ON THE PERIODICITY
OF SOME FARHI ARITHMETICAL FUNCTIONS

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Abstract. Let \( k \in \mathbb{N} \). Let \( f(x) \in \mathbb{Z}[x] \) be any polynomial such that \( f(x) \) and \( f(x+1)f(x+2) \cdots f(x+k) \) are coprime in \( \mathbb{Q}[x] \). We call
\[
g_{k,f}(n) := \frac{|f(n)f(n+1) \cdots f(n+k)|}{\text{lcm}(f(n), f(n+1), \cdots, f(n+k))}
\]
a Farhi arithmetic function. In this paper, we prove that \( g_{k,f} \) is periodic. This generalizes the previous results of Farhi and Kane, and Hong and Yang.

1. Introduction

Throughout this paper, let \( \mathbb{Q} \), \( \mathbb{Z} \) and \( \mathbb{N} \) denote the field of rational numbers, the ring of rational integers and the set of nonnegative integers. Let \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). As usual, let \( v_p \) denote the normalized \( p \)-adic valuation of \( \mathbb{Q} \), i.e., \( v_p(a) = b \) if \( p^b | a \).

It is known that an equivalent variation of the Prime Number Theorem states that \( \log \text{lcm}(1, 2, \cdots, n) \sim n \) as \( n \) tends to infinity (see, e.g., [5]). One thus expects that a better understanding of the function \( \text{lcm}(1, 2, \cdots, n) \) may entail a deeper understanding of the distribution of the prime numbers. Some progress has been made towards this direction. Before we state our main theorems, let us first give a short account on the recent results in this subject.

In his pioneering paper [2], Farhi introduced the following arithmetic functions:
\[
g_k(n) := \frac{n(n+1) \cdots (n+k)}{\text{lcm}(n, n+1, \cdots, n+k)}, \quad n \in \mathbb{N}^*.
\]

Farhi proved that the sequence \( (g_k)_{k \in \mathbb{N}} \) satisfies the recursion relation:
\[
g_k(n) = \gcd(k!, (n+k)g_{k-1}(n)), \quad \forall n \in \mathbb{N}^*.
\]
Using this relation, Farhi proved

Theorem 1.1 ([2]). The function \( g_k \) (\( k \in \mathbb{N} \)) is periodic and \( k! \) is a period of \( g_k \).
An interesting question is how to determine the least period of \( g_k \) (see [2]). In [6], by using (1) and \( g_k(1)| g_k(n) \) for any positive integer \( n \), Hong and Yang gave a partial answer to this question. A complete solution to the question was given by Farhi and Kane in [3]. They proved

**Theorem 1.2** ([3], Theorem 3.2). The least period \( T_k \) of \( g_k \) is given by

\[
T_k = \prod_{p \text{ prime}, p \leq k} p^{\delta_p(k)},
\]

where

\[
\delta_p(k) = \begin{cases} 
0, & \text{if } v_p(k+1) \geq \max_{1 \leq i \leq k} \{v_p(i)\}, \\
\max_{1 \leq i \leq k} \{v_p(i)\}, & \text{otherwise}.
\end{cases}
\]

Let \( g(n) \) be an arithmetic function defined on the set \( \mathbb{Z}\setminus A \), where \( A \) is a finite subset of \( \mathbb{Z} \). If there exists an integer \( T \) such that \( g(n) = g(n+T) \) for all \( n, n+T \in \mathbb{Z}\setminus A \), then it is clear that the arithmetic function \( g(n) \) can be extended to a periodic function defined on all the integers \( \mathbb{Z} \).

Throughout this paper, let \( k \) be a nonnegative integer and \( f(x) \in \mathbb{Z}[x] \) such that \( \gcd(f(x), f(x+1)f(x+2) \cdots f(x+k)) = 1 \) in \( \mathbb{Q}[x] \). Set

\[
Z_{k,f} := \{ n \in \mathbb{Z} \mid f(n+i) = 0 \text{ for some } 0 \leq i \leq k \}.
\]

Then \( Z_{k,f} \) is a finite subset of \( \mathbb{Z} \). Set

\[
g_{k,f}(n) = \frac{|f(n)f(n+1) \cdots f(n+k)|}{\lcm(f(n), f(n+1), \ldots, f(n+k))},
\]

for \( n \in \mathbb{Z}\setminus Z_{k,f} \). We call \( g_{k,f}(n) \) a Farhi arithmetic function. In §3, we will prove

**Theorem 1.3.** Let \( k \) be a nonnegative integer and \( f(x) \in \mathbb{Z}[x] \) such that \( \gcd(f(x), f(x+1)f(x+2) \cdots f(x+k)) = 1 \) in \( \mathbb{Q}[x] \). Then the arithmetic function \( g_{k,f} \) can be extended to a periodic arithmetic function defined on all the integers.

By assumption of \( f(x) \) in Theorem 1.3, for any \( 1 \leq i \leq k \), there exist polynomials \( a_i(x), b_i(x) \in \mathbb{Z}[x] \) and the smallest positive integer \( C_i \) such that

\[
a_i(x)f(x) + b_i(x)f(x+i) = C_i.
\]

Let \( C \) be the least common multiple of the \( C_i \)'s, i.e.,

\[
C = \lcm(C_1, C_2, \ldots, C_k).
\]

In the proof of Theorem 1.3 we will prove

**Theorem 1.4.** Let \( T_{k,f} \) denote the least period of \( g_{k,f} \). Then \( T_{k,f} | C \).

Let \( p \) be a prime. Define the arithmetic function \( h_{k,f,p} \) by

\[
h_{k,f,p}(n) := v_p(g_{k,f}(n)).
\]

If \( p \nmid C \), using the definition of \( g_{k,f} \), then we have \( h_{k,f,p}(n) = 0 \) for any \( n \in \mathbb{Z} \). If \( p \mid C \), then \( h_{k,f,p} \) is a periodic function by Theorem 1.3. Set

\[
S_n := \{ n, n+1, \ldots, n+k \}, \quad n \in \mathbb{Z}
\]

and

\[
e_p := \max\{v_p(\gcd(|f(n)|, |f(n+i)|)) \mid 1 \leq n \leq p^{v_p(C)}, 1 \leq i \leq k \}.
\]
In §4, we will prove

**Theorem 1.5.** For any prime $p$, let $T_{k,f,p}$ be the least period of the arithmetic function $h_{k,f,p}$. Then

(i) $p^r$ is a period of $h_{k,f,p}$ and $T_{k,f,p}|p^r$.

(ii) $T_{k,f,p} = 1$ if and only if for any $1 \leq n \leq p^r$, we have

\[ v_p(\gcd(|f(n)|, |f(n + k + 1)|)) \geq \max_{1 \leq i \leq k} \{v_p(f(n + i))\} \]

or

\[ v_p(f(n)) = v_p(f(n + k + 1)) < \max_{1 \leq i \leq k} \{v_p(f(n + i))\}. \]

(iii) Let $1 \leq e \leq e_p$. Suppose that $p^r$ is a period of $h_{k,f,p}$. Then $T_{k,f,p} = p^r$ if and only if there exists an integer $n_0 : 1 \leq n_0 \leq p^r$ such that the following inequality holds:

\[ \sum_{t = 0}^{\epsilon_p} \max\{0, \#\{m \in S_{n_0} \mid p^t|f(m)\} - 1\} \neq \sum_{t = e}^{\epsilon_p} \max\{0, \#\{m \in S_{n_0} \mid p^t|f(m + p^{e-1})\} - 1\}. \]

In particular, $T_{k,f,p} = p^r$ if and only if there exists an integer $n_0 : 1 \leq n_0 \leq p^r$ such that the following inequality holds:

\[ \#\{m \in S_{n_0} \mid p^r|f(m)\} \neq \#\{m \in S_{n_0} \mid p^r|f(m + p^{r-1})\}. \]

**Remark.** Let $T_{k,f,p}$ be the least period of $h_{k,f,p}$ for any prime $p$. Then

\[ T_{k,f} = \prod_p T_{k,f,p}. \]

(This infinite product is meaningful, for almost all its terms are equal to 1.)

As an application of Theorem 1.5 in §5 we will give a new different proof of Theorem 3.2 of [3].

**Corollary 1.6.** Let $k \in \mathbb{N}$ and $f(x) = x$. Then the least period $T_{k,f}$ of the Farhi arithmetic function $g_{k,f}$ is given by the formula 6.

Let $a, b \in \mathbb{Z}$ be any integer such that $\gcd(a, b) = 1$ and $a > 0$. Let $f(x) = ax + b$. By Theorem 1.3 we know that the Farhi arithmetic function

\[ g_{k,ax+b}(n) = \frac{|(an+b)(a(n+1)+b)\cdots(a(n+k)+b)|}{\text{lcm}(an+b,a(n+1)+b,\ldots,a(n+k)+b)} \]

can be extended to a periodic arithmetic function defined on all the integers. Now we define the arithmetical function $g_{k,a}$ by

\[ g_{k,a}(n) = \frac{|n(n+a)\cdots(n+ka)|}{\text{lcm}(n+a,\ldots,n+ka)}. \]

When $a = 1$, the arithmetical function $g_{k,1}$ is the arithmetical function $g_k$ defined by Farhi. It is clear that

\[ g_{k,ax+b}(n) = g_{k,a}(na+b). \]

Hence the function $g_{k,a}$ can also be extended to a periodic arithmetic function defined on all the integers. In §6, we shall prove the following results:
Theorem 1.7. Let \( a, k \) be any two positive integers. Then the following assertions hold:

(i) The positive integer \( a \cdot \text{lcm}(1, 2, \cdots, k) \) is a period of \( g_{k,a} \).

(ii) A positive integer \( S \) is a period of \( g_{k,a} \) if and only if \( S = aT \), where \( T \) is a period of \( g_k \).

(iii) Consequently, the least period \( T_k(a) \) of \( g_{k,a} \) is \( aT_k(1) = aT_k \), where \( T_k(1) = T_k \) is the least period of \( g_k \).

By (8) and Theorem 1.7 we have the following result:

Corollary 1.8. Let \( a, k \) be any two positive integers and let \( b \in \mathbb{Z} \) be any integer such that \( \gcd(a, b) = 1 \). Then the least period \( T_{k,ax+b} \) of the Farhi arithmetic function \( g_{k,ax+b} \) is given by the following formula:

\[
T_{k,ax+b} = \prod_{p \text{ prime}, p \leq k} p^\delta_p(k),
\]

where

\[
\delta_p(k) = \begin{cases} 0, & \text{if } v_p(k + 1) \geq \max_{1 \leq i \leq k} \{v_p(i)\} \text{ or } p|a, \\ \max_{1 \leq i \leq k} \{v_p(i)\}, & \text{otherwise}. \end{cases}
\]

In §7, we will give some examples.

2. Two basic lemmas

Lemma 2.1. Let \( a_1, a_2, \cdots, a_n \) and \( b_1, b_2, \cdots, b_n \) be any \( 2n \) positive integers. If \( \gcd(a_i, a_j) = \gcd(b_i, b_j) \) for any \( 1 \leq i, j \leq n \), then

\[
\frac{a_1a_2\cdots a_n}{\text{lcm}(a_1, a_2, \cdots, a_n)} = \frac{b_1b_2\cdots b_n}{\text{lcm}(b_1, b_2, \cdots, b_n)}. \tag{10}
\]

Proof. Let \( p \) be any prime. It suffices to show that the following equality holds:

\[
v_p\left(\frac{a_1a_2\cdots a_n}{\text{lcm}(a_1, a_2, \cdots, a_n)}\right) = v_p\left(\frac{b_1b_2\cdots b_n}{\text{lcm}(b_1, b_2, \cdots, b_n)}\right), \tag{11}
\]

i.e.,

\[
\sum_{i=1}^{n} v_p(a_i) - \max_{1 \leq i \leq n} \{v_p(a_i)\} = \sum_{i=1}^{n} v_p(b_i) - \max_{1 \leq i \leq n} \{v_p(b_i)\}. \tag{12}
\]

By symmetry, it suffices to show that

\[
\sum_{i=1}^{n} v_p(a_i) - \max_{1 \leq i \leq n} \{v_p(a_i)\} \leq \sum_{i=1}^{n} v_p(b_i) - \max_{1 \leq i \leq n} \{v_p(b_i)\}. \tag{13}
\]

Without loss of generality, we assume that \( v_p(a_1) \leq v_p(a_2) \leq \cdots \leq v_p(a_{n-1}) \leq v_p(a_n) \). Then for any \( 1 \leq i \leq n - 1 \), we have

\[
v_p(a_i) = v_p(\gcd(a_i, a_n)) = v_p(\gcd(b_i, b_n)) \leq \min\{v_p(b_i), v_p(b_n)\}. \]

Hence for any \( 1 \leq i \leq n - 1 \), we have \( v_p(a_i) \leq v_p(b_i) \) and \( v_p(a_i) \leq v_p(b_n) \). Let \( v_p(b_k) = \max_{1 \leq i \leq n} \{v_p(b_i)\} \). Then

\[
\sum_{i=1}^{n-1} v_p(a_i) \leq v_p(b_1) + \cdots + v_p(b_{k-1}) + v_p(b_n) + v_p(b_{k+1}) + \cdots + v_p(b_{n-1}).
\]

So \(13\) is true. This completes the proof of Lemma 2.1. \qed
Lemma 2.2. Let $k$ be a positive integer and $f(x) \in \mathbb{Z}[x]$ such that
\[ \gcd(f(x), f(x+1)f(x+2)\cdots f(x+k)) = 1 \]
in $\mathbb{Q}[x]$. Then $d_i(n) = \gcd(|f(n)|, |f(n+i)|)$ is periodic for any $1 \leq i \leq k$.

**Proof.** By assumption, for any $1 \leq i \leq k$, $f(x)$ and $f(x+i)$ are coprime in $\mathbb{Q}[x]$; hence there exist $a_i(x), b_i(x) \in \mathbb{Z}[x]$ and the smallest positive integer $C_i$ such that
\[ a_i(x)f(x) + b_i(x)f(x+i) = C_i. \]
Hence for all $m \in \mathbb{Z}$, we have
\[ a_i(m)f(m) + b_i(m)f(m+i) = C_i, \]
In the following, we will prove
\[ d_i(n) = d_i(n+C_i), \quad n \in \mathbb{Z}. \]
Let $d_i = d_i(n)$ and $d_i' = d_i(n+C_i)$. Then $d_i|f(n)$ and $d_i|f(n+i)$. By (16), we have $d_i|C_i$. Hence, by the Taylor formula of $f$, we obtain that $d_i|f(n+C_i)$ and $d_i|f(n+i+C_i)$. Therefore $d_i|d_i'$. Similarly, we have $d_i'|d_i$. Hence $d_i = d_i'$, i.e., (16) is true. This completes the proof of Lemma 2.2.

3. The proofs of Theorem 1.3 and Theorem 1.4

**Proof of Theorem 1.3.** By the definition (13), $Z_{k,f}$ is a finite set and $g_{k,f}$ is well defined on the set $\mathbb{Z} \setminus Z_{k,f}$. First we prove that $g_{k,f}$ is periodic on the set $\mathbb{Z} \setminus Z_{k,f}$. For $1 \leq i \leq k$, by Lemma 2.2 $d_i(n) = \gcd(|f(n)|, |f(n+i)|)$ is periodic. Let $T_i$ be the least period of $d_i$. Then
\[ d_i(n) = d_i(n+T_i), \quad \text{for any } n \in \mathbb{Z}. \]
Hence by the proof of Lemma 2.2 we have that $T_i|C_i$, where $C_i$ is defined by (14). Denote by $T$ (resp. $C$) the least common multiple of the $T_i$’s (resp. $C_i$’s), $i = 1, 2, \cdots, k$. Then $T|C$ and for any $1 \leq i \leq k$, we have
\[ d_i(n) = d_i(n+T) \quad \text{for } n \in \mathbb{Z}. \]
Hence for any $0 \leq i < j \leq k$, we have
\[ d_{j-i}(n+i) = d_{j-i}(n+i+T) \quad \text{for } n \in \mathbb{Z}, \]
that is,
\[ \gcd(|f(n+i)|, |f(n+j)|) = \gcd(|f(n+i+T)|, |f(n+j+T)|). \]
So by Lemma 2.1 and the definition of $g_{k,f}$, we obtain that $g_{k,f}(n) = g_{k,f}(n+T)$ for any $n$ and $n+T \in \mathbb{Z} \setminus Z_{k,f}$. Hence $g_{k,f}(n)$ is periodic and $T$ is a period of $g_{k,f}$.

If $n \in Z_{k,f}$, then there exist a positive integer $a$ such that $n+aT \notin Z_{k,f}$. Hence the function $g_{k,f}$ can be extended to $g_{k,f} : \mathbb{Z} \to \mathbb{Z}$, defined at $n \in Z_{k,f}$, by
\[ g_{k,f}(n) = g_{k,f}(n+aT). \]
This completes the proof of Theorem 1.3.

**Proof of Theorem 1.4.** It is obvious that the property $T|C$ implies that $C$ is a multiple of the least period $T_{k,f}$ of $g_{k,f}$. This completes the proof of Theorem 1.4.
4. The proof of Theorem 1.5

We use the same notation as in previous sections.

Proof. (i) By the definitions of $h_{k,f,p}$ and $g_{k,f}$, it suffices to show that $v_p(g_{k,f}(n)) = v_p(g_{k,f}(n + p^{e_p}))$ for any $n \in \mathbb{Z} \setminus \mathbb{Z}_{k,f}$, i.e.,

$$
\sum_{i=0}^{k} v_p(f(n+i)) - \max_{0 \leq i \leq k} \{ v_p(f(n+i)) \}
$$

$$
= \sum_{i=0}^{k} v_p(f(n+i+p^{e_p})) - \max_{0 \leq i \leq k} \{ v_p(f(n+i+p^{e_p})) \}.
$$

Let $e_{ij} = v_p(\gcd(|f(n+i)|, |f(n+j)|))$

and

$e'_{ij} = v_p(\gcd(|f(n+i+p^{e_p})|, |f(n+j+p^{e_p})|))$

for any $0 \leq i < j \leq k$. By the proof of Lemma 2.1, it suffices to show that

$$(17) \quad e_{ij} = e'_{ij}.$$  

By the assumption of $f(x)$, we have

$$a_{j-i}(m)f(m) + b_{j-i}(m)f(m + j - i) = C_{j-i}, \quad m \in \mathbb{Z}.$$  

Let $m = n+i$. We have $p^{e_{ij}} | f(n+i)$ and $p^{e_{ij}} | f(n+j)$, so $p^{e_{ij}} | C_{j-i}$. Hence $e_{ij} \leq e_p$ by the definition of $e_p$. So $p^{e_{ij}} | f(n+i+p^{e_p})$, $p^{e_{ij}} | f(n+j+p^{e_p})$. Therefore $e_{ij} \leq e'_{ij}$. Similarly, we have $e'_{ij} \leq e_{ij}$. Hence (17) is true. It is easy to see that $T_{k,f,p} | p^{e_p}$.

(ii) By (i) of Theorem 1.5, we know that $h_{k,f,p}$ is periodic and $p^{e_p}$ is a period. So $T_{k,f,p} = 1$ if and only if $h_{k,f,p}(n) = h_{k,f,p}(n+1)$ for any $1 \leq n \leq p^{e_p}$. By the definition of $g_{k,f}$, we have $T_{k,f,p} = 1$ if and only if for any $1 \leq n \leq p^{e_p}$,

$$
\sum_{i=0}^{k} v_p(f(n+i)) - \max_{0 \leq i \leq k} \{ v_p(f(n+i)) \}
$$

$$
= \sum_{i=1}^{k+1} v_p(f(n+i)) - \max_{1 \leq i \leq k+1} \{ v_p(f(n+i)) \}.
$$

Hence $T_{k,f,p} = 1$ if and only if for any $1 \leq n \leq p^{e_p}$,

$$v_p(\gcd(|f(n)|, |f(n+k+1)|)) \geq \max_{1 \leq i \leq k} \{ v_p(f(n+i)) \}$$

or

$$v_p(f(n)) = v_p(f(n+k+1)) < \max_{1 \leq i \leq k} \{ v_p(f(n+i)) \}.$$  

(iii) Let $1 \leq e \leq e_p$. Suppose that $p^e$ is a period of $h_{k,f,p}$. Hence $p^e$ is the least period of $h_{k,f,p}$ if and only if $p^{e-1}$ is not a period of $h_{k,f,p}$. Therefore $p^e$ is the least period of $h_{k,f,p}$ if and only if there exists an integer $n_0 : 1 \leq n_0 \leq p^e$ such that the following inequality holds:

$$h_{k,f,p}(n_0) \neq h_{k,f,p}(n_0 + p^{e-1}).$$
By definition (5), we have

$$h_{k,f,p}(n_0) = \sum_{i=0}^k v_p(f(n_0 + i)) - \max_{0 \leq i \leq k} \{v_p(f(n_0 + i))\}$$

(18)

and only if

$$= \sum_{i=1}^\infty \max\{0, \#\{m \in S_{n_0} \mid p^i | f(m)\} - 1\}$$

(19) $$h_{k,f,p}(n_0 + p^{e-1}) = \sum_{i=1}^\infty \max\{0, \#\{m \in S_{n_0} \mid p^i | f(m + p^{e-1})\} - 1\}.$$  

Remark that the infinite sums of (18) and (19) are meaningful, for almost all their terms are equal to 0. On the other hand, when $$t \leq e - 1$$, we know that $$p^i | f(m)$$ if and only if $$p^i | f(m + p^{e-1})$$. Hence by the definition of $$e_p$$, the inequality $$h_{k,f,p}(n_0) \neq h_{k,f,p}(n_0 + p^{e-1})$$ holds if and only if the inequality

$$\sum_{t=e}^{e_p} \max\{0, \#\{m \in S_{n_0} \mid p^i | f(m)\} - 1\}$$


$$\neq \sum_{t=e}^{e_p} \max\{0, \#\{m \in S_{n_0} \mid p^i | f(m + p^{e-1})\} - 1\}$$

holds. In particular, $$T_{k,f,p} = p^{e_p}$$ if and only if there exists an integer $$n_0 : 1 \leq n_0 \leq p^{e_p}$$ such that the following inequality holds:

$$\#\{m \in S_{n_0} \mid p^i | f(m)\} \neq \#\{m \in S_{n_0} \mid p^i | f(m + p^{e_p-1})\}.$$  

This completes the proof of Theorem 1.5. \qed

5. The proof of Corollary 1.6

Proof. When $$k = 0$$, then $$g_{0,f} = 1$$. Let $$k \geq 1$$. For $$1 \leq i \leq k$$, we have $$C_i = i$$ and $$C = \text{lcm}(1,2,\cdots,k)$$. Hence we obtain

$$T_{k,f} = \prod_{\text{prime}, p \leq k} T_{k,f,p}.$$  

Letting $$p \leq k$$ be a prime, it suffices to prove the following statements:

(I) $$T_{k,f,p} = 1$$ if $$v_p(k+1) \geq \max_{1 \leq i \leq k} \{v_p(i)\}$$.

(II) $$T_{k,f,p} = p^{e_p(C)}$$ if $$v_p(k+1) < \max_{1 \leq i \leq k} \{v_p(i)\}$$.

We first prove (I). As $$e_p = v_p(C) = \max_{1 \leq i \leq k} \{v_p(i)\}$$, by assumption $$v_p(k+1) \geq e_p$$, we have $$v_p(k+1) = e_p$$ or $$e_p + 1$$.

Case (a): $$1 \leq n \leq p^{e_p} - 1$$; then $$e = v_p(n) < e_p$$. Hence $$v_p(n) = v_p(n+k+1)$$ and $$n = p^{e_p} n_1$$, $$p \nmid n_1$$. Set $$i = p^{e_p} i_0$$, $$1 \leq i_0 \leq p - 1$$ such that $$p | (n_1 + i_0)$$. Then $$1 \leq i \leq k$$ and $$v_p(n+i) > v_p(n)$$. Hence

$$v_p(n) = v_p(n+k+1) < \max_{1 \leq i \leq k} \{v_p(n+i)\}.$$  

Case (b1): $$n = p^{e_p}$$ and $$v_p(k+1) = e_p + 1$$. We have $$k+1 = p^{e_p+1}$$. Let $$i = p^{e_p} (p - 1)$$. Then $$1 \leq i \leq k$$ and $$v_p(n+i) = e_p + 1 > e_p$$. Hence

$$v_p(n) = v_p(n+k+1) < \max_{1 \leq i \leq k} \{v_p(n+i)\}.$$  

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Case (b2): \( n = p^r p \) and \( v_p(k+1) = e_p \). We have \( k+1 = p^r u \), where \( 2 \leq u \leq p-1 \). Hence \( k = up^r - 1 \). If \( i > 0 \) and \( v_p(n+i) > e_p \), then \( i \geq p^r(p-1) > k \). Therefore

\[
\max_{1 \leq i \leq k} \{ v_p(n+i) \} \leq e_p.
\]

By (20), (21), (22) and using (ii) of Theorem 1.5, we have \( T_{k,f,p} = 1 \).

(II) Note that \( k = a_0 + a_1 p + \cdots + a_{e_p} p^r, \quad 0 \leq a_i \leq p-1, \quad i = 0, 1, \cdots, e_p, \quad a_{e_p} \neq 0 \).

It is easy to show that the inequality \( v_p(k+1) \geq e_p \) holds if and only if \( a_0 = a_1 = \cdots = a_{e_p-1} = p-1 \).

Assume that the inequality \( v_p(k+1) < e_p = v_p(C) = \max_{1 \leq i \leq k} v_p(i) \) holds. Then there exists an integer \( r : 0 \leq r \leq e_p - 1 \) such that the following conditions hold:

\[
0 \leq a_r \leq p-2 \quad \text{and} \quad a_{r+1} = \cdots = a_{e_p-1} = p-1.
\]

Set

\[
n_0 = \left\{ \begin{array}{ll}
  p^r, & \text{if } r = e_p - 1; \\
  (p-1-a_r)p^r, & \text{if } 0 \leq r \leq e_p - 2.
\end{array} \right.
\]

Then we have

\[
\# \{ m \in S_{n_0} \mid p^r\mid m \} = \left\{ \begin{array}{ll}
  1, & \text{if } r = e_p - 1, \\
  a_{e_p}, & \text{if } 0 \leq r \leq e_p - 2;
\end{array} \right.
\]

and

\[
\# \{ m \in S_{n_0} \mid p^r\mid (m + p^{r-1}) \} = \left\{ \begin{array}{ll}
  a_{e_p}, & \text{if } r = e_p - 1, \\
  1, & \text{if } 0 \leq r \leq e_p - 2.
\end{array} \right.
\]

By (iii) of Theorem 1.5 we know that \( p^r = p^{r_p}(C) \) is the least period of \( h_{k,f,p} \).

This completes the proof of Corollary 1.6. \( \square \)

6. The proof of Theorem 1.7

Proof. (i) Set \( S = a \cdot \text{lcm}(1, 2, \cdots, k) \). Let \( n \) be any positive integer. For any \( 0 \leq i < j \leq k \), it is clear that \( \gcd(n + ia, n + ja) = \gcd(n + S + ia, n + S + ja) \).

Hence \( g_{k,a}(n+S) = g_{k,a}(n) \) follows from Lemma 2.1.

(ii) Suppose \( S \) is a period of \( g_{k,a} \). Then \( g_{k,a}(n) = g_{k,a}(n+S) \) for all \( n \in \mathbb{N}^+ \). In particular, we have \( g_{k,a}(na) = g_{k,a}(na+S) \).

Since

\[
g_{k,a}(na) = \frac{na \cdot (na+a) \cdots (na+ka) \cdot a^k}{\text{lcm}(na, na+a, \cdots, na+ka)} = \frac{n(n+1) \cdots (n+k) \cdot a^k}{\text{lcm}(n, n+1, \cdots, n+k)},
\]

and

\[
g_{k,a}(na+S) = \frac{(na+S)(na+a+S) \cdots (na+ka+S) \cdot a^k}{\text{lcm}(na+S, na+a+S, \cdots, na+ka+S)},
\]

we have

\[
g_{k,a}(na+S) = \frac{(na+S)(na+a+S) \cdots (na+ka+S) \cdot a^k}{\text{lcm}(na+S, na+a+S, \cdots, na+ka+S)}.
\]

Thus

\[
g_{k,n} \cdot a^k = g_{k,a}(na) = g_{k,a}(na+S)
\]

and

\[
g_{k,n} \cdot a^k = \frac{(na+S)(na+a+S) \cdots (na+ka+S)}{\text{lcm}(na+S, na+a+S, \cdots, na+ka+S)}.
\]
Let \( (27) \)

Hence, without loss of generality, we assume that \((28)\). By using \((24)\), we have

\[
\gcd(a_1, a, 1 + a, 1 + a) \cdots (na_1 + ka + S_1).
\]

Because \(\gcd(a_1, na_1 + ia_1 + S_1) = 1\) for any \(0 \leq i \leq k\), we have \(a_1 = 1\). Hence \(a|S\).

Let \(S = aT\). Then using \((23)\), we have

\[
g_k(n) \cdot a^k = g_{k,a}(na) = g_{k,a}(na + aT) = g_k(n + T) \cdot a^k.
\]

Hence by using \((25)\), we have

\[
\gcd(a, g_{k,a}(n)) = 1, \quad \gcd(a, g_{k,a}(n + aT)) = 1.
\]

Hence by using \((25)\), we have

\[
g_{k,a}(n) = g_{k,a}(n + aT)
\]

if and only if

\[
v_p(g_{k,a}(n)) = v_p(g_{k,a}(n + aT)),
\]

for any prime \(p \nmid a\).

Let \(p\) be any prime such that \(p \nmid a\) and let \(N\) be a positive integer greater than \(v_p(k!)\). Then there exists a unique positive integer \(m\) such that \(1 \leq m < p^N\) and

\[
ma \equiv 1 \pmod{p^N}.
\]

Let \(p, n, a, m\) be as above and \(0 \leq i < j \leq k\). Then for any integer \(l\), there are

\[
v_p(\gcd(n + al + ai, (j - i)a)) = v_p(\gcd(mn + l + i, j - i)).
\]

Let

\[
\gcd(n + al + ai, (j - i)a) = p^{x_{ij}}w, \quad p \nmid w
\]

and

\[
\gcd(mn + l + i, j - i) = p^{y_{ij}}u, \quad p \nmid u.
\]

Then by \((23)\), there exists \(s_1, t_1 \in \mathbb{Z}\) such that \(\gcd(n + al + ai)\).

Multiplying by \(m\) on both sides, we have \((mn + l + i) = p^{x_{ij}}w\).

Using \((27)\), we have \((mn + l + i) = p^{y_{ij}}u\).

By \((30)\), we have \(y_{ij} \leq x_{ij}\). Conversely, by \((29)\), there exist \(s_2, t_2 \in \mathbb{Z}\) such that \((mn + l + i) = p^{x_{ij}}w\).

Similarly, we have \(x_{ij} \leq y_{ij}\). So \(x_{ij} = y_{ij}\) and \((28)\) is true. Let \(l = 0\) and \(T\). By using \((23)\), we have

\[
v_p(g_{k,a}(n + ai, n + aj)) = v_p(g_{k,a}(mn + i, mn + j))
\]

and

\[
v_p(g_{k,a}(n + aT + ai, n + aT + aj)) = v_p(g_{k,a}(mn + T + i, mn + T + j))
\]

for any \(0 \leq i < j \leq k\). By the proof of Lemma \((23)\), we have

\[
v_p(g_{k,a}(n)) = v_p(g_{k,a}(mn)), \quad v_p(g_{k,a}(n + aT)) = v_p(g_{k,a}(mn + T)).
\]
The least period $T_{k,f}$ of $g_{k,f}$ with $f(x) = x^2 + b$

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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<tbody>
<tr>
<td>$x^2 + 1$</td>
<td>5</td>
<td>5</td>
<td>2 · 5</td>
<td>2 · 3 · 5 · 13</td>
<td>2 · 3 · 5 · 13</td>
<td>2 · 3 · 5 · 13 · 29</td>
<td>2 · 3 · 5 · 13 · 29</td>
</tr>
<tr>
<td>$x^2 + 2$</td>
<td>3²</td>
<td>3²</td>
<td>2 · 3²</td>
<td>2 · 3² · 17</td>
<td>2 · 3² · 17</td>
<td>2 · 3² · 5 · 11 · 17</td>
<td>2 · 3² · 5 · 11 · 17</td>
</tr>
<tr>
<td>$x^2 + 3$</td>
<td>13</td>
<td>13</td>
<td>2 · 3 · 7 · 13</td>
<td>2 · 3 · 7 · 13</td>
<td>2 · 3 · 5 · 7 · 13 · 37</td>
<td>2 · 3 · 5 · 7 · 13 · 37</td>
<td></td>
</tr>
<tr>
<td>$x^2 + 4$</td>
<td>17</td>
<td>17</td>
<td>2 · 5 · 17</td>
<td>2 · 3 · 5² · 17</td>
<td>2 · 3 · 5² · 17</td>
<td>2 · 3 · 5² · 5 · 17 · 41</td>
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</tr>
<tr>
<td>$x^2 + 5$</td>
<td>3 · 7</td>
<td>3 · 7</td>
<td>2 · 3 · 7 · 29</td>
<td>2 · 3 · 7 · 29</td>
<td>2 · 3 · 5 · 7 · 29</td>
<td>2 · 3 · 5 · 7 · 29</td>
<td></td>
</tr>
<tr>
<td>$x^2 + 6$</td>
<td>5²</td>
<td>5²</td>
<td>2 · 5 · 7 · 11</td>
<td>2 · 3 · 5 · 7 · 11</td>
<td>2 · 3 · 5 · 7 · 2² · 11</td>
<td>2 · 3 · 5 · 7 · 2² · 11</td>
<td></td>
</tr>
</tbody>
</table>

So using $g_k(mn) = g_k(mn + T)$, we have $v_p(g_{k,a}(n + aT)) = v_p(g_{k,a}(n))$ for any prime $p$ such that $p \nmid a$. Hence $g_{k,a}(n + aT) = g_{k,a}(n)$ and $aT$ is a period of $g_{k,a}(n)$.

The proof of (iii) follows from (ii). This completes the proof of Theorem 1.7. □

Proof of Corollary 1.8 (i) Assume that $p | a$. Then it is clear that the equality $v_p(g_{k,ax+b}(n)) = 0$ holds for any integer $n$ when $g_{k,ax+b}(n)$ is well defined.

(ii) Assume that $p$ is not a prime factor of $a$. By the formula 5, we have that $T_p$ is a period of $v_p(g_{k,ax+b}(n))$ if and only if $aT_p$ is a period of $v_p(g_{k,a}(n))$. Hence, by Theorem 1.7 we have that $T_p$ is a period of $v_p(g_{k,ax+b}(n))$ if and only if $T_p$ is a period of $v_p(g_{k,a}(n))$. Therefore Corollary 1.8 is obtained by Theorem 1.2. □

7. EXAMPLES

Lemma 7.1. Let $f_1(x) = f_2(x)^r$, where $r \geq 1$ is an integer. Then $T_{k,f_1} = T_{k,f_2}$.

Proof. By 4, we have $g_{k,f_1}(n) = g_{k,f_2}(n)^r$. Hence the result is obvious. □

Example. Let $f(x) = x^r$, $r \geq 1$. Then by Lemma 7.1 we have $T_{k,x^r} = T_{k,x}$, where $T_{k,x}$ is given by the formula 4.

Example. Let $f(x) = x^2 + b$. For $1 \leq i \leq k$, we have

\[
(2x + 3i)(x^2 + b) + (-2x + i)((x + i)^2 + b) = i(i^2 + 4b), \text{ if } i \text{ is odd,}
\]
\[
(x + 3j)(x^2 + b) + (-x + j)((x + 2j)^2 + b) = 4j(j^2 + b), \text{ if } i = 2j.
\]

Hence

\[
C_i = \begin{cases} 
  i(i^2 + 4b), & \text{if } i \text{ is odd,} \\
  4j(j^2 + b), & \text{if } i = 2j.
\end{cases}
\]

Hence, given any $k \in \mathbb{N}$ and $b \in \mathbb{Z}$, by Theorem 1.5 we can determine the least period $T_{k,f}$ of the arithmetic function $g_{k,f}$. For $1 \leq k \leq 6$ and $1 \leq b \leq 6$, Table 1 gives the $T_{k,f}$'s.

Example. Let $f(x) = x^3 + b$. For $1 \leq i \leq k$, we have

\[
a_i(x)(x^3 + b) + b_i(x)((x + i)^3 + b) = C_i,
\]

where

\[
a_i(x) = \begin{cases} 
  6i^2x^2 + (15i^3 - 9)x + 10i^4 - 18i, & \text{if } 3 \nmid i; \\
  6j^2x^2 + (45j^3 - 1)x + 90i^4 - 6i, & \text{if } i = 3j.
\end{cases}
\]
Table 2

The least period $T_{k,f}$ of $g_{k,f}$ with $f(x) = x^3 + b$

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3+1$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·7·13·17·19·31·43</td>
<td>2·3·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3+2$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·7·13·17·19·31·43</td>
<td>2·3·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3+3$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·7·13·17·19·31·43</td>
<td>2·3·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3+4$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·7·13·17·19·31·43</td>
<td>2·3·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3+5$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
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<td>2·3·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3+6$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·7·13·17·19·31·43</td>
<td>2·3·7·13·17·19·31·43</td>
</tr>
</tbody>
</table>

$$b_i(x) = \begin{cases} -6i^2x^2 + (3i^3 + 9)x - i^4 - 9i, & \text{if } 3 \nmid i; \\ -6j^2x^2 + (9j^3 + 1)x - 9j^4 - 3j, & \text{if } i = 3j. \end{cases}$$

$$C_i = \begin{cases} -i^7 - 27i, & \text{if } 3 \nmid i; \\ -35j^7 - 9j, & \text{if } i = 3j. \end{cases}$$

Hence, given any $k \in \mathbb{N}$ and $b \in \mathbb{Z}$, by Theorem 1.5, we can determine the least period $T_{k,f}$ of the arithmetic function $g_{k,f}$. For $1 \leq k \leq 6$ and $1 \leq b \leq 6$, Table 2 gives the $T_{k,f}$'s.

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