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ON THE PERIODICITY OF SOME FARHI ARITHMETICAL FUNCTIONS

QING-ZHONG JI AND CHUN-GANG JI

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Abstract. Let $k \in \mathbb{N}$. Let $f(x) \in \mathbb{Z}[x]$ be any polynomial such that $f(x)$ and $f(x+1)f(x+2)\cdots f(x+k)$ are coprime in $\mathbb{Q}[x]$. We call

$$g_{k,f}(n) := \frac{|f(n)f(n+1)\cdots f(n+k)|}{\operatorname{lcm}(f(n), f(n+1), \cdots, f(n+k))}$$

a Farhi arithmetic function. In this paper, we prove that $g_{k,f}$ is periodic. This generalizes the previous results of Farhi and Kane, and Hong and Yang.

1. Introduction

Throughout this paper, let $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the field of rational numbers, the ring of rational integers and the set of nonnegative integers. Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. As usual, let v_p denote the normalized p-adic valuation of $\mathbb{Q}$, i.e., $v_p(a) = b$ if $p^b || a$.

It is known that an equivalent variation of the Prime Number Theorem states that $\log \operatorname{lcm}(1, 2, \cdots, n) \sim n$ as n tends to infinity (see, e.g., [5]). One thus expects that a better understanding of the function $\operatorname{lcm}(1, 2, \cdots, n)$ may entail a deeper understanding of the distribution of the prime numbers. Some progress has been made towards this direction. Before we state our main theorems, let us first give a short account on the recent results in this subject.

In his pioneering paper [2], Farhi introduced the following arithmetic functions:

$$g_k(n) := \frac{n(n+1)\cdots(n+k)}{\operatorname{lcm}(n, n+1, \cdots, n+k)}, \quad n \in \mathbb{N}^*.$$

Farhi proved that the sequence $(g_k)_{k\in\mathbb{N}}$ satisfies the recursion relation:

$$g_k(n) = \gcd(k!, (n+k)g_{k-1}(n)), \quad \forall n \in \mathbb{N}^*. \tag{1}$$

Using this relation, Farhi proved

Theorem 1.1 ([2]). *The function g_k ($k \in \mathbb{N}$) is periodic and $k!$ is a period of g_k.*

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An interesting question is how to determine the least period of g_k (see [2]). In [6], by using (1) and $g_k(1)|g_k(n)$ for any positive integer n, Hong and Yang gave a partial answer to this question. A complete solution to the question was given by Farhi and Kane in [3]. They proved

Theorem 1.2 ([3], Theorem 3.2)**.** *The least period T_k of g_k is given by*

$$T_k = \prod_{p\ prime,\ p\leq k} p^{\delta_p(k)}, \tag{2}$$

where

$$\delta_p(k) = \begin{cases} 0, & \text{if } v_p(k+1) \geq \max\limits_{1\leq i\leq k}\{v_p(i)\}, \\ \max\limits_{1\leq i\leq k}\{v_p(i)\}, & \text{otherwise.} \end{cases}$$

Let $g(n)$ be an arithmetic function defined on the set $\mathbb{Z}\backslash A$, where A is a finite subset of $\mathbb{Z}$. If there exists an integer T such that $g(n) = g(n+T)$ for all $n,\ n+T \in \mathbb{Z}\backslash A$, then it is clear that the arithmetic function $g(n)$ can be extended to a periodic function defined on all the integers $\mathbb{Z}$.

Throughout this paper, let k be a nonnegative integer and $f(x) \in \mathbb{Z}[x]$ such that

$$\gcd(f(x), f(x+1)f(x+2)\cdots f(x+k)) = 1$$

in $\mathbb{Q}[x]$. Set

$$Z_{k,f} := \{n \in \mathbb{Z} \mid f(n+i) = 0 \text{ for some } 0 \leq i \leq k\}. \tag{3}$$

Then $Z_{k,f}$ is a finite subset of $\mathbb{Z}$. Set

$$g_{k,f}(n) = \frac{|f(n)f(n+1)\cdots f(n+k)|}{\operatorname{lcm}(f(n), f(n+1), \cdots, f(n+k))}, \tag{4}$$

for $n \in \mathbb{Z}\backslash Z_{k,f}$. We call $g_{k,f}(n)$ a Farhi arithmetic function. In §3, we will prove

Theorem 1.3. *Let k be a nonnegative integer and $f(x) \in \mathbb{Z}[x]$ such that $\gcd(f(x), f(x+1)f(x+2)\cdots f(x+k)) = 1$ in $\mathbb{Q}[x]$. Then the arithmetic function $g_{k,f}$ can be extended to a periodic arithmetic function defined on all the integers.*

By assumption of $f(x)$ in Theorem 1.3, for any $1 \leq i \leq k$, there exist polynomials $a_i(x), b_i(x) \in \mathbb{Z}[x]$ and the smallest positive integer C_i such that

$$a_i(x)f(x) + b_i(x)f(x+i) = C_i.$$

Let C be the least common multiple of the C_i's, i.e.,

$$C = \operatorname{lcm}(C_1, C_2, \cdots, C_k).$$

In the proof of Theorem 1.3, we will prove

Theorem 1.4. *Let $T_{k,f}$ denote the least period of $g_{k,f}$. Then $T_{k,f}|C$.*

Let p be a prime. Define the arithmetic function $h_{k,f,p}$ by

$$h_{k,f,p}(n) := v_p(g_{k,f}(n)). \tag{5}$$

If $p \nmid C$, using the definition of $g_{k,f}$, then we have $h_{k,f,p}(n) = 0$ for any $n \in \mathbb{Z}$. If $p|C$, then $h_{k,f,p}$ is a periodic function by Theorem 1.3. Set

$$S_n := \{n, n+1, \cdots, n+k\},\ n \in \mathbb{Z} \tag{6}$$

and

$$e_p := \max\{v_p(\gcd(|f(n)|, |f(n+i)|)) \mid 1 \leq n \leq p^{v_p(C)}, 1 \leq i \leq k\}. \tag{7}$$

In §4, we will prove

Theorem 1.5. *For any prime p, let $T_{k,f,p}$ be the least period of the arithmetic function $h_{k,f,p}$. Then*

(i) *p^{e_p} is a period of $h_{k,f,p}$ and $T_{k,f,p} | p^{e_p}$.*

(ii) *$T_{k,f,p} = 1$ if and only if for any $1 \le n \le p^{e_p}$, we have*

$$v_p(\gcd(|f(n)|, |f(n+k+1)|)) \ge \max_{1\le i\le k}\{v_p(f(n+i))\}$$

or

$$v_p(f(n)) = v_p(f(n+k+1)) < \max_{1\le i\le k}\{v_p(f(n+i))\}.$$

(iii) *Let $1 \le e \le e_p$. Suppose that p^e is a period of $h_{k,f,p}$. Then $T_{k,f,p} = p^e$ if and only if there exists an integer $n_0 :\ 1 \le n_0 \le p^e$ such that the following inequality holds*:

$$\sum_{t=e}^{e_p} \max\{0, \#\{m \in S_{n_0} \mid p^t | f(m)\} - 1\}$$
$$\ne \sum_{t=e}^{e_p} \max\{0, \#\{m \in S_{n_0} \mid p^t | f(m + p^{e-1})\} - 1\}.$$

In particular, $T_{k,f,p} = p^{e_p}$ if and only if there exists an integer $n_0 :\ 1 \le n_0 \le p^{e_p}$ such that the following inequality holds:

$$\#\{m \in S_{n_0} \mid p^{e_p} | f(m)\} \ne \#\{m \in S_{n_0} \mid p^{e_p} | f(m + p^{e_p - 1})\}.$$

Remark. Let $T_{k,f,p}$ be the least period of $h_{k,f,p}$ for any prime p. Then

$$T_{k,f} = \prod_p T_{k,f,p}.$$

(This infinite product is meaningful, for almost all its terms are equal to 1.)

As an application of Theorem 1.5, in §5 we will give a new different proof of Theorem 3.2 of [3].

Corollary 1.6. *Let $k \in \mathbb{N}$ and $f(x) = x$. Then the least period $T_{k,f}$ of the* Farhi *arithmetic function $g_{k,f}$ is given by the formula* (9).

Let $a, b \in \mathbb{Z}$ be any integer such that $\gcd(a,b) = 1$ and $a > 0$. Let $f(x) = ax + b$. By Theorem 1.3, we know that the Farhi arithmetic function

$$g_{k,ax+b}(n) = \frac{|(an+b)(a(n+1)+b)\cdots(a(n+k)+b)|}{\operatorname{lcm}(an+b, a(n+1)+b, \cdots, a(n+k)+b)}$$

can be extended to a periodic arithmetic function defined on all the integers. Now we define the arithmetical function $g_{k,a}$ by

$$g_{k,a}(n) = \frac{|n(n+a)\cdots(n+ka)|}{\operatorname{lcm}(n, n+a, \cdots, n+ka)}.$$

When $a = 1$, the arithmetical function $g_{k,1}$ is the arithmetical function g_k defined by Farhi. It is clear that

$$g_{k,ax+b}(n) = g_{k,a}(na+b). \tag{8}$$

Hence the function $g_{k,a}$ can also be extended to a periodic arithmetic function defined on all the integers. In §6, we shall prove the following results:

Theorem 1.7. *Let a, k be any two positive integers. Then the following assertions hold:*

(i) *The positive integer $a \cdot lcm(1, 2, \cdots, k)$ is a period of $g_{k,a}$.*

(ii) *A positive integer S is a period of $g_{k,a}$ if and only if $S = aT$, where T is a period of g_k.*

(iii) *Consequently, the least period $T_k(a)$ of $g_{k,a}$ is $aT_k(1) = aT_k$, where $T_k(1) = T_k$ is the least period of g_k.*

By (8) and Theorem 1.7, we have the following result:

Corollary 1.8. *Let a, k be any two positive integers and let $b \in \mathbb{Z}$ be any integer such that $\gcd(a, b) = 1$. Then the least period $T_{k,ax+b}$ of the Farhi arithmetic function $g_{k,ax+b}$ is given by the following formula:*

$$T_{k,ax+b} = \prod_{p\ prime,\ p \leq k} p^{\delta_p(k)}, \tag{9}$$

where

$$\delta_p(k) = \begin{cases} 0, & \text{if } v_p(k+1) \geq \max\limits_{1 \leq i \leq k} \{v_p(i)\} \text{ or } p|a, \\ \max\limits_{1 \leq i \leq k} \{v_p(i)\}, & \text{otherwise.} \end{cases}$$

In §7, we will give some examples.

2. Two basic lemmas

Lemma 2.1. *Let $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ be any $2n$ positive integers. If $\gcd(a_i, a_j) = \gcd(b_i, b_j)$ for any $1 \leq i < j \leq n$, then*

$$\frac{a_1 a_2 \cdots a_n}{\text{lcm}(a_1, a_2, \cdots, a_n)} = \frac{b_1 b_2 \cdots b_n}{\text{lcm}(b_1, b_2, \cdots, b_n)}. \tag{10}$$

Proof. Let p be any prime. It suffices to show that the following equality holds:

$$v_p\left(\frac{a_1 a_2 \cdots a_n}{\text{lcm}(a_1, a_2, \cdots, a_n)}\right) = v_p\left(\frac{b_1 b_2 \cdots b_n}{\text{lcm}(b_1, b_2, \cdots, b_n)}\right), \tag{11}$$

i.e.,

$$\sum_{i=1}^{n} v_p(a_i) - \max_{1 \leq i \leq n} \{v_p(a_i)\} = \sum_{i=1}^{n} v_p(b_i) - \max_{1 \leq i \leq n} \{v_p(b_i)\}. \tag{12}$$

By symmetry, it suffices to show that

$$\sum_{i=1}^{n} v_p(a_i) - \max_{1 \leq i \leq n} \{v_p(a_i)\} \leq \sum_{i=1}^{n} v_p(b_i) - \max_{1 \leq i \leq n} \{v_p(b_i)\}. \tag{13}$$

Without loss of generality, we assume that $v_p(a_1) \leq v_p(a_2) \leq \cdots \leq v_p(a_{n-1}) \leq v_p(a_n)$. Then for any $1 \leq i \leq n-1$, we have

$$v_p(a_i) = v_p(\gcd(a_i, a_n)) = v_p(\gcd(b_i, b_n)) \leq \min\{v_p(b_i), v_p(b_n)\}.$$

Hence for any $1 \leq i \leq n-1$, we have $v_p(a_i) \leq v_p(b_i)$ and $v_p(a_i) \leq v_p(b_n)$. Let $v_p(b_k) = \max_{1 \leq i \leq n}\{v_p(b_i)\}$. Then

$$\sum_{i=1}^{n-1} v_p(a_i) \leq v_p(b_1) + \cdots + v_p(b_{k-1}) + v_p(b_n) + v_p(b_{k+1}) + \cdots + v_p(b_{n-1}).$$

So (13) is true. This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let k be a positive integer and $f(x) \in \mathbb{Z}[x]$ such that*

$$\gcd(f(x), f(x+1)f(x+2)\cdots f(x+k)) = 1$$

in $\mathbb{Q}[x]$. Then $d_i(n) = \gcd(|f(n)|, |f(n+i)|)$ is periodic for any $1 \le i \le k$.

Proof. By assumption, for any $1 \le i \le k$, $f(x)$ and $f(x+i)$ are coprime in $\mathbb{Q}[x]$; hence there exist $a_i(x), b_i(x) \in \mathbb{Z}[x]$ and the smallest positive integer C_i such that

$$a_i(x)f(x) + b_i(x)f(x+i) = C_i. \tag{14}$$

Hence for all $m \in \mathbb{Z}$, we have

$$a_i(m)f(m) + b_i(m)f(m+i) = C_i. \tag{15}$$

In the following, we will prove

$$d_i(n) = d_i(n+C_i), \quad n \in \mathbb{Z}. \tag{16}$$

Let $d_i = d_i(n)$ and $d_i' = d_i(n+C_i)$. Then $d_i|f(n)$ and $d_i|f(n+i)$. By (15), we have $d_i|C_i$. Hence, by the Taylor formula of f, we obtain that $d_i|f(n+C_i)$ and $d_i|f(n+i+C_i)$. Therefore $d_i|d_i'$. Similarly, we have $d_i'|d_i$. Hence $d_i = d_i'$, i.e., (16) is true. This completes the proof of Lemma 2.2. □

3. The proofs of Theorem 1.3 and Theorem 1.4

Proof of Theorem 1.3. By the definition (3), $Z_{k,f}$ is a finite set and $g_{k,f}$ is well defined on the set $\mathbb{Z} \setminus Z_{k,f}$. First we prove that $g_{k,f}$ is periodic on the set $\mathbb{Z} \setminus Z_{k,f}$. For $1 \le i \le k$, by Lemma 2.2, $d_i(n) = \gcd(|f(n)|, |f(n+i)|)$ is periodic. Let T_i be the least period of d_i. Then

$$d_i(n) = d_i(n+T_i), \quad \text{for any } n \in \mathbb{Z}.$$

Hence by the proof of Lemma 2.2, we have that $T_i|C_i$, where C_i is defined by (14). Denote by T (resp. C) the least common multiple of the T_i's (resp. C_i's), $i = 1, 2, \cdots, k$. Then $T|C$ and for any $1 \le i \le k$, we have

$$d_i(n) = d_i(n+T) \quad \text{for } n \in \mathbb{Z}.$$

Hence for any $0 \le i < j \le k$, we have

$$d_{j-i}(n+i) = d_{j-i}(n+i+T) \quad \text{for } n \in \mathbb{Z},$$

that is,

$$\gcd(|f(n+i)|, |f(n+j)|) = \gcd(|f(n+i+T)|, |f(n+j+T)|).$$

So by Lemma 2.1 and the definition of $g_{k,f}$, we obtain that $g_{k,f}(n) = g_{k,f}(n+T)$ for any n and $n+T \in \mathbb{Z}\backslash Z_{k,f}$. Hence $g_{k,f}(n)$ is periodic and T is a period of $g_{k,f}$.

If $n \in Z_{k,f}$, then there exist a positive integer a such that $n + aT \notin Z_{k,f}$. Hence the function $g_{k,f}$ can be extended to $g_{k,f} : \mathbb{Z} \longrightarrow \mathbb{Z}$, defined at $n \in Z_{k,f}$, by

$$g_{k,f}(n) = g_{k,f}(n+aT).$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. It is obvious that the property $T|C$ implies that C is a multiple of the least period $T_{k,f}$ of $g_{k,f}$. This completes the proof of Theorem 1.4. □

4. The proof of Theorem 1.5

We use the same notation as in previous sections.

Proof. (i) By the definitions of $h_{k,f,p}$ and $g_{k,f}$, it suffices to show that $v_p(g_{k,f}(n)) = v_p(g_{k,f}(n+p^{e_p}))$ for any $n \in \mathbb{Z} \setminus Z_{k,f}$, i.e.,

$$\sum_{i=0}^{k} v_p(f(n+i)) - \max_{0\le i\le k}\{v_p(f(n+i))\}$$
$$= \sum_{i=0}^{k} v_p(f(n+i+p^{e_p})) - \max_{0\le i\le k}\{v_p(f(n+i+p^{e_p}))\}.$$

Let

$$e_{ij} = v_p(\gcd(|f(n+i)|, |f(n+j)|))$$

and

$$e'_{ij} = v_p(\gcd(|f(n+i+p^{e_p})|, |f(n+j+p^{e_p})|))$$

for any $0 \le i < j \le k$. By the proof of Lemma 2.1, it suffices to show that

$$e_{ij} = e'_{ij}. \tag{17}$$

By the assumption of $f(x)$, we have

$$a_{j-i}(m)f(m) + b_{j-i}(m)f(m+j-i) = C_{j-i}, \quad m \in \mathbb{Z}.$$

Let $m = n+i$. We have $p^{e_{ij}} | f(n+i)$ and $p^{e_{ij}} | f(n+j)$, so $p^{e_{ij}} | C_{j-i}$. Hence $e_{ij} \le e_p$ by the definition of e_p. So $p^{e_{ij}} | f(n+i+p^{e_p})$, $p^{e_{ij}} | f(n+j+p^{e_p})$. Therefore $e_{ij} \le e'_{ij}$. Similarly, we have $e'_{ij} \le e_{ij}$. Hence (17) is true. It is easy to see that $T_{k,f,p} | p^{e_p}$.

(ii) By (i) of Theorem 1.5, we know that $h_{k,f,p}$ is periodic and p^{e_p} is a period. So $T_{k,f,p} = 1$ if and only if $h_{k,f,p}(n) = h_{k,f,p}(n+1)$ for any $1 \le n \le p^{e_p}$. By the definition of $g_{k,f}$, we have $T_{k,f,p} = 1$ if and only if for any $1 \le n \le p^{e_p}$,

$$\sum_{i=0}^{k} v_p(f(n+i)) - \max_{0\le i\le k}\{v_p(f(n+i))\}$$
$$= \sum_{i=1}^{k+1} v_p(f(n+i)) - \max_{1\le i\le k+1}\{v_p(f(n+i))\}.$$

Hence $T_{k,f,p} = 1$ if and only if for any $1 \le n \le p^{e_p}$,

$$v_p(\gcd(|f(n)|, |f(n+k+1)|)) \ge \max_{1\le i\le k}\{v_p(f(n+i))\}$$

or

$$v_p(f(n)) = v_p(f(n+k+1)) < \max_{1\le i\le k}\{v_p(f(n+i))\}.$$

(iii) Let $1 \le e \le e_p$. Suppose that p^e is a period of $h_{k,f,p}$. Hence p^e is the least period of $h_{k,f,p}$ if and only if p^{e-1} is not a period of $h_{k,f,p}$. Therefore p^e is the least period of $h_{k,f,p}$ if and only if there exists an integer $n_0 :\ 1 \le n_0 \le p^e$ such that the following inequality holds:

$$h_{k,f,p}(n_0) \ne h_{k,f,p}(n_0 + p^{e-1}).$$

By definition (5), we have

$$h_{k,f,p}(n_0) = \sum_{i=0}^{k} v_p(f(n_0+i)) - \max_{0\le i\le k}\{v_p(f(n_0+i))\} = \sum_{t=1}^{\infty} \max\{0,\ \#\{m\in S_{n_0} \mid p^t|f(m)\}-1\} \tag{18}$$

and

$$h_{k,f,p}(n_0+p^{e-1}) = \sum_{t=1}^{\infty} \max\{0,\ \#\{m\in S_{n_0} \mid p^t|f(m+p^{e-1})\}-1\}. \tag{19}$$

Remark that the infinite sums of (18) and (19) are meaningful, for almost all their terms are equal to 0. On the other hand, when $t\le e-1$, we know that $p^t|f(m)$ if and only if $p^t|f(m+p^{e-1})$. Hence by the definition of e_p, the inequality $h_{k,f,p}(n_0)\ne h_{k,f,p}(n_0+p^{e-1})$ holds if and only if the inequality

$$\sum_{t=e}^{e_p} \max\{0, \#\{m\in S_{n_0} \mid p^t|f(m)\}-1\} \ne \sum_{t=e}^{e_p} \max\{0, \#\{m\in S_{n_0} \mid p^t|f(m+p^{e-1})\}-1\}$$

holds. In particular, $T_{k,f,p}=p^{e_p}$ if and only if there exists an integer $n_0:\ 1\le n_0\le p^{e_p}$ such that the following inequality holds:

$$\#\{m\in S_{n_0} \mid p^{e_p}|f(m)\} \ne \#\{m\in S_{n_0} \mid p^{e_p}|f(m+p^{e_p-1})\}.$$

This completes the proof of Theorem 1.5. □

5. The proof of Corollary 1.6

Proof. When $k=0$, then $g_{0,f}=1$. Let $k\ge 1$. For $1\le i\le k$, we have $C_i=i$ and $C=\operatorname{lcm}(1,2,\cdots,k)$. Hence we obtain

$$T_{k,f} = \prod_{p\ prime, p\le k} T_{k,f,p}.$$

Letting $p\le k$ be a prime, it suffices to prove the following statements:

(I) $T_{k,f,p}=1$ if $v_p(k+1)\ge \max_{1\le i\le k}\{v_p(i)\}$.

(II) $T_{k,f,p}=p^{v_p(C)}$ if $v_p(k+1)< \max_{1\le i\le k}\{v_p(i)\}$.

We first prove (I). As $e_p=v_p(C)=\max_{1\le i\le k}\{v_p(i)\}$, by assumption $v_p(k+1)\ge e_p$, we have $v_p(k+1)=e_p$ or e_p+1.

Case (a): $1\le n\le p^{e_p}-1$; then $e=v_p(n)<e_p$. Hence $v_p(n)=v_p(n+k+1)$ and $n=p^e n_1$, $p\nmid n_1$. Set $i=p^e i_0$, $1\le i_0\le p-1$ such that $p|(n_1+i_0)$. Then $1\le i\le k$ and $v_p(n+i)>v_p(n)$. Hence

$$v_p(n)=v_p(n+k+1)<\max_{1\le i\le k}\{v_p(n+i)\}. \tag{20}$$

Case (b1): $n=p^{e_p}$ and $v_p(k+1)=e_p+1$. We have $k+1=p^{e_p+1}$. Let $i=p^{e_p}(p-1)$. Then $1\le i\le k$ and $v_p(n+i)=e_p+1>e_p$. Hence

$$v_p(n)=v_p(n+k+1)<\max_{1\le i\le k}\{v_p(n+i)\}. \tag{21}$$

Case (b2): $n = p^{e_p}$ and $v_p(k+1) = e_p$. We have $k+1 = p^{e_p}u$, where $2 \leq u \leq p-1$. Hence $k = up^{e_p} - 1$. If $i > 0$ and $v_p(n+i) > e_p$, then $i \geq p^{e_p}(p-1) > k$. Hence $\max_{1\leq i\leq k}\{v_p(n+i)\} \leq e_p$. Therefore

$$v_p(\gcd(|n|, |n+k+1|)) \geq \max_{1\leq i\leq k}\{v_p(n+i)\}. \tag{22}$$

By (20), (21), (22) and using (ii) of Theorem 1.5, we have $T_{k,f,p} = 1$.

(II) Note that $e_p = v_p(C)$. Hence

$$k = a_0 + a_1 p + \cdots + a_{e_p} p^{e_p}, \ \ 0 \leq a_i \leq p-1, \ \ i = 0, 1, \cdots, e_p, \ a_{e_p} \neq 0.$$

It is easy to show that the inequality $v_p(k+1) \geq e_p$ holds if and only if $a_0 = a_1 = \cdots = a_{e_p - 1} = p - 1$.

Assume that the inequality $v_p(k+1) < e_p = v_p(C) = \max_{1\leq i\leq k} v_p(i)$ holds. Then there exists an integer $r:\ 0 \leq r \leq e_p - 1$ such that the following conditions hold:

$$0 \leq a_r \leq p-2 \text{ and } a_{r+1} = \cdots = a_{e_p-1} = p-1.$$

Set

$$n_0 = \begin{cases} p^{e_p}, & \text{if } r = e_p - 1; \\ (p-1-a_r)p^r, & \text{if } 0 \leq r \leq e_p - 2. \end{cases}$$

Then we have

$$\#\{m \in S_{n_0} \ | \ p^{e_p} | m\} = \begin{cases} a_{e_p} + 1, & \text{if } r = e_p - 1, \\ a_{e_p}, & \text{if } 0 \leq r \leq e_p - 2; \end{cases}$$

and

$$\#\{m \in S_{n_0} \ | \ p^{e_p} | (m + p^{e_p - 1})\} = \begin{cases} a_{e_p}, & \text{if } r = e_p - 1, \\ a_{e_p} + 1, & \text{if } 0 \leq r \leq e_p - 2. \end{cases}$$

By (iii) of Theorem 1.5, we know that $p^{e_p} = p^{v_p(C)}$ is the least period of $h_{k,f,p}$. This completes the proof of Corollary 1.6. □

6. The proof of Theorem 1.7

Proof. (i) Set $S = a \cdot \text{lcm}(1, 2, \cdots, k)$. Let n be any positive integer. For any $0 \leq i < j \leq k$, it is clear that $\gcd(n + ia, n + ja) = \gcd(n + S + ia, n + S + ja)$. Hence $g_{k,a}(n + S) = g_{k,a}(n)$ follows from Lemma 2.1.

(ii) Suppose S is a period of $g_{k,a}$. Then $g_{k,a}(n) = g_{k,a}(n + S)$ for all $n \in \mathbb{N}^*$. In particular, we have $g_{k,a}(na) = g_{k,a}(na + S)$. Since

$$g_{k,a}(na) = \frac{na \cdot (na + a) \cdot \cdots \cdot (na + ka)}{\text{lcm}(na, na + a, \cdots, na + ka)} = \frac{n(n+1)\cdots(n+k)}{\text{lcm}(n, n+1, \cdots, n+k)} \cdot a^k$$

and

$$g_{k,a}(na + S) = \frac{(na + S)(na + a + S)\cdots(na + ka + S)}{\text{lcm}(na + S, na + a + S, \cdots, na + ka + S)},$$

we have

$$g_k(n) \cdot a^k = g_{k,a}(na) = g_{k,a}(na + S) \tag{23}$$

and

$$g_k(n) \cdot a^k = \frac{(na + S)(na + a + S)\cdots(na + ka + S)}{\text{lcm}(na + S, na + a + S, \cdots, na + ka + S)}. \tag{24}$$

We claim that $a|S$. Let $\gcd(a,S)=d$, $a=a_1d$, $S=S_1d$. Then $\gcd(a_1,S_1)=1$. By using (24), we have

$$a_1^k \mid (na_1+S_1)(na_1+a_1+S_1)\cdots(na_1+ka_1+S_1).$$

Because $\gcd(a_1, na_1+ia_1+S_1)=1$ for any $0\le i\le k$, we have $a_1=1$. Hence $a|S$. Let $S=aT$. Then using (23), we have

$$g_k(n)\cdot a^k = g_{k,a}(na) = g_{k,a}(na+aT) = g_k(n+T)\cdot a^k.$$

Hence $g_k(n+T)=g_k(n)$ for all $n\in\mathbb{N}^*$; i.e., T is a period of $g_k(n)$.

Conversely, suppose T is a period of $g_k(n)$. Let n be any positive integer. If $d=\gcd(n,a)$, $n=n_1d$, $a=a_1d$, then $\gcd(n_1,a_1)=1$ and

$$g_{k,a}(n)=g_{k,a_1}(n_1)\cdot d^k, \quad g_{k,a}(n+aT)=g_{k,a_1}(n_1+a_1T)\cdot d^k.$$

Hence, without loss of generality, we assume that $(n,a)=1$. Therefore

$$\gcd(a, g_{k,a}(n))=1, \quad \gcd(a, g_{k,a}(n+aT))=1. \tag{25}$$

Hence by using (25), we have

$$g_{k,a}(n)=g_{k,a}(n+aT)$$

if and only if

$$v_p(g_{k,a}(n)) = v_p(g_{k,a}(n+aT)), \tag{26}$$

for any prime $p\nmid a$.

Let p be any prime such that $p\nmid a$ and let N be a positive integer greater than $v_p(k!)$. Then there exists a unique positive integer m such that $1\le m<p^N$ and

$$ma\equiv 1 (\mathrm{mod}\ p^N). \tag{27}$$

Let p, n, a, m be as above and $0\le i<j\le k$. Then for any integer l, there are

$$v_p(\gcd(n+al+ai,(j-i)a)) = v_p(\gcd(mn+l+i, j-i)). \tag{28}$$

Let

$$\gcd(n+al+ai,(j-i)a)=p^{x_{ij}}w, \quad p\nmid w \tag{29}$$

and

$$\gcd(mn+l+i, j-i)=p^{y_{ij}}u, \quad p\nmid u. \tag{30}$$

Then by (29), there exists s_1, $t_1\in\mathbb{Z}$ such that $(n+al+ai)s_1+(j-i)at_1=p^{x_{ij}}w$. Multiplying by m on both sides, we have $(mn+aml+ami)s_1+(j-i)mat_1=p^{x_{ij}}mw$. Using (27), we have $(mn+l+i)s_1+(j-i)t_1=p^{x_{ij}}mw-p^N\delta$. By (30), we have $y_{ij}\le x_{ij}$. Conversely, by (30), there exist s_2, $t_2\in\mathbb{Z}$ such that $(mn+l+i)s_2+(j-i)t_2=p^{y_{ij}}u$. Multiplying by a on both sides, we have $(mna+al+ai)s_2+(j-i)at_2=p^{y_{ij}}au$. Similarly, we have $x_{ij}\le y_{ij}$. So $x_{ij}=y_{ij}$ and (28) is true. Let $l=0$ and T. By using (28) we have

$$v_p(\gcd(n+ai,n+aj)) = v_p(\gcd(mn+i,mn+j))$$

and

$$v_p(\gcd(n+aT+ai,n+aT+aj)) = v_p(\gcd(mn+T+i,mn+T+j))$$

for any $0\le i<j\le k$. By the proof of Lemma 2.1, we have

$$v_p(g_{k,a}(n)) = v_p(g_k(mn)), \quad v_p(g_{k,a}(n+aT)) = v_p(g_k(mn+T)).$$

TABLE 1

The least period $T_{k,f}$ of $g_{k,f}$ with $f(x) = x^2 + b$

$f(x)$ \ k	1	2	3	4	5	6
x^2+1	5	$2\cdot5$	$2\cdot3\cdot5\cdot13$	$2\cdot3\cdot5\cdot13$	$2\cdot3\cdot5\cdot13\cdot29$	$2\cdot3\cdot5\cdot13\cdot29$
x^2+2	3^2	$2\cdot3^2$	$2\cdot3^2\cdot17$	$2\cdot3^2\cdot17$	$2\cdot3^2\cdot5\cdot11\cdot17$	$2\cdot3^2\cdot5\cdot11\cdot17$
x^2+3	13	$2\cdot13$	$2\cdot3\cdot7\cdot13$	$2\cdot3\cdot7\cdot13$	$2\cdot3\cdot5\cdot7\cdot13\cdot37$	$2\cdot3\cdot5\cdot7\cdot13\cdot37$
x^2+4	17	$2\cdot5\cdot17$	$2\cdot3\cdot5^2\cdot17$	$2^2\cdot3\cdot5^2\cdot17$	$2^2\cdot3\cdot5^2\cdot17\cdot41$	$2^2\cdot3\cdot5^2\cdot13\cdot17\cdot41$
x^2+5	$3\cdot7$	$2\cdot3\cdot7$	$2\cdot3\cdot7\cdot29$	$2\cdot3^2\cdot7\cdot29$	$2\cdot3^2\cdot5\cdot7\cdot29$	$2\cdot3^2\cdot5\cdot7\cdot29$
x^2+6	5^2	$2\cdot5^2\cdot7$	$2\cdot3\cdot5^2\cdot7\cdot11$	$2\cdot3\cdot5^2\cdot7\cdot11$	$2\cdot3\cdot5^2\cdot7^2\cdot11$	$2\cdot3\cdot5^2\cdot7^2\cdot11$

So using $g_k(mn) = g_k(mn+T)$, we have $v_p(g_{k,a}(n+aT)) = v_p(g_{k,a}(n))$ for any prime p such that $p \nmid a$. Hence $g_{k,a}(n+aT) = g_{k,a}(n)$ and aT is a period of $g_{k,a}(n)$.

The proof of (iii) follows from (ii). This completes the proof of Theorem 1.7. □

Proof of Corollary 1.8. (i) Assume that $p|a$. Then it is clear that the equality $v_p(g_{k,ax+b}(n)) = 0$ holds for any integer n when $g_{k,ax+b}(n)$ is well defined.

(ii) Assume that p is not a prime factor of a. By the formula (8), we have that T_p is a period of $v_p(g_{k,ax+b}(n))$ if and only if aT_p is a period of $v_p(g_{k,a}(n))$. Hence, by Theorem 1.7, we have that T_p is a period of $v_p(g_{k,ax+b}(n))$ if and only if T_p is a period of $v_p(g_k(n))$. Therefore Corollary 1.8 is obtained by Theorem 1.2. □

7. EXAMPLES

Lemma 7.1. *Let $f_1(x) = f_2(x)^r$, where $r \geq 1$ is an integer. Then $T_{k,f_1} = T_{k,f_2}$.*

Proof. By (4), we have $g_{k,f_1}(n) = g_{k,f_2}(n)^r$. Hence the result is obvious. □

Example. Let $f(x) = x^r$, $r \geq 1$. Then by Lemma 7.1, we have $T_{k,x^r} = T_{k,x}$, where $T_{k,x}$ is given by the formula (9).

Example. Let $f(x) = x^2 + b$. For $1 \leq i \leq k$, we have

$$(2x+3i)(x^2+b) + (-2x+i)((x+i)^2+b) = i(i^2+4b), \text{ if } i \text{ is odd},$$
$$(x+3j)(x^2+b) + (-x+j)((x+2j)^2+b) = 4j(j^2+b), \text{ if } i = 2j.$$

Hence

$$C_i = \begin{cases} i(i^2+4b), & \text{if } i \text{ is odd}, \\ 4j(j^2+b), & \text{if } i = 2j. \end{cases}$$

Hence, given any $k \in \mathbb{N}$ and $b \in \mathbb{Z}$, by Theorem 1.5, we can determine the least period $T_{k,f}$ of the arithmetic function $g_{k,f}$. For $1 \leq k \leq 6$ and $1 \leq b \leq 6$, Table 1 gives the $T_{k,f}$'s.

Example. Let $f(x) = x^3 + b$. For $1 \leq i \leq k$, we have

$$a_i(x)(x^3+b) + b_i(x)((x+i)^3+b) = C_i,$$

where

$$a_i(x) = \begin{cases} 6i^2x^2 + (15i^3-9)x + 10i^4 - 18i, & \text{if } 3 \nmid i; \\ 6j^2x^2 + (45j^3-1)x + 90i^4 - 6i, & \text{if } i = 3j. \end{cases}$$

TABLE 2

The least period $T_{k,f}$ of $g_{k,f}$ with $f(x) = x^3 + b$

$f(x)$ \ k	1	2	3	4	5	6
x^3+1	$2\cdot 7$	$2\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 13$	$2^2\cdot 3\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31$	$2^2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 31\cdot 43$	$2^2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43$
x^3+2	$2\cdot 7$	$2\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31$	$2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 31\cdot 43$	$2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43$
x^3+3	$2\cdot 7$	$2\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 13$	$2^2\cdot 3\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31$	$2^2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 31\cdot 43$	$2^2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43$
x^3+4	$2\cdot 7$	$2\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31$	$2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 31\cdot 43$	$2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43$
x^3+5	$2\cdot 7$	$2\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 13$	$2^2\cdot 3\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31$	$2^2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 31\cdot 43$	$2^2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43$
x^3+6	$2\cdot 7$	$2\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 13$	$2\cdot 3\cdot 7\cdot 11\cdot 13\cdot 17\cdot 31$	$2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 31\cdot 43$	$2\cdot 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 31\cdot 43$

$$b_i(x) = \begin{cases} -6i^2x^2 + (3i^3+9)x - i^4 - 9i, & \text{if } 3 \nmid i; \\ -6j^2x^2 + (9j^3+1)x - 9j^4 - 3j, & \text{if } i = 3j. \end{cases}$$

$$C_i = \begin{cases} -i^7 - 27i, & \text{if } 3 \nmid i; \\ -3^5j^7 - 9j, & \text{if } i = 3j. \end{cases}$$

Hence, given any $k \in \mathbb{N}$ and $b \in \mathbb{Z}$, by Theorem 1.5, we can determine the least period $T_{k,f}$ of the arithmetic function $g_{k,f}$. For $1 \le k \le 6$ and $1 \le b \le 6$, Table 2 gives the $T_{k,f}$'s.

REFERENCES

[1] B. Farhi, Minorations non triviales du plus petit commun multiple de certaines suites finies d'entiers *C. R. Math. Acad. Sci. Paris,* **341**(2005), 469-474. MR2180812 (2006g:11006)

[2] B. Farhi, Nontrivial lower bounds for the least common multiple of some finite sequences of integers, *J. Number Theory*, **125**(2007), 393-411. MR2332595 (2008i:11001)

[3] B. Farhi and D. Kane, New results on the least common multiple of consecutive integers, *Proc. Amer. Math. Soc.*, **137**(2009), 1933-1939. MR2480273 (2010a:11004)

[4] B. Green and T. Tao, The primes contain arbitrarily long arithmetic progressions, *Ann. of Math. (2)* **167**(2008), 481-548. MR2415379 (2009e:11181)

[5] G. H. Hardy and E. M. Wright, Theory of Numbers, fifth ed., Oxford Univ. Press, London, 1979. MR568909 (81i:10002)

[6] S. Hong and Y. Yang, On the periodicity of an arithmetical function, *C. R. Math. Acad. Sci. Paris,* **346**(2008), 717-721. MR2427068 (2009e:11007)

DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, PEOPLE'S REPUBLIC OF CHINA

E-mail address: qingzhji@nju.edu.cn

DEPARTMENT OF MATHEMATICS, NANJING NORMAL UNIVERSITY, NANJING 210097, PEOPLE'S REPUBLIC OF CHINA

E-mail address: cgji@njnu.edu.cn