\Psi(\Omega)$, denoted $W^{1,A}(\Omega) \subset L^\Psi(\Omega)$.*

The following two corollaries follow immediately from the theorem.

Corollary 3.2. *If a bounded embedding $W^{1,A}(\Omega) \subset L^\Phi(\Omega)$ is optimal in the category of Orlicz spaces, then it is not compact.*

For sharp results regarding embeddings and compact embeddings into Orlicz spaces in the case in which Ω is a bounded domain with Lipschitz boundary, see [2], [7, Theorem 5.6].

Nečas [8, Théorème 1.4] proved that if $\Omega \subset \mathbb{R}^n$ is a bounded domain with continuous boundary (i.e. the boundary is locally a graph of a continuous function), then the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$ is compact. As an immediate consequence of this result and Theorem 3.1 we obtain

Corollary 3.3. *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with continuous boundary, then there is a Young function Φ that grows essentially faster at infinity than t^2 such that $W^{1,2}(\Omega) \subset L^\Phi(\Omega)$.*

A more precise description of the function Φ can be obtained from the information about the modulus of continuity of the functions used to represent the boundary as a graph, but it is interesting to observe that our argument implies the existence of Φ without any careful investigation of the structure of the boundary.

We will prove Theorem 3.1 from more general results formulated for abstract normed spaces. Theorem 3.1 is a direct consequence of Theorem 3.4 and Theorem 3.8 below. The first result, which is a common generalization of some results in [1], [5], proves the implication from (b) to (a) in Theorem 3.1.

Theorem 3.4. *Let $W(X)$ be a normed space of measurable functions on (X, μ), $\mu(X) < \infty$, with the property that every bounded sequence in $W(X)$ has a subsequence that is convergent a.e. If Ψ is a Young function such that the embedding $W(X) \subset L^\Psi(X)$ is bounded, then for every Young function Φ such that $\Psi \succ\succ \Phi$, the embedding $W(X) \Subset L^\Phi(X)$ is compact.*

Proof. Since the embedding $W \subset L^\Psi$ is bounded, there is a constant $C > 0$ such that $\|f\|_\Psi \le C\|f\|_W$ for all $f \in W$. Let $\{f_i\} \subset W$ be a bounded sequence, $\|f_i\|_W \le M$. By our assumptions, f_i has a subsequence f_{i_j} that is convergent a.e. It suffices to prove that f_{i_j} is a Cauchy sequence in L^Φ. Fix $\varepsilon > 0$ and let $u_i = f_i/\varepsilon$. Then

$$\|u_i - u_j\|_\Psi \le C\|u_i - u_j\|_W \le 2CM\varepsilon^{-1}$$

and hence

$$\int_X \Psi\left(\frac{|u_i - u_j|}{2CM\varepsilon^{-1}}\right) d\mu \le 1 \quad \text{for all } i, j.$$

Since Ψ grows essentially faster than Φ, there is $t_0 > 0$ such that

$$\Phi(t) \le \frac{1}{4}\,\Psi\left(\frac{t}{2CM\varepsilon^{-1}}\right) \quad \text{for } t > t_0.$$

On the set $\{|u_i - u_j| \le t_0\}$ we have $\Phi(|u_i - u_j|) \le \Phi(t_0)$. Let $\delta = (4\Phi(t_0))^{-1}$. If $E \subset X$ is such that $\mu(X \setminus E) < \delta$, then

$$\begin{aligned}
\int_{X\setminus E} \Phi(|u_i - u_j|)\, d\mu &\le \int_{\{|u_i - u_j| > t_0\}} \Phi(|u_i - u_j|)\, d\mu + \int_{X\setminus E} \Phi(t_0)\, d\mu \\
&\le \frac{1}{4}\int_X \Psi\left(\frac{|u_i - u_j|}{2CM\varepsilon^{-1}}\right) d\mu + \frac{\Phi(t_0)}{4\Phi(t_0)} \\
&\le \frac{1}{2} \quad \text{for all } i, j.
\end{aligned}$$

By our assumptions, u_{i_j} is convergent a.e. According to the Egorov theorem there is a measurable set $E \subset X$ such that $\mu(X \setminus E) < \delta$ and u_{i_j} converges uniformly on E. Hence there is N such that

$$|u_{i_j}(x) - u_{i_k}(x)| \le \Phi^{-1}\left(\frac{1}{2\mu(X)}\right) \quad \text{for all } x \in E \text{ and } j, k \ge N.$$

Then for $j, k \ge N$ we have

$$\begin{aligned}
\int_X \Phi\left(\frac{|f_{i_j} - f_{i_k}|}{\varepsilon}\right) d\mu &= \int_E \Phi(|u_{i_j} - u_{i_k}|)\, d\mu + \int_{X\setminus E} \Phi(|u_{i_j} - u_{i_k}|)\, d\mu \\
&\le \frac{\mu(X)}{2\mu(X)} + \frac{1}{2} = 1
\end{aligned}$$

and hence

$$\|f_{i_j} - f_{i_k}\|_\Phi \le \varepsilon \quad \text{for all } j, k \ge N.$$

The proof is complete. □

The next result is a version of the implication from (a) to (b) in Theorem 3.1 formulated in the setting of normed spaces $W(X)$. Observe that in the statement we require that the Young function Φ satisfies the Δ_2 condition, and Theorem 3.6 shows that it is not possible to avoid the Δ_2 condition. On the other hand, no Δ_2 condition is needed in Theorem 3.1. This will be explained in Theorem 3.8, where we will show what additional property of $W(X)$ (satisfied by $W^{1,A}(\Omega)$) allows us to remove the Δ_2 condition from the statement.

Theorem 3.5. *Let $W(X)$ be a normed space of measurable functions on (X, μ), $\mu(X) < \infty$. Let Φ be a Young function that satisfies the Δ_2 condition near infinity. If $W(X)$ is compactly embedded into $L^\Phi(X)$, $W(X) \Subset L^\Phi(X)$, then there is a*

Young function $\Psi \succ\succ \Phi$ such that $W(X)$ is continuously embedded into $L^\Psi(X)$, $W(X) \subset L^\Psi(X)$.

Proof. Suppose that $W(X) \Subset L^\Phi$ and Φ satisfies the Δ_2 condition near infinity. Let $C > 0$ be such that $\|f\|_\Phi \le C\|f\|_W$ for $f \in W$. Consider the unit sphere in W,

$$S = \{f \in W : \|f\|_W = 1\}.$$

We claim that the family

$$\mathcal{F} = \{\Phi(|f|/C) : f \in S\}$$

is bounded and equi-integrable in $L^1(X)$. Boundedness follows from the definition of the Luxemburg norm. Indeed, $\|f\|_\Phi \le C$ for $f \in S$ and hence

$$\int_X \Phi(|f|/C)\, d\mu \le 1\,.$$

Thus $\mathcal{F}$ is contained in the unit ball in $L^1(X)$. On the contrary, suppose that $\mathcal{F}$ is not equi-integrable. Then there is $\varepsilon > 0$ and two sequences $E_n \subset X$, $f_n \in S$ such that $\mu(E_n) < 1/n$, while

$$\int_{E_n} \Phi\left(\frac{|f_n|}{C}\right) d\mu \ge \varepsilon\,. \tag{3.1}$$

The sequence $2f_n/C$ is bounded in W and since the embedding $W \Subset L^\Phi$ is compact, the sequence has a subsequence (still denoted by $2f_n/C$) convergent in L^Φ to some function $g \in L^\Phi$. The convergence in mean (2.1) gives

$$\int_X \Phi\left(\left|\frac{2f_n}{C} - g\right|\right) d\mu < \varepsilon \quad \text{for } n \ge n_1.$$

Since $g \in L^\Phi$ and Φ satisfies the Δ_2 condition near infinity, $\int_X \Phi(|g|)\, d\mu < \infty$, and hence there is n_2 such that

$$\int_{E_n} \Phi(|g|)\, d\mu < \varepsilon \quad \text{for } n \ge n_2$$

by absolute continuity of the integral. For $n > \max\{n_1, n_2\}$, convexity of Φ gives

$$\begin{aligned}\int_{E_n} \Phi\left(\frac{|f_n|}{C}\right) d\mu &\le \int_{E_n} \Phi\left(\frac{1}{2}\left|\frac{2f_n}{C} - g\right| + \frac{1}{2}|g|\right) d\mu \\ &\le \frac{1}{2}\int_{E_n} \Phi\left(\left|\frac{2f_n}{C} - g\right|\right) d\mu + \frac{1}{2}\int_{E_n} \Phi(|g|)\, d\mu < \varepsilon,\end{aligned}$$

which contradicts (3.1). We proved that the family $\mathcal{F}$ satisfies the assumptions of the de la Vallée Poussin theorem, and hence there is a Young function η such that $\eta(t)/t \to \infty$ as $t \to \infty$ and

$$\sup_{f \in S} \int_X \eta\left(\Phi\left(\frac{|f|}{C}\right)\right) d\mu \le 1\,.$$

Hence for all $0 \ne f \in W$ and $\Psi = \eta \circ \Phi$,

$$\int_X \Psi\left(\frac{|f|}{C\|f\|_W}\right) d\mu \le 1,$$

which proves boundedness of the embedding $W \subset L^\Psi$ with the same constant $\|f\|_\Psi \le C\|f\|_W$. It remains to observe that $\Psi \succ\succ \Phi$. Indeed, for any $k > 0$,

$$\lim_{t\to\infty} \frac{\Psi(t)}{\Phi(kt)} = \lim_{t\to\infty} \frac{\eta(\Phi(t))}{\Phi(t)} \frac{\Phi(t)}{\Phi(kt)} = \infty$$

since $\Phi(t)/\Phi(kt)$ is bounded away from 0 by the Δ_2 condition. □

The following example shows that we cannot avoid the Δ_2 condition in Theorem 3.5. In particular it shows that if we do not assume the Δ_2 condition, the optimal embedding for the space $W(X)$ in the category of Orlicz spaces can be compact, different from the case of Corollary 3.2.

Theorem 3.6. *There is a Banach space W of measurable functions on $[0,1]$ with the following properties:*

(a) *Every bounded sequence in W has a subsequence convergent a.e.*
(b) *$W \Subset L^\Phi([0,1])$ for $\Phi(t) = \frac{2}{\pi}(e^t - 1)$.*
(c) *There is no Young function $\Psi \succ\succ \Phi$ such that $W \subset L^\Psi([0,1])$.*

Remark 3.7. We do not even require in (c) that the embedding $W \subset L^\Psi([0,1])$ is bounded. We only assume that every function in W belongs to $L^\Psi([0,1])$.

Proof. First we will define auxiliary functions that will be used to construct the space W. Let

$$f(x) = -\log(x + x\log^2 x), \quad x \in (0,1]\,.$$

Note that f is strictly decreasing from ∞ to 0. We have

$$\int_0^1 \Phi(|f(x)|)\,dx = \frac{2}{\pi}\int_0^1 \left(e^{f(x)} - 1\right)\,dx = \frac{2}{\pi}\left(\frac{\pi}{2} - 1\right) < 1$$

since the antiderivative of $e^{f(x)} = (x + x\log^2 x)^{-1}$ is $\arctan\log x$. It is easy to see that for any $0 < k < 1$,

$$\int_0^1 \Phi\left(\frac{|f(x)|}{k}\right)\,dx = \infty$$

and hence $\|f\|_\Phi = 1$. For $n \ge 2$ we define

$$g_n = c_n\chi_{[0,\frac{1}{n}]}, \quad \text{where } c_n = -\log\left(\frac{\pi}{2} + \arctan\log\frac{1}{n}\right)\,.$$

Observe that $c_n > 0$ for $n \ge 2$. Finally let $f_n = f + g_n$. We have

$$\begin{aligned}\int_0^1 \Phi(|f_n(x)|)\,dx &= \frac{2}{\pi}\left(\int_0^{1/n}\left(e^{f(x)}e^{c_n} - 1\right)\,dx + \int_{1/n}^1 \left(e^{f(x)} - 1\right)\,dx\right)\\ &\le \frac{2}{\pi}\left(1 - \frac{1}{n} + \frac{\pi}{2} - 1\right) < 1\end{aligned}$$

and for $0 < k < 1$,

$$\int_0^1 \Phi\left(\frac{|f_n(x)|}{k}\right)\,dx > \int_0^1 \Phi\left(\frac{|f(x)|}{k}\right)\,dx = \infty\,,$$

so $\|f_n\|_\Phi = 1$. Note also that $f_n \to f$ in L^Φ as $n \to \infty$. Indeed, for every $\varepsilon > 0$,

$$\lim_{n\to\infty}\int_0^1 \Phi\left(\frac{|f_n - f|}{\varepsilon}\right)\,dx = \lim_{n\to\infty}\frac{2}{\pi}\frac{1}{n}\left(e^{c_n/\varepsilon} - 1\right) = 0$$

by a simple application of the l'Hospital rule.

Let $f_1 = f$ and define a Banach space W of measurable functions on $[0,1]$ as

$$W = \left\{ h = \sum_{i=1}^{\infty} a_i f_i : (a_i)_{i=1}^{\infty} \in \ell^1 \right\}$$

with the norm

$$\|h\|_W = \Big\| \sum_{i=1}^{\infty} a_i f_i \Big\|_W := \sum_{i=1}^{\infty} |a_i| \,.$$

Since for every $x \in (0,1]$, $f_i(x) = f(x)$ for all sufficiently large i, the series $\sum_{i=1}^{\infty} a_i f_i(x)$ converges at every $x \in (0,1]$, and hence it defines a measurable function. Considering intervals $(1/(n+1), 1/n]$, $n = 1, 2, 3, \ldots$ one can easily check by induction that if $\sum_{i=1}^{\infty} a_i f_i = 0$ a.e., then $a_i = 0$ for all i, so the coefficients a_i are uniquely determined and hence $\|\cdot\|_W$ is a well-defined norm. Now it is obvious that W is isometric to ℓ^1 and hence W is a Banach space.

The partial sums of the series $\sum_{i=1}^{\infty} a_i f_i$ form a Cauchy sequence in L^Φ because

$$\Big\| \sum_{i=k}^{\ell} a_i f_i \Big\|_\Phi \le \sum_{i=k}^{\ell} |a_i| \|f_i\|_\Phi = \sum_{i=k}^{\ell} |a_i|,$$

and hence the series converges in the Banach space L^Φ. This also shows that W is continuously embedded into L^Φ,

$$\|h\|_\Phi = \Big\| \sum_{i=1}^{\infty} a_i f_i \Big\|_\Phi \le \sum_{i=1}^{\infty} |a_i| \|f_i\|_\Phi = \sum_{i=1}^{\infty} |a_i| = \|h\|_W \,,$$

but what is more interesting, the embedding is compact, $W \Subset L^\Phi([0,1])$. Before we prove this fact, observe that compactness of the embedding implies that every bounded sequence in W has a subsequence that is convergent a.e., which is the property (a).

Recall that $f_n \to f$ in L^Φ as $n \to \infty$, and hence the set

$$F = \{f_i\}_{i=1}^{\infty} \subset L^\Phi, \quad \text{where } f_1 = f,$$

is compact. Then also the family of functions

$$K = \{x \mapsto t f_i(x) : t \in [-M, M],\ i \ge 1\} \subset L^\Phi$$

is compact. Indeed, K is the image of a continuous mapping defined on a compact set

$$\lambda : [-M, M] \times F \to L^\Phi, \quad \lambda(t, f_i) = t f_i, \quad \lambda([-M, M] \times F) = K \,.$$

According to Mazur's theorem [3, Theorem 4.8], the convex hull $co(K)$ is relatively compact in L^Φ. With this introduction we can complete the proof of (b) as follows.

Let $h_n = \sum_{i=1}^{\infty} a_i^n f_i \in W$ be a bounded sequence and let $\tilde{h}_n = \sum_{i=1}^{k(n)} a_i^n f_i$ be such that $\|h_n - \tilde{h}_n\|_W < 1/n$. The sequence $\tilde{h}_n$ is bounded, say

$$\|\tilde{h}_n\|_W = \sum_{i=1}^{k(n)} |a_i^n| \le M \,.$$

Then

$$\tilde{h}_n = \sum_{i=1}^{k(n)} \frac{|a_i^n|}{\|\tilde{h}_n\|_W} \Big(\operatorname{sgn}(a_i^n) \|\tilde{h}_n\|_W f_i \Big) \in co(K),$$

and hence $\tilde{h}_n$ has a subsequence convergent in L^Φ. This also implies that h_n has a subsequence convergent in L^Φ to the same limit.

We are left with the proof of (c). Suppose that there is $\Psi \succ\succ \Phi$ such that $W \subset L^\Psi$. It follows from the closed graph theorem that the embedding is bounded. Indeed, if $h_n \to h$ in W and $h_n \to g$ in L^Ψ, then from the boundedness of the embedding into L^Φ, $h_n \to h$ in L^Φ and hence $g = h$.

Since $\|f_n\|_W = 1$, the sequence f_n is bounded in L^Ψ, say $\|f\|_\Psi \le C$, so

$$\int_0^1 \Psi\left(\frac{|f_n(x)|}{C}\right) dx \le 1\,. \tag{3.2}$$

Note that

$$\inf_{x\in[0,1/n]} f_n(x) \ge f(1/n) \to \infty, \quad \text{as } n \to \infty,$$

and therefore the condition $\Psi \succ\succ \Phi$ implies that

$$A_n = \inf_{x\in[0,1/n]} \frac{\Psi\left(\frac{|f_n(x)|}{C}\right)}{\Phi(|f_n(x)|)} \to \infty \quad \text{as } n \to \infty.$$

Thus

$$\begin{aligned}\int_0^1 \Psi\left(\frac{|f_n(x)|}{C}\right) dx &\ge A_n \int_0^{1/n} \Phi(|f_n(x)|)\, dx \\ &= \frac{2}{\pi} A_n \int_0^{1/n} \left(e^{f_n(x)} - 1\right) dx \\ &= \frac{2}{\pi} A_n \left(1 - \frac{1}{n}\right) \to \infty,\end{aligned}$$

which contradicts (3.2). The proof is complete. □

The following result shows what additional property of the space $W(X)$ allows us to remove the Δ_2 condition from the statement of Theorem 3.5.

We say that a normed space $W(X)$ of measurable functions has the *truncation property* if for very $f \in W(X)$ and $t > 0$, the truncated function at the level t,

$$f_t(x) = \begin{cases} t & \text{if } f(x) \ge t, \\ f(x) & \text{if } -t \le f(x) \le t, \\ -t & \text{if } f(x) \le -t, \end{cases}$$

belongs to $W(X)$ and

$$\|f_t\|_W \le \|f\|_W\,.$$

Clearly the space $W^{1,A}(\Omega)$ has the truncation property, while the space W constructed in Theorem 3.6 has not. Thus the theorem below shows the implication from (a) to (b) in Theorem 3.1.

Theorem 3.8. *Let $W(X)$ be a normed space with the truncation property on (X, μ), $\mu(X) < \infty$. If Φ is a Young function such that the embedding $W(X) \Subset L^\Phi(X)$ is compact, then there is a Young function $\Psi \succ\succ \Phi$ such that the embedding $W(X) \subset L^\Psi(X)$ is bounded.*

Proof. Suppose that the embedding given by the identity mapping $e : W(X) \to L^\Phi(X)$ is compact. We first claim that essentially bounded functions are dense in the embedding range $e(W(X)) \subset L^\Phi(X)$; i.e. $e(W(X) \cap L^\infty)$ is dense in $e(W(X))$.

Indeed, let $f_n(x)$ be the truncation of f at the level $t = n$, as defined above. Then $f_n \in W(X) \cap L^\infty$ and $\|f_n\|_W \le \|f\|_W$. Hence $\{f_n\}$ is a bounded sequence in $W(X)$ and thus it has a subsequence convergent to some $f_0 \in L^\Phi(X)$. Since $f_n \to f$ a.e., it easily follows that $f_0 = f$. This also implies that $e(W(X) \cap L^\infty)$ is dense in the closure $\overline{e(W(X))}$ in $L^\Phi(X)$.

Our second claim is that

$$\int_X \Phi(k|f|)\, d\mu < \infty, \quad \text{for every } f \in \overline{e(W(X))} \text{ and } k > 0.$$

Indeed, given $f \in \overline{e(W(X))}$ and $k > 0$, by the first claim we can find a sequence $f_n \in W(X) \cap L^\infty$ such that $f_n \to f$ in $L^\Phi(X)$. Let n be so large that $\|f - f_n\|_\Phi < (2k)^{-1}$. Then

$$\begin{aligned}
\int_X \Phi(k|f|)\, d\mu &\le \frac{1}{2}\int_X \Phi(2k|f - f_n|)\, d\mu + \frac{1}{2}\int_X \Phi(2k|f_n|)\, d\mu \\
&\le \frac{1}{2}\|2k(f - f_n)\|_\Phi \int_X \Phi\left(\frac{2k|f - f_n|}{\|2k(f - f_n)\|_\Phi}\right) d\mu + \frac{1}{2}\int_X \Phi(2k|f_n|)\, d\mu \\
&\le \frac{1}{2}\|2k(f - f_n)\|_\Phi + \frac{1}{2}\int_X \Phi(2k|f_n|)\, d\mu < \infty\,.
\end{aligned}$$

We used here the inequality $\Phi(x) \le \varepsilon\Phi(x/\varepsilon)$, $0 < \varepsilon \le 1$, along with the estimate $\|2k(f - f_n)\|_\Phi < 1$, and the last inequality follows from the fact that $2k|f_n|$ is bounded.

Consider the unit sphere in $W(X)$:

$$S = \{f \in W(X) : \|f\|_W = 1\}\,.$$

Fix $k > 0$. We claim that the family

$$\mathcal{F} = \{\Phi(k|f|) : f \in S\}$$

is bounded and equi-integrable in $L^1(X)$. Let $f_n \in S$ be a sequence that is convergent in $L^\Phi(X)$ to some $f \in L^\Phi(X)$, so $f \in \overline{e(W(X))}$. Let $0 < \varepsilon \le 1$ and let n be so large that $\|f - f_n\|_\Phi < \varepsilon(2k)^{-1} \le (2k)^{-1}$. Then the same argument as the one used above shows that for any measurable set $E \subset X$,

$$\int_E \Phi(k|f_n|)\, d\mu \le \frac{1}{2}\|2k(f - f_n)\|_\Phi + \frac{1}{2}\int_E \Phi(2k|f|)\, d\mu\,. \tag{3.3}$$

Note that the last integral is finite, because $\Phi(2k|f|)$ is integrable by the second claim. Suppose that $\mathcal{F}$ is not bounded in $L^1(X)$. Then there is a sequence $f_n \in S$ such that $\|\Phi(k|f_n|)\|_1 \to \infty$ as $n \to \infty$. Since the sequence $\{f_n\}$ is bounded in $W(X)$, it has a subsequence (still denoted by $\{f_n\}$) convergent to some $f \in L^\Phi(X)$ (by compactness of the embedding). Hence (3.3) with $E = X$ implies that $\|\Phi(k|f_n|)\|_1$ is bounded, which is a contradiction. Thus $\mathcal{F}$ is bounded in $L^1(X)$. A similar argument along with the absolute continuity of the integral $\int_E \Phi(2k|f|)\, d\mu$ implies equi-integrability of $\mathcal{F}$.

We proved that the family $\mathcal{F}$ satisfies the assumptions of the de la Vallée Poussin theorem, and hence for each $k > 0$ there is a Young function η_k such that $\eta_k(t)/t \to \infty$ as $t \to \infty$ and

$$\sup_{f \in S} \int_X \eta_k\left(\Phi(k|f|)\right)\, d\mu \le 1\,.$$

If follows that

$$\sup_{f\in S}\sum_{k=1}^{\infty}\int_X 2^{-k}\eta_k\left(\Phi(k|f|)\right)\,d\mu\le 1.$$

We can find a decreasing sequence $0<a_k\le 2^{-k}$ of positive numbers convergent to 0 so fast that the series

$$\Psi(t):=\sum_{k=1}^{\infty}a_k\eta_k(\Phi(kt))$$

converges uniformly on compact sets and hence defines a continuous function. Since Φ is convex, Ψ is a Young function. Moreover

$$\sup_{f\in S}\int_X \Psi\left(|f|\right)\,d\mu\le 1,$$

so $\|f\|_\Psi\le\|f\|_W$ and

$$\lim_{t\to\infty}\frac{\Psi(t)}{\Phi(kt)}=\infty$$

for all $k>0$. The proof is complete. □

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