ON THE PERIODICITY
OF SOME FARHI ARITHMETICAL FUNCTIONS

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Abstract. Let \( k \in \mathbb{N} \). Let \( f(x) \in \mathbb{Z}[x] \) be any polynomial such that \( f(x) \) and \( f(x + 1)f(x + 2) \cdots f(x + k) \) are coprime in \( \mathbb{Q}[x] \). We call 
\[
g_{k, f}(n) := \frac{\lcm(f(n), f(n + 1), \cdots, f(n + k))}{\lcm(f(n), f(n + 1), \cdots, f(n + k))}
\]
a Farhi arithmetic function. In this paper, we prove that \( g_{k, f} \) is periodic. This generalizes the previous results of Farhi and Kane, and Hong and Yang.

1. Introduction

Throughout this paper, let \( \mathbb{Q}, \mathbb{Z} \) and \( \mathbb{N} \) denote the field of rational numbers, the ring of rational integers and the set of nonnegative integers. Let \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). As usual, let \( v_p \) denote the normalized \( p \)-adic valuation of \( \mathbb{Q} \), i.e., \( v_p(a) = b \) if \( p^b \mid |a| \).

It is known that an equivalent variation of the Prime Number Theorem states that 
\[
\log \lcm(1, 2, \cdots, n) \sim n
\]
as \( n \) tends to infinity (see, e.g., [5]). One thus expects that a better understanding of the function \( \lcm(1, 2, \cdots, n) \) may entail a deeper understanding of the distribution of the prime numbers. Some progress has been made towards this direction. Before we state our main theorems, let us first give a short account on the recent results in this subject.

In his pioneering paper [2], Farhi introduced the following arithmetic functions: 
\[
g_k(n) := \frac{n(n + 1) \cdots (n + k)}{\lcm(n, n + 1, \cdots, n + k)}, \quad n \in \mathbb{N}^*.
\]
Farhi proved that the sequence \( (g_k)_{k \in \mathbb{N}} \) satisfies the recursion relation:
\[
g_k(n) = \gcd(k!, (n + k)g_{k-1}(n)), \quad \forall n \in \mathbb{N}^*.
\]
Using this relation, Farhi proved

**Theorem 1.1** ([2]). The function \( g_k \) \((k \in \mathbb{N})\) is periodic and \( k! \) is a period of \( g_k \).
An interesting question is how to determine the least period of $g_k$ (see [3]). In [6], by using (1) and $g_k(1)|g_k(n)$ for any positive integer $n$, Hong and Yang gave a partial answer to this question. A complete solution to the question was given by Farhi and Kane in [3]. They proved

**Theorem 1.2** ([3], Theorem 3.2). The least period $T_k$ of $g_k$ is given by

$$T_k = \prod_{p \text{ prime }, p \leq k} p^{\delta_p(k)},$$

where

$$\delta_p(k) = \begin{cases} 0, & \text{if } v_p(k+1) \geq \max_{1 \leq i \leq k} \{v_p(i)\}, \\ \max_{1 \leq i \leq k} \{v_p(i)\}, & \text{otherwise}. \end{cases}$$

Let $g(n)$ be an arithmetic function defined on the set $\mathbb{Z}\setminus A$, where $A$ is a finite subset of $\mathbb{Z}$. If there exists an integer $T$ such that $g(n) = g(n+T)$ for all $n, n+T \in \mathbb{Z}\setminus A$, then it is clear that the arithmetic function $g(n)$ can be extended to a periodic function defined on all the integers $\mathbb{Z}$.

Throughout this paper, let $k$ be a nonnegative integer and $f(x) \in \mathbb{Z}[x]$ such that $\gcd(f(x), f(x+1)f(x+2) \cdots f(x+k)) = 1$

in $\mathbb{Q}[x]$. Set

$$Z_{k,f} := \{n \in \mathbb{Z} \mid f(n+i) = 0 \text{ for some } 0 \leq i \leq k\}.$$ 

Then $Z_{k,f}$ is a finite subset of $\mathbb{Z}$. Set

$$g_{k,f}(n) = \frac{|f(n)f(n+1) \cdots f(n+k)|}{\operatorname{lcm}(f(n), f(n+1), \ldots, f(n+k))},$$

for $n \in \mathbb{Z}\setminus Z_{k,f}$. We call $g_{k,f}(n)$ a Farhi arithmetic function. In §3, we will prove

**Theorem 1.3.** Let $k$ be a nonnegative integer and $f(x) \in \mathbb{Z}[x]$ such that $\gcd(f(x), f(x+1)f(x+2) \cdots f(x+k)) = 1$ in $\mathbb{Q}[x]$. Then the arithmetic function $g_{k,f}$ can be extended to a periodic arithmetic function defined on all the integers $\mathbb{Z}$.

By assumption of $f(x)$ in Theorem 1.3 for any $1 \leq i \leq k$, there exist polynomials $a_i(x), b_i(x) \in \mathbb{Z}[x]$ and the smallest positive integer $C_i$ such that

$$a_i(x)f(x) + b_i(x)f(x+i) = C_i.$$

Let $C$ be the least common multiple of the $C_i$’s, i.e.,

$$C = \operatorname{lcm}(C_1, C_2, \ldots, C_k).$$

In the proof of Theorem 1.3, we will prove

**Theorem 1.4.** Let $T_{k,f}$ denote the least period of $g_{k,f}$. Then $T_{k,f} | C$.

Let $p$ be a prime. Define the arithmetic function $h_{k,f,p}$ by

$$h_{k,f,p}(n) := v_p(g_{k,f}(n)).$$

If $p | C$, using the definition of $g_{k,f}$, then we have $h_{k,f,p}(n) = 0$ for any $n \in \mathbb{Z}$. If $p | C$, then $h_{k,f,p}$ is a periodic function by Theorem 1.3. Set

$$S_n := \{n, n+1, \ldots, n+k\}, \ n \in \mathbb{Z}$$

and

$$e_p := \max\{v_p(\gcd(|f(n)|, |f(n+i)|)) \mid 1 \leq n \leq p^\nu(C), 1 \leq i \leq k\}.$$
In §4, we will prove

**Theorem 1.5.** For any prime \( p \), let \( T_{k,f,p} \) be the least period of the arithmetic function \( h_{k,f,p} \). Then

(i) \( p^{e_p} \) is a period of \( h_{k,f,p} \) and \( T_{k,f,p} \mid p^{e_p} \).

(ii) \( T_{k,f,p} = 1 \) if and only if for any \( 1 \leq n \leq p^{e_p} \), we have

\[
v_p(\gcd(|f(n)|, |f(n + k + 1)|)) \geq \max_{1 \leq i \leq k} \{v_p(f(n + i))\}
\]

or

\[
v_p(f(n)) = v_p(f(n + k + 1)) < \max_{1 \leq i \leq k} \{v_p(f(n + i))\}.
\]

(iii) Let \( 1 \leq e \leq e_p \). Suppose that \( p^e \) is a period of \( h_{k,f,p} \). Then \( T_{k,f,p} = p^e \) if and only if there exists an integer \( n_0 : 1 \leq n_0 \leq p^e \) such that the following inequality holds:

\[
\sum_{i=0}^{e_p} \max\{0, \#\{m \in S_{n_0} \mid p^i|f(m)\} - 1\} \neq \sum_{i=0}^{e_p} \max\{0, \#\{m \in S_{n_0} \mid p^i|f(m + p^{e-1})\} - 1\}.
\]

In particular, \( T_{k,f,p} = p^{e_p} \) if and only if there exists an integer \( n_0 : 1 \leq n_0 \leq p^{e_p} \) such that the following inequality holds:

\[
\#\{m \in S_{n_0} \mid p^e|f(m)\} \neq \#\{m \in S_{n_0} \mid p^e|f(m + p^{e-1})\}.
\]

**Remark.** Let \( T_{k,f,p} \) be the least period of \( h_{k,f,p} \) for any prime \( p \). Then

\[
T_{k,f} = \prod_{p} T_{k,f,p}.
\]

(This infinite product is meaningful, for almost all its terms are equal to 1.)

As an application of Theorem 1.5 in §5 we will give a new different proof of Theorem 3.2 of [3].

**Corollary 1.6.** Let \( k \in \mathbb{N} \) and \( f(x) = x \). Then the least period \( T_{k,f} \) of the Farhi arithmetic function \( g_{k,f} \) is given by the formula (9).

Let \( a, b \in \mathbb{Z} \) be any integer such that \( \gcd(a, b) = 1 \) and \( a > 0 \). Let \( f(x) = ax + b \).

By Theorem 1.3 we know that the Farhi arithmetic function

\[
g_{k,ax+b}(n) = \frac{|(an+b)(a(n+1)+b)\cdots(a(n+k)+b)|}{\operatorname{lcm}(an+b,a(n+1)+b,\ldots,a(n+k)+b)}
\]

can be extended to a periodic arithmetic function defined on all the integers. Now we define the arithmetical function \( g_{k,a} \) by

\[
g_{k,a}(n) = \frac{|n(n+a)\cdots(n+ka)|}{\operatorname{lcm}(n,n+a,\ldots,n+ka)}.
\]

When \( a = 1 \), the arithmetical function \( g_{k,1} \) is the arithmetical function \( g_k \) defined by Farhi. It is clear that

\[
g_{k,ax+b}(n) = g_{k,a}(na + b).
\]

Hence the function \( g_{k,a} \) can also be extended to a periodic arithmetic function defined on all the integers. In §6, we shall prove the following results:
Theorem 1.7. Let \( a, k \) be any two positive integers. Then the following assertions hold:

(i) The positive integer \( a \cdot \gcd(1, 2, \cdots, k) \) is a period of \( g_{k,a} \).

(ii) A positive integer \( S \) is a period of \( g_{k,a} \) if and only if \( S = aT \), where \( T \) is a period of \( g_k \).

(iii) Consequently, the least period \( T_k(a) \) of \( g_{k,a} \) is \( aT_k(1) = aT_k \), where \( T_k(1) = T_k \) is the least period of \( g_k \).

By (8) and Theorem 1.7, we have the following result:

Corollary 1.8. Let \( a, k \) be any two positive integers and let \( b \in \mathbb{Z} \) be any integer such that \( \gcd(a, b) = 1 \). Then the least period \( T_{k,ax+b} \) of the Farhi arithmetic function \( g_{k,ax+b} \) is given by the following formula:

\[
T_{k,ax+b} = \prod_{p \text{ prime}, p \leq k} p^{\delta_p(k)},
\]

where

\[
\delta_p(k) = \begin{cases} 
0, & \text{if } v_p(k+1) \geq \max_{1 \leq i \leq k} \{v_p(i)\} \text{ or } p|a, \\
\max_{1 \leq i \leq k} \{v_p(i)\}, & \text{otherwise}.
\end{cases}
\]

In §7, we will give some examples.

2. Two Basic Lemmas

Lemma 2.1. Let \( a_1, a_2, \cdots, a_n \) and \( b_1, b_2, \cdots, b_n \) be any \( 2n \) positive integers. If \( \gcd(a_i, a_j) = \gcd(b_i, b_j) \) for any \( 1 \leq i < j \leq n \), then

\[
\frac{a_1a_2 \cdots a_n}{\gcd(a_1, a_2, \cdots, a_n)} = \frac{b_1b_2 \cdots b_n}{\gcd(b_1, b_2, \cdots, b_n)}.
\]

Proof. Let \( p \) be any prime. It suffices to show that the following equality holds:

\[
v_p \left( \frac{a_1a_2 \cdots a_n}{\gcd(a_1, a_2, \cdots, a_n)} \right) = v_p \left( \frac{b_1b_2 \cdots b_n}{\gcd(b_1, b_2, \cdots, b_n)} \right),
\]

i.e.,

\[
\sum_{i=1}^{n} v_p(a_i) - \max_{1 \leq i \leq n} \{v_p(a_i)\} = \sum_{i=1}^{n} v_p(b_i) - \max_{1 \leq i \leq n} \{v_p(b_i)\}.
\]

By symmetry, it suffices to show that

\[
\sum_{i=1}^{n} v_p(a_i) - \max_{1 \leq i \leq n} \{v_p(a_i)\} \leq \sum_{i=1}^{n} v_p(b_i) - \max_{1 \leq i \leq n} \{v_p(b_i)\}.
\]

Without loss of generality, we assume that \( v_p(a_1) \leq v_p(a_2) \leq \cdots \leq v_p(a_{n-1}) \leq v_p(a_n) \). Then for any \( 1 \leq i \leq n-1 \), we have

\[
v_p(a_i) = v_p(\gcd(a_i, a_n)) = v_p(\gcd(b_i, b_n)) \leq \min \{v_p(b_1), v_p(b_n)\}.
\]

Hence for any \( 1 \leq i \leq n-1 \), we have \( v_p(a_i) \leq v_p(b_1) \) and \( v_p(a_i) \leq v_p(b_n) \). Let \( v_p(b_k) = \max_{1 \leq i \leq n} \{v_p(b_i)\} \). Then

\[
\sum_{i=1}^{n-1} v_p(a_i) \leq v_p(b_1) + \cdots + v_p(b_{k-1}) + v_p(b_n) + v_p(b_{k+1}) + \cdots + v_p(b_{n-1}).
\]

So (13) is true. This completes the proof of Lemma 2.1. \qed
Lemma 2.2. Let \( k \) be a positive integer and \( f(x) \in \mathbb{Z}[x] \) such that 
\[
\gcd(f(x), f(x+1)f(x+2) \cdots f(x+k)) = 1
\]
in \( \mathbb{Q}[x] \). Then \( d_i(n) = \gcd(|f(n)|, |f(n+i)|) \) is periodic for any \( 1 \leq i \leq k \).

Proof. By assumption, for any \( 1 \leq i \leq k \), \( f(x) \) and \( f(x+i) \) are coprime in \( \mathbb{Q}[x] \); hence there exist \( a_i(x), b_i(x) \in \mathbb{Z}[x] \) and the smallest positive integer \( C_i \) such that
\[
(14) \quad a_i(x)f(x) + b_i(x)f(x+i) = C_i.
\]
Hence for all \( m \in \mathbb{Z} \), we have
\[
(15) \quad a_i(m)f(m) + b_i(m)f(m+i) = C_i.
\]
In the following, we will prove
\[
(16) \quad d_i(n) = d_i(n+C_i), \quad n \in \mathbb{Z}.
\]
Let \( d_i = d_i(n) \) and \( d'_i = d_i(n+C_i) \). Then \( d_i|f(n) \) and \( d_i|f(n+i) \). By (15), we have \( d_i|C_i \). Hence, by the Taylor formula of \( f \), we obtain that \( d_i|f(n+C_i) \) and \( d_i|f(n+i+C_i) \). Therefore \( d_i|d'_i \). Similarly, we have \( d'_i|d_i \). Hence \( d_i = d'_i \), i.e., (16) is true. This completes the proof of Lemma 2.2.

3. The proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. By the definition (12), \( Z_{k,f} \) is a finite set and \( g_{k,f} \) is well defined on the set \( \mathbb{Z} \setminus Z_{k,f} \). First we prove that \( g_{k,f} \) is periodic on the set \( \mathbb{Z} \setminus Z_{k,f} \). For \( 1 \leq i \leq k \), by Lemma 2.2 \( d_i(n) = \gcd(|f(n)|, |f(n+i)|) \) is periodic. Let \( T_i \) be the least period of \( d_i \). Then 
\[
d_i(n) = d_i(n+T_i), \quad \text{for any } n \in \mathbb{Z}.
\]
Hence by the proof of Lemma 2.2 we have that \( T_i|C_i \), where \( C_i \) is defined by (14). Denote by \( T \) (resp. \( C \)) the least common multiple of the \( T_i \)'s (resp. \( C_i \)'s), \( i = 1, 2, \cdots, k \). Then \( T|C \) and for any \( 1 \leq i \leq k \), we have 
\[
d_i(n) = d_i(n+T) \quad \text{for } n \in \mathbb{Z}.
\]
Hence for any \( 0 \leq i < j \leq k \), we have 
\[
d_j-i(n+i) = d_j-i(n+i+T) \quad \text{for } n \in \mathbb{Z},
\]
that is,
\[
\gcd(|f(n+i)|, |f(n+j)|) = \gcd(|f(n+i+T)|, |f(n+j+T)|).
\]
So by Lemma 2.1 and the definition of \( g_{k,f} \), we obtain that \( g_{k,f}(n) = g_{k,f}(n+T) \) for any \( n \) and \( n+T \in \mathbb{Z} \setminus Z_{k,f} \). Hence \( g_{k,f}(n) \) is periodic and \( T \) is a period of \( g_{k,f} \).

If \( n \in Z_{k,f} \), then there exist a positive integer \( a \) such that \( n+aT \not\in Z_{k,f} \). Hence the function \( g_{k,f} \) can be extended to \( g_{k,f} : \mathbb{Z} \rightarrow \mathbb{Z} \), defined at \( n \in Z_{k,f} \), by 
\[
g_{k,f}(n) = g_{k,f}(n+aT).
\]
This completes the proof of Theorem 1.3.

Proof of Theorem 1.4. It is obvious that the property \( T|C \) implies that \( C \) is a multiple of the least period \( T_{k,f} \) of \( g_{k,f} \). This completes the proof of Theorem 1.4.
4. The proof of Theorem 1.5

We use the same notation as in previous sections.

Proof. (i) By the definitions of $h_{k,f,p}$ and $g_{k,f}$, it suffices to show that $v_p(g_{k,f}(n)) = v_p(g_{k,f}(n + p^e))$ for any $n \in \mathbb{Z} \setminus \mathbb{Z}_{k,f}$, i.e.,

$$
\sum_{i=0}^{k} v_p(f(n + i)) - \max_{0 \leq i \leq k} \{ v_p(f(n + i)) \}
= \sum_{i=0}^{k} v_p(f(n + i + p^e)) - \max_{0 \leq i \leq k} \{ v_p(f(n + i + p^e)) \}.
$$

Let

$$
e_{ij} = v_p(\gcd(|f(n+i)|, |f(n+j)|))
$$

and

$$
e'_{ij} = v_p(\gcd(|f(n+i+p^e)|, |f(n+j+p^e)|))
$$

for any $0 \leq i < j \leq k$. By the proof of Lemma 2.1, it suffices to show that

$$
(17) \quad e_{ij} = e'_{ij}.
$$

By the assumption of $f(x)$, we have

$$
a_{j-i}(m) f(m) + b_{j-i}(m) f(m + j - i) = C_{j-i}, \quad m \in \mathbb{Z}.
$$

Let $m = n + i$. We have $p^e | f(n+i)$ and $p^e | f(n+j)$, so $p^e | C_{j-i}$. Hence $e_{ij} \leq e_p$ by the definition of $e_p$. So $p^e | f(n+i+p^e)$, $p^e | f(n+j+p^e)$. Therefore $e_{ij} \leq e'_{ij}$. Similarly, we have $e'_{ij} \leq e_{ij}$. Hence (17) is true. It is easy to see that $T_{k,f,p} | p^e$.

(ii) By (i) of Theorem 1.5, we know that $h_{k,f,p}$ is periodic and $p^e$ is a period. So $T_{k,f,p} = 1$ if and only if $h_{k,f,p}(n) = h_{k,f,p}(n + 1)$ for any $1 \leq n \leq p^e$. By the definition of $g_{k,f}$, we have $T_{k,f,p} = 1$ if and only if for any $1 \leq n \leq p^e$,

$$
\sum_{i=0}^{k} v_p(f(n + i)) - \max_{0 \leq i \leq k} \{ v_p(f(n + i)) \}
= \sum_{i=1}^{k+1} v_p(f(n + i)) - \max_{1 \leq i \leq k+1} \{ v_p(f(n + i)) \}.
$$

Hence $T_{k,f,p} = 1$ if and only if for any $1 \leq n \leq p^e$,

$$
v_p(\gcd(|f(n)|, |f(n + k + 1)|)) \geq \max_{1 \leq i \leq k} \{ v_p(f(n + i)) \}
$$

or

$$
v_p(f(n)) = v_p(f(n + k + 1)) < \max_{1 \leq i \leq k} \{ v_p(f(n + i)) \}.
$$

(iii) Let $1 \leq e \leq e_p$. Suppose that $p^e$ is a period of $h_{k,f,p}$. Hence $p^e$ is the least period of $h_{k,f,p}$ if and only if $p^{e-1}$ is not a period of $h_{k,f,p}$. Therefore $p^e$ is the least period of $h_{k,f,p}$ if and only if there exists an integer $n_0 : 1 \leq n_0 \leq p^e$ such that the following inequality holds:

$$
h_{k,f,p}(n_0) \neq h_{k,f,p}(n_0 + p^{e-1}).$$
By definition (5), we have
\[
 h_{k,f,p}(n_0) = \sum_{i=0}^{k} v_p(f(n_0 + i)) - \max_{0 \leq i \leq k} \{ v_p(f(n_0 + i)) \}
\]
and only if
\[
 C_p = \sum_{i=1}^{\infty} \max\{0, \# \{ m \in S_{n_0} \mid p^i | f(m) \} - 1 \}
\]
This completes the proof of Theorem 1.5.

Remark that the infinite sums of (18) and (19) are meaningful, for almost all their
(19)
\[
 h_{k,f,p}(n_0 + p^{e-1}) = \sum_{i=1}^{\infty} \max\{0, \# \{ m \in S_{n_0} \mid p^i | f(m + p^{e-1}) \} - 1 \}.
\]
holds. In particular, \( T_{k,f,p} = p^{e_p} \) if and only if there exists an integer \( n_0 : 1 \leq n_0 \leq p^{e_p} \) such that the following inequality holds:
\[
 \# \{ m \in S_{n_0} \mid p^{e_p} | f(m) \} \neq \# \{ m \in S_{n_0} \mid p^{e_p} | f(m + p^{e_p-1}) \}.
\]
This completes the proof of Theorem 1.5.

5. The proof of Corollary 1.6

Proof. When \( k = 0 \), then \( g_{0,f} = 1 \). Let \( k \geq 1 \). For \( 1 \leq i \leq k \), we have \( C_i = i \) and \( C = \text{lcm}(1,2,\cdots,k) \). Hence we obtain
\[
 T_{k,f} = \prod_{p \text{ prime}, p \leq k} T_{k,f,p}.
\]
Letting \( p \leq k \) be a prime, it suffices to prove the following statements:
(I) \( T_{k,f,p} = 1 \) if \( v_p(k+1) \geq \max_{1 \leq i \leq k} \{ v_p(i) \} \).

(II) \( T_{k,f,p} = p^{e_p(C)} \) if \( v_p(k+1) < \max_{1 \leq i \leq k} \{ v_p(i) \} \).

We first prove (I). As \( e_p = v_p(C) = \max_{1 \leq i \leq k} \{ v_p(i) \} \), by assumption \( v_p(k+1) \geq e_p \),
we have \( v_p(k+1) = e_p \) or \( e_p + 1 \).

Case (a): \( 1 \leq n \leq p^{e_p} - 1 \); then \( e = v_p(n) < e_p \). Hence \( v_p(n) = v_p(n + k + 1) \) and \( n = p^{e} n_1, p \nmid n_1 \). Set \( i = p^{e} i_0, 1 \leq i_0 \leq p - 1 \) such that \( p | (n_1 + i_0) \). Then \( 1 \leq i \leq k \) and \( v_p(n+i) > v_p(n) \). Hence
\[
 v_p(n) = v_p(n+k+1) < \max_{1 \leq i \leq k} \{ v_p(n+i) \}.
\]

Case (b1): \( n = p^{e_p} \) and \( v_p(k+1) = e_p + 1 \). We have \( k+1 = p^{e_p+1} \). Let \( i = p^{e_p}(p-1) \). Then \( 1 \leq i \leq k \) and \( v_p(n+i) = e_p + 1 > e_p \). Hence
\[
 v_p(n) = v_p(n+k+1) < \max_{1 \leq i \leq k} \{ v_p(n+i) \}.
\]
Case (b2): \( n = p^r s \) and \( v_p(k+1) = e_p \). We have \( k+1 = p^r u \), where \( 2 \leq u \leq p-1 \). Hence \( k = u p^r - 1 \). If \( i > 0 \) and \( v_p(n+i) > e_p \), then \( i \geq p^r(p-1) > k \). Hence
\[
\max_{1 \leq i \leq k} \{ v_p(n+i) \} \leq e_p.
\]
Therefore
\[
v_p(gcd(|n|,|n+k+1|)) \geq \max_{1 \leq i \leq k} \{ v_p(n+i) \}.
\]
By (20), (21), (22) and using (ii) of Theorem 1.5, we have \( T_{k,f,p} = 1 \).

(ii) Note that \( e_p = v_p(C) \). Hence
\[
k = a_0 + a_1 p + \cdots + a_{e_p} p^r, \quad 0 \leq a_i \leq p-1, \quad i = 0, 1, \cdots, e_p, \quad a_{e_p} \neq 0.
\]
It is easy to show that the inequality \( v_p(k+1) > e_p \) holds if and only if \( a_0 = a_1 = \cdots = a_{e_p-1} = 1 \).

Assume that the inequality \( v_p(k+1) < e_p = v_p(C) = \max_{1 \leq i \leq k} v_p(i) \) holds. Then there exists an integer \( r : 0 \leq r \leq e_p - 1 \) such that the following conditions hold:
\[
0 \leq a_r \leq p-2 \quad \text{and} \quad a_{r+1} = \cdots = a_{e_p-1} = p-1.
\]
Set
\[
n_0 = \begin{cases} p^r, & \text{if } r = e_p - 1; \\ (p-1-a_r)p^r, & \text{if } 0 \leq r \leq e_p - 2. \end{cases}
\]
Then we have
\[
\# \{ m \in S_{n_0} \mid p^r \mid m \} = \begin{cases} a_{e_p} + 1, & \text{if } r = e_p - 1, \\ a_{e_p}, & \text{if } 0 \leq r \leq e_p - 2. \end{cases}
\]
and
\[
\# \{ m \in S_{n_0} \mid p^r \mid (m+p^{r-1}) \} = \begin{cases} a_{e_p}, & \text{if } r = e_p - 1, \\ a_{e_p} + 1, & \text{if } 0 \leq r \leq e_p - 2. \end{cases}
\]
By (iii) of Theorem 1.5 we know that \( p^r = p^r(C) \) is the least period of \( h_{k,f,p} \).
This completes the proof of Corollary 1.6. \(
\)

6. The Proof of Theorem 1.7

Proof. (i) Set \( S = a \cdot \text{lcm}(1,2,\cdots,k) \). Let \( n \) be any positive integer. For any \( 0 \leq i < j \leq k \), it is clear that \( \text{gcd}(n + ia, n + ja) = \text{gcd}(n + S + ia, n + S + ja) \).
Hence \( g_{k,a}(n + S) = g_{k,a}(n) \) follows from Lemma 2.1.

(ii) Suppose \( S \) is a period of \( g_{k,a} \). Then \( g_{k,a}(n) = g_{k,a}(n + S) \) for all \( n \in \mathbb{N}^* \). In particular, we have \( g_{k,a}(na) = g_{k,a}(na + S) \). Since
\[
g_{k,a}(na) = \frac{na \cdot (na + a) \cdots (na + ka)}{\text{lcm}(na, na + a, \cdots, na + ka)} = \frac{n(n+1) \cdots (n+k)}{\text{lcm}(n, n+1, \cdots, n+k)} \cdot a^k
\]
and
\[
g_{k,a}(na + S) = \frac{(na + S)(na + a + S) \cdots (na + ka + S)}{\text{lcm}(na + S, na + a + S, \cdots, na + ka + S)},
\]
we have
\[
g_{k,a}(n) \cdot a^k = g_{k,a}(na) = g_{k,a}(na + S)
\]
and
\[
g_{k,a}(n) \cdot a^k = \frac{(na + S)(na + a + S) \cdots (na + ka + S)}{\text{lcm}(na + S, na + a + S, \cdots, na + ka + S)}.
\]
We claim that \( a | S \). Let \( \gcd(a, S) = d, a = a_1 d, S = S_1 d \). Then \( \gcd(a_1, S_1) = 1 \).

By using (24), we have
\[
a_1^k | (na_1 + S_1)(na_1 + a_1 + S_1) \cdots (na_1 + ka_1 + S_1).
\]
Because \( \gcd(a_1, na_1 + ia_1 + S_1) = 1 \) for any \( 0 \leq i \leq k \), we have \( a_1 = 1 \). Hence \( a | S \).

Let \( S = a T \). Then using (28), we have
\[
g_k(n) \cdot a^k = g_{k,a}(n a) = g_{k,a}(n a + aT) = g_k(n + T) \cdot a^k.
\]
Hence \( g_k(n + T) = g_k(n) \) for all \( n \in \mathbb{N}^* \); i.e., \( T \) is a period of \( g_k(n) \).

Conversely, suppose \( T \) is a period of \( g_k(n) \). Let \( n \) be any positive integer. If \( d = \gcd(n, a), n = n_1 d, a = a_1 d \), then \( \gcd(n_1, a_1) = 1 \) and
\[
g_{k,a}(n) = g_{k,a_1}(n_1) \cdot d^k,
\]
\[
g_{k,a}(n + aT) = g_{k,a_1}(n_1 + a_1 T) \cdot d^k.
\]
Hence, without loss of generality, we assume that \( (n, a) = 1 \). Therefore
\[
(25) \quad \gcd(a, g_{k,a}(n)) = 1, \quad \gcd(a, g_{k,a}(n + aT)) = 1.
\]
Hence by using (24), we have
\[
g_{k,a}(n) = g_{k,a}(n + aT)
\]
if and only if
\[
(26) \quad v_p(g_{k,a}(n)) = v_p(g_{k,a}(n + aT)),
\]
for any prime \( p \nmid a \).

Let \( p \) be any prime such that \( p \nmid a \) and let \( N \) be a positive integer greater than \( v_p(k!) \). Then there exists a unique positive integer \( m \) such that \( 1 \leq m < p^N \) and
\[
(27) \quad ma \equiv 1(\text{mod } p^N).
\]
Let \( p, n, a, m \) be as above and \( 0 \leq i < j \leq k \). Then for any integer \( l \), there are
\[
(28) \quad v_p(\gcd(n + al + ai, (j - i)a)) = v_p(\gcd(mn + l + i, j - i)).
\]
Let
\[
(29) \quad \gcd(n + al + ai, (j - i)a) = p^{r_{ij}} w, \quad p \nmid w
\]
and
\[
(30) \quad \gcd(mn + l + i, j - i) = p^{y_{ij}} u, \quad p \nmid u.
\]
Then by (29), there exists \( s_1, t_1 \in \mathbb{Z} \) such that \( (n + al + ai)s_1 + (j - i)t_1 = p^{r_{ij}} w \).

Multiplying by \( n \) on both sides, we have \( (mn + \alpha + am)s_1 + (j - i)mat_1 = p^{r_{ij}} mw \).

Using (27), we have \( (mn + l + i)s_1 + (j - i)t_1 = p^{x_{ij}} mw - p^N \delta \). By (30), we have \( y_{ij} \leq x_{ij} \). Conversely, by (30), there exist \( s_2, t_2 \in \mathbb{Z} \) such that \( (mn + l + i)s_2 + (j - i)t_2 = p^{y_{ij}} u \).

Using (27), we have \( y_{ij} \leq x_{ij} \). So \( x_{ij} = y_{ij} \) and (28) is true. Let \( l = 0 \) and \( T \). By using (29), we have
\[
v_p(\gcd(n + ai, n + aj)) = v_p(\gcd(mn + i, mn + j))
\]
and
\[
v_p(\gcd(n + aT + ai, n + aT + aj)) = v_p(\gcd(mn + T + i, mn + T + j))
\]
for any \( 0 \leq i < j \leq k \). By the proof of Lemma 2.1, we have
\[
v_p(g_{k,a}(n)) = v_p(g_k(mn)), \quad v_p(g_{k,a}(n + aT)) = v_p(g_k(mn + T)).
\]
Table 1

The least period $T_{k,f}$ of $g_{k,f}$ with $f(x) = x^2 + b$

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^2 + 1$</td>
<td>5</td>
<td>2 · 5</td>
<td>2 · 3 · 5 · 13</td>
<td>2 · 3 · 5 · 13</td>
<td>2 · 3 · 5 · 13 · 29</td>
<td>2 · 3 · 5 · 13 · 29</td>
</tr>
<tr>
<td>$x^2 + 2$</td>
<td>3 · 2</td>
<td>2 · 3 · 17</td>
<td>2 · 3 · 17</td>
<td>2 · 3 · 5 · 11 · 17</td>
<td>2 · 3 · 5 · 11 · 17</td>
<td>2 · 3 · 5 · 11 · 17</td>
</tr>
<tr>
<td>$x^2 + 3$</td>
<td>13</td>
<td>2 · 3 · 7 · 13</td>
<td>2 · 3 · 7 · 13</td>
<td>2 · 3 · 5 · 7 · 13 · 37</td>
<td>2 · 3 · 5 · 7 · 13 · 37</td>
<td>2 · 3 · 5 · 7 · 13 · 37</td>
</tr>
<tr>
<td>$x^2 + 4$</td>
<td>17</td>
<td>2 · 3 · 5 · 17</td>
<td>2 · 3 · 5 · 17</td>
<td>2 · 3 · 5 · 17 · 41</td>
<td>2 · 3 · 5 · 17 · 41</td>
<td>2 · 3 · 5 · 17 · 41</td>
</tr>
<tr>
<td>$x^2 + 5$</td>
<td>3 · 7</td>
<td>2 · 3 · 7 · 29</td>
<td>2 · 3 · 7 · 29</td>
<td>2 · 3 · 5 · 7 · 29</td>
<td>2 · 3 · 5 · 7 · 29</td>
<td>2 · 3 · 5 · 7 · 29</td>
</tr>
<tr>
<td>$x^2 + 6$</td>
<td>5 · 2</td>
<td>2 · 3 · 5 · 17</td>
<td>2 · 3 · 5 · 17</td>
<td>2 · 3 · 7 · 29</td>
<td>2 · 3 · 7 · 29</td>
<td>2 · 3 · 7 · 29</td>
</tr>
</tbody>
</table>

So using $g_k(mn) = g_k(mn + T)$, we have $v_p(g_{k,a}(n + aT)) = v_p(g_{k,a}(n))$ for any prime $p$ such that $p 
mid a$. Hence $g_{k,a}(n + aT) = g_{k,a}(n)$ and $aT$ is a period of $g_{k,a}(n)$.

The proof of (iii) follows from (ii). This completes the proof of Theorem 1.7.


Proof of Corollary 1.8 (i) Assume that $p 
mid a$. Then it is clear that the equality $v_p(g_{k,a+b}(n)) = 0$ holds for any integer $n$ when $g_{k,a+b}$ is well defined.

(ii) Assume that $p$ is not a prime factor of $a$. By the formula (5), we have that $T_p$ is a period of $v_p(g_{k,a+b}(n))$ if and only if $aT_p$ is a period of $v_p(g_{k,a+b}(n))$. Hence, by Theorem 1.7 we have that $T_p$ is a period of $v_p(g_{k,a+b}(n))$ if and only if $T_p$ is a period of $v_p(g_{k,a+b}(n))$. Therefore Corollary 1.8 is obtained by Theorem 1.2.


7. Examples

Lemma 7.1. Let $f_1(x) = f_2(x)^r$, where $r \geq 1$ is an integer. Then $T_{k,f_1} = T_{k,f_2}$.

Proof. By (4), we have $g_{k,f_1}(n) = g_{k,f_2}(n)^r$. Hence the result is obvious.

Example. Let $f(x) = x^r$, $r \geq 1$. Then by Lemma 7.1 we have $T_{k,x^r} = T_{k,x}$, where $T_{k,x}$ is given by the formula (9).

Example. Let $f(x) = x^2 + b$. For $1 \leq i \leq k$, we have

$(2x + 3i)(x^2 + b) + (-2x + i)(x + i)^2 + b = i(i^2 + 4b)$, if $i$ is odd,

$(x + 3j)(x^2 + b) + (-x + j)(x + 2j)^2 + b = 4j(j^2 + b)$, if $i = 2j$.

Hence

$C_i = \begin{cases} 
  i(i^2 + 4b), & \text{if } i \text{ is odd}, \\
  4j(j^2 + b), & \text{if } i = 2j.
\end{cases}$

Hence, given any $k \in \mathbb{N}$ and $b \in \mathbb{Z}$, by Theorem 1.5 we can determine the least period $T_{k,f}$ of the arithmetic function $g_{k,f}$. For $1 \leq k \leq 6$ and $1 \leq b \leq 6$, Table 1 gives the $T_{k,f}$'s.

Example. Let $f(x) = x^3 + b$. For $1 \leq i \leq k$, we have

$a_i(x)(x^3 + b) + b_i(x)((x + i)^3 + b) = C_i$,

where

$a_i(x) = \begin{cases} 
  6i^2x^2 + (15i^3 - 9)x + 10i^4 - 18i, & \text{if } 3 \nmid i; \\
  6j^2x^2 + (45j^3 - 1)x + 90i^4 - 6i, & \text{if } i = 3j.
\end{cases}$
The periodicity of some Farhi arithmetical functions

Table 2

The least period $T_{k,f}$ of $g_{k,f}$ with $f(x) = x^3 + b$

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^3 + 1$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·5·7·13·17·31·43</td>
<td>2·3·5·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3 + 2$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·5·7·13·17·31·43</td>
<td>2·3·5·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3 + 3$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·5·7·13·17·31·43</td>
<td>2·3·5·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3 + 4$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·5·7·13·17·31·43</td>
<td>2·3·5·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3 + 5$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·5·7·13·17·31·43</td>
<td>2·3·5·7·13·17·19·31·43</td>
</tr>
<tr>
<td>$x^3 + 6$</td>
<td>2·7</td>
<td>2·7·13</td>
<td>2·3·7·13</td>
<td>2·3·7·11·13·17·31</td>
<td>2·3·5·7·13·17·31·43</td>
<td>2·3·5·7·13·17·19·31·43</td>
</tr>
</tbody>
</table>

$b_i(x) = \begin{cases} 
-6i^2x^2 + (3i^3 + 9)x - i^4 - 9i, & \text{if } 3 \mid i; \\
-6j^2x^2 + (9j^3 + 1)x - 9j^4 - 3j, & \text{if } i = 3j. 
\end{cases}$

$C_i = \begin{cases} 
-i^2 - 27i, & \text{if } 3 \mid i; \\
-35j^2 - 9j, & \text{if } i = 3j. 
\end{cases}$

Hence, given any $k \in \mathbb{N}$ and $b \in \mathbb{Z}$, by Theorem 1.5, we can determine the least period $T_{k,f}$ of the arithmetic function $g_{k,f}$. For $1 \leq k \leq 6$ and $1 \leq b \leq 6$, Table 2 gives the $T_{k,f}$'s.

References


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