PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 138, Number 9, September 2010, Pages 3187–3202
S 0002-9939(10)10522-X
Article electronically published on May 12, 2010

SPECTRUM OF THE COMPLEX LAPLACIAN ON PRODUCT DOMAINS

DEBRAJ CHAKRABARTI

(Communicated by Franc Forstneric)

Abstract. We show that the spectrum of the complex Laplacian $\Box$ on a product of Hermitian manifolds is the Minkowski sum of the spectra of the complex Laplacians on the factors. We use this to show that the range of the Cauchy-Riemann operator $\overline{\partial}$ is closed on a product manifold, provided it is closed on each factor manifold.

1. Introduction

The study of the $\overline{\partial}$ and $\overline{\partial}$-Neumann problems on product domains raises a series of interesting questions, which have been studied by many authors [8, 9, 10, 1, 3, 12]. In [12], the method of separation of variables was used to compute the spectrum of the complex Laplacian $\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ on a polydisc, and for each eigenvalue, the corresponding eigenspace was identified. In this paper, inspired by [12], we use the same technique to compute the spectrum of $\Box$ on an arbitrary product manifold in terms of the spectra of $\Box$ on the factors, and we give a description of the spectral representation of $\Box$. As a consequence, we obtain a strengthened version (see Theorem 1.2 below) of the closed-range theorem for the $\overline{\partial}$-operator on products.

In this paper an *operator* A on a Hilbert space H (always assumed separable) is a linear map with target H and source a linear subspace $\mathrm{dom}(A)$ of H. The $\Box$-Laplacian on a Hermitian manifold Ω is a densely defined selfadjoint linear operator on the Hilbert space $L^2_*(\Omega)$ of L^2 differential forms on Ω. Further, $\Box$ maps the subspace $L^2_{p,q}(\Omega)$ of forms of bidegree (p,q) to itself, and we denote the restriction of $\Box$ to $L^2_{p,q}(\Omega)$ by $\Box_{p,q}$. More details on $\Box$ may be found in §3 below.

Recall that the *spectrum* of A consists of those $z \in \mathbb{C}$ such that the operator $A - zI$ does not have a bounded inverse. We denote by $\sigma(\Omega)$ (resp. $\sigma_{p,q}(\Omega)$) the spectrum of the operator $\Box$ on $L^2_*(\Omega)$ (resp. $\Box_{p,q}$ on $L^2_{p,q}(\Omega)$). Since $\Box$ is a nonnegative selfadjoint operator, it follows that each of $\sigma(\Omega)$ and $\sigma_{p,q}(\Omega)$ is a closed nonempty subset of the set of nonnegative real numbers. For noncompact Ω, the set $\sigma(\Omega) \subset \mathbb{R}$ need not be discrete: for example, Lemma 4.1 below implies that for the "bumped shell" domain described on pp. 75-76 of [11], $0 \in \mathbb{R}$ is a point of accumulation of the spectrum.

For sets of numbers $P_1, \ldots, P_N$ we denote by $P_1 + \cdots + P_N$ their *Minkowski sum*, i.e. the set $\{p_1 + \cdots + p_N \mid p_1 \in P_1, \ldots, p_N \in P_N\}$. We have the following:

Received by the editors November 17, 2009.
2010 *Mathematics Subject Classification.* Primary 32W05; Secondary 35P10, 35N15.

©2010 American Mathematical Society
Reverts to public domain 28 years from publication

Theorem 1.1. *For $j = 1, \dots, N$, let Ω_j be a Hermitian manifold, and let $\Omega = \Omega_1 \times \cdots \times \Omega_N$ be the product manifold with the product Hermitian metric. Then*

$$\sigma(\Omega) = \sigma(\Omega_1) + \cdots + \sigma(\Omega_N). \tag{1}$$

Further, we have

$$\sigma_{\boldsymbol{p},\boldsymbol{q}}(\Omega) = \bigcup_{\substack{\sum_{j=1}^N p_j = \boldsymbol{p} \\ \sum_{j=1}^N q_j = \boldsymbol{q}}} \left(\sigma_{p_1,q_1}(\Omega_1) + \cdots + \sigma_{p_N,q_N}(\Omega_N)\right). \tag{2}$$

The computation of the spectrum of Ω described above can be used to deduce properties of Ω from those of $\Omega_1, \dots, \Omega_N$. We consider one example of this approach. A very important property that the operator $\overline{\partial} : L^2_*(\Omega) \to L^2_*(\Omega)$ might possess is that of having closed range, i.e., $\mathrm{img}(\overline{\partial}) \subset L^2_*(\Omega)$ is a closed subspace. This is equivalent to the solvability of the $\overline{\partial}$-equation $\overline{\partial} u = f$ (where $\overline{\partial} f = 0$, and f is orthogonal to the harmonic forms) in the L^2-sense (see [6] for details). The closed range property holds under suitable convexity assumptions, e.g. when Ω is pseudoconvex. We can define the L^2-Dolbeault cohomology space

$$H^*_{L^2}(\Omega) = \frac{\ker(\overline{\partial})}{\mathrm{img}(\overline{\partial})}.$$

When the closed-range property holds, this is a Hilbert space with the quotient norm. This space represents the obstruction to solving the $\overline{\partial}$-problem in the L^2-sense on Ω. We have the following:

Theorem 1.2. *Let $\Omega_1, \dots, \Omega_N$ and Ω be as in Theorem 1.1. Suppose that for each j, the $\overline{\partial}$-operator on Ω_j in the L^2-sense has closed range in $L^2_*(\Omega_j)$. Then $\overline{\partial} : L^2_*(\Omega) \to L^2_*(\Omega)$ also has closed range. Furthermore, the Künneth formula holds for the L^2 cohomology:*

$$H^*_{L^2}(\Omega) = H^*_{L^2}(\Omega_1) \widehat{\otimes} \dots \widehat{\otimes} H^*_{L^2}(\Omega_N), \tag{3}$$

where $\widehat{\otimes}$ denotes the Hilbert space tensor product (cf. §2.3 below).

The analog of (3) for the L^2 de Rham cohomology, in the special case when the cohomology spaces are finite dimensional, goes back to the work of Cheeger (see [4] and especially [5, p. 614]). Another approach, using an explicit solution of the d-equation on a product domain was given by Zucker in [18]. This can be extended to the $\overline{\partial}$-equation (see [3]), or, in another direction, to abstractly defined products of "Hilbert Complexes" (see [2]). A crucial feature in this approach is the assumption that the d- or $\overline{\partial}$-operator on the product satisfies a "Leibniz rule" in the strict operator sense (cf. assumption (i) in the statement of [18, Theorem 2.29]). In practice, this means that some boundary-smoothness or completeness assumptions must be made on the factors Ω_j in order to get a handle on the domain of the d- or $\overline{\partial}$-operator (via Friedrichs' lemma when the Ω_j have boundaries). A version of Theorem 1.2 is proved in [3], where it is assumed that each of the factors Ω_j has Lipschitz boundary. Consequently, using the results of [3], we cannot conclude, for example, that $\overline{\partial}$ has closed range on the product domain in $\mathbb{C}^4$ given as

$$\Omega = \{z \in \mathbb{C}^2 \mid 0 < |z_1| < |z_2| < 1\} \times \{z \in \mathbb{C}^2 \mid 1 < |z| < 2\},$$

although, in both factors, $\overline{\partial}$ has closed range: the first (the "Hartogs triangle") is pseudoconvex (but has a singular non-Lipschitz boundary), and for the second

we can see [15, 16]. Our result above shows that $\overline{\partial}$ has closed range on Ω, since Theorem 1.2 involves no assumptions on the factor manifolds Ω_j except the closed-range property for $\overline{\partial}$.

2. Some results from functional analysis

We recount here some facts from functional analysis which will be used in the proofs of Theorems 1.1 and 1.2. Most of what we need can be found in e.g. [14], but we discuss the required results to set up notation and for completeness.

2.1. **Multiplication operators.** Consider the measure space $(X, \mu) = (X, \mathcal{S}, \mu)$ (we systematically suppress the σ-algebra from the notation from now on), and a real-valued measurable function h on X which is finite a.e. (with respect to μ; this is the last time we will mention the measure with "a.e"). Define the multiplication operator T_h to be the operator on $L^2(X, \mu)$ on the domain

$$\mathrm{dom}(T_h) = \{f \in L^2(X, \mu) \mid hf \in L^2(X, \mu)\}$$

defined by $T_h f = hf$. Note that if $h = \tilde{h}$ a.e., then the operators T_h and $T_{\tilde{h}}$ on $L^2(X, \mu)$ are identical. If μ is a finite measure, T_h is densely defined. In fact, it is not difficult to see (cf. [14, Prop.2, p.260]) that if the function $h \in L^p(X, \mu)$, then any dense linear subspace of $L^q(X, \mu)$ is a core of T_h, provided $p^{-1} + q^{-1} = \frac{1}{2}$. (Recall that for an operator A on a Hilbert space H, a *core* of A is a linear subspace of $\mathrm{dom}(A)$ which is dense in the graph norm $x \mapsto \|x\|_{\mathsf{H}} + \|Ax\|_{\mathsf{H}}$ in $\mathrm{dom}(A)$.)

It is not difficult to see that the operator T_h is selfadjoint, and the spectrum $\mathrm{spec}(T_h)$ of the operator T_h is identical to the *essential range* of the function h. Recall that the *essential range* of a real-valued function h on (X, μ) is the set of $\lambda \in \mathbb{R}$ such that for all $\epsilon > 0$, we have

$$\mu\{x \in X \mid \lambda - \epsilon < h(x) < \lambda + \epsilon\} > 0.$$

Clearly, the essential range is a closed subset of the real line. Further, λ is an eigenvalue of T_h if and only if $\mu(h^{-1}(\lambda)) > 0$. The corresponding eigenspace of T_h is the closed subspace of $L^2(X, \mu)$ consisting of functions which vanish a.e. outside the set $h^{-1}(\lambda) \subset X$. Considering the special case when $\lambda = 0$, we obtain a natural identification

$$\ker(T_h) \cong L^2(h^{-1}(0), \mu) \hookrightarrow L^2(X, \mu), \tag{4}$$

where the measure space $(h^{-1}(0), \mu)$ is defined by restriction, and the inclusion of $L^2(h^{-1}(0), \mu)$ in $L^2(X, \mu)$ is induced by extension of functions by 0 from $h^{-1}(0)$ to X. When $h^{-1}(0)$ is the empty set, we will define $L^2(h^{-1}(0), \mu)$ to be the trivial vector space $\{0\}$. With this understanding, (4) is correct for all multiplication operators T_h.

2.2. **The spectral theorem.** The spectral theorem, a structure theorem for selfadjoint operators on a Hilbert space, can be stated in various equivalent forms (see [14, Theorems VIII.4, VIII.5 and VIII.6, pp. 260-264]; for bounded operators, see the masterly exposition [13]). We will use it in the following form: *the multiplication operators T_h defined in* §2.1 *are* (up to an isometric identification of Hilbert spaces) *the* **only** *examples of selfadjoint operators.* More precisely, let A be a selfadjoint operator on a Hilbert space H with domain $\mathrm{dom}(A)$. Then there is a measure space (X, μ) with μ a *finite* measure, a unitary operator (i.e. isometry of Hilbert spaces) $U : \mathsf{H} \to L^2(X, \mu)$ and a real-valued h on X finite a.e., so that

$\mathrm{dom}(A) = U^{-1}(\mathrm{dom}(T_h))$, and $A = U^{-1}T_hU$. Note that there is no uniqueness here for the space X, the measure μ or the multiplying function h. We will refer to T_h as a representation of A by a multiplication.

We note that we can without loss of generality assume that the function h in the conclusion of the spectral theorem belongs to $L^p(X,\mu)$ for every $p \geq 1$. Indeed, if this is not already the case, we replace the measure space (X,μ) by a new measure space (Y,ν), where $Y = X$, and $d\nu = e^{-h^2}d\mu$. Note that $f \mapsto e^{\frac{h^2}{2}}f$ defines an isometry from $L^2(X,\mu)$ to $L^2(Y,\nu)$. On $Y = X$, the operator A is still represented by multiplication by the same function h, but now $h \in L^p(Y,\nu)$ for each $p \geq 1$.

2.3. **Tensor products.** Let H_1 and H_2 be complex vector spaces. We denote by $\mathsf{H}_1 \otimes \mathsf{H}_2$ the *algebraic* tensor product (over $\mathbb{C}$) of H_1 and H_2. Then $\mathsf{H}_1 \otimes \mathsf{H}_2$ can be thought of as the space of finite sums of elements of the type $x \otimes y$, where $x \in \mathsf{H}_1$ and $y \in \mathsf{H}_2$, where $\otimes : \mathsf{H}_1 \times \mathsf{H}_2 \to \mathsf{H}_1 \otimes \mathsf{H}_2$ is the canonical bilinear map. If H_1, H_2 are vector spaces of functions defined on spaces X_1, X_2 respectively, then the algebraic tensor product $\mathsf{H}_1 \otimes \mathsf{H}_2$ can be concretely realized as a space of functions on $X_1 \times X_2$ by the correspondence

$$(f \otimes g)(x_1, x_2) = f(x_1)g(x_2), \tag{5}$$

followed by linear extension. We will always make this identification.

When H_1 and H_2 are Hilbert spaces we can define an inner product on $\mathsf{H}_1 \otimes \mathsf{H}_2$ by setting

$$(x \otimes y, z \otimes w) = (x,z)_{\mathsf{H}_1}(y,w)_{\mathsf{H}_2}$$

and extending bilinearly. This is well-defined thanks to the bilinearity of $\otimes$. This makes $\mathsf{H}_1 \otimes \mathsf{H}_2$ into a pre-Hilbert space, and its completion is a Hilbert space denoted by $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$, the *Hilbert tensor product* of the spaces H_1 and H_2. The algebraic tensor product $\mathsf{H}_1 \otimes \mathsf{H}_2$ sits inside $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ as a dense subspace.

For $j = 1, 2$, let $\mathsf{H}_j = L^2(X_j, \mu)$, and let $\mu_1 \otimes \mu_2$ denote the product measure on $X_1 \times X_2$. Then the injective map $L^2(X_1,\mu_1) \otimes L^2(X_2,\mu_2) \hookrightarrow L^2(X_1 \times X_2, \mu_1 \otimes \mu_2)$ given by (5) extends to an isometric isomorphism of Hilbert spaces

$$L^2(X_1,\mu_1) \widehat{\otimes} L^2(X_2,\mu_2) \cong L^2(X_1 \times X_2, \mu_1 \otimes \mu_2), \tag{6}$$

and we will always make this identification.

If S_1 and S_2 are operators on H_1 and H_2 with dense domains $\mathrm{dom}(S_1)$ and $\mathrm{dom}(S_2)$, we can define an operator $S_1 \otimes S_2$ on $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ with dense domain the algebraic tensor product $\mathrm{dom}(S_1) \otimes \mathrm{dom}(S_2)$ by setting on the simple tensors $x \otimes y$:

$$(S_1 \otimes S_2)(x \otimes y) = S_1 x \otimes S_2 y \tag{7}$$

and extending bilinearly.

2.4. **The operator $A_1 \otimes I_2 + I_1 \otimes A_2$.** Let H_1 and H_2 be separable Hilbert spaces, A_1, A_2 be densely defined selfadjoint operators on $\mathsf{H}_1, \mathsf{H}_2$ respectively, and let I_1, I_2 respectively denote the identity maps on $\mathsf{H}_1, \mathsf{H}_2$. We recall here (see [14, Theorem VIII.3] or [17, Theorem 4.14]) the spectral representation of the operator B on $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$, given by

$$B = A_1 \otimes I_2 + I_1 \otimes A_2,$$

which is densely defined with domain $\mathrm{dom}(A_1) \otimes \mathrm{dom}(A_2)$. For $j = 1, 2$ let (X_j, μ_j) be measure spaces and $U_j : \mathsf{H}_j \to L^2(X_j, \mu_j)$ be unitary isomorphisms of Hilbert spaces given by the spectral theorem of §2.2, such that $A_j = U_j^{-1} T_{h_j} U_j$, where h_j

is a real-valued function on X_j, such that for each $p \geq 1$, we have $h_j \in L^p(X_j, \mu_j)$. Let $U = U_1 \widehat{\otimes} U_2$, so that U is a unitary isomorphism from $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$ onto $L^2(X_1 \times X_2, \mu_1 \otimes \mu_2)$. Consider the operator A on $\mathsf{H}_1 \otimes \mathsf{H}_2$ defined by

$$A = U^{-1} T_h U, \tag{8}$$

where h is the function on $X_1 \times X_2$ defined by $h(x_1, x_2) = h_1(x_1) + h_2(x_2)$. Observe that, thanks to the hypotheses on h_1 and h_2, we have for each $p \geq 1$, that $h \in L^p(X_1 \times X_2, \mu_1 \otimes \mu_2)$. Then by the results recalled in §2.1, A is a selfadjoint operator on $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$.

Lemma 2.1. *The restriction of A to $\operatorname{dom}(A_1) \otimes \operatorname{dom}(A_2)$ coincides with B.*

Proof. Note that by linearity, we only need to check this on simple tensor products of the type $f \otimes g$. The proof is completed by a direct computation, using the fact that $U^{-1} = U_1^{-1} \otimes U_2^{-1}$ on the algebraic tensor product $L^2(X_1, \mu_1) \otimes L^2(X_2, \mu_2)$. $\square$

Several important consequences follow from the lemma above:

1° *The operator B is essentially selfadjoint.*

Recall that an *essentially selfadjoint* operator is one whose closure is selfadjoint. It is easy to see that such an operator has a unique selfadjoint extension, namely, its closure. It follows that *the operator A is the closure of the operator B.*

Since A is a selfadjoint extension of B, to show that B is essentially selfadjoint we need to show that $\operatorname{dom}(B) = \operatorname{dom}(A_1) \otimes \operatorname{dom}(A_2)$ is a core of A. Using (8), and translating the problem to the representation by multiplication operators, we need to show that $\operatorname{dom}(T_{h_1}) \otimes \operatorname{dom}(T_{h_2})$ is a core of T_h. Since the function $h \in L^4(X_1 \times X_2, \mu_1 \otimes \mu_2)$ (in fact $h \in L^p$ for each $p \geq 1$), it follows (cf. [14, Prop. 2, p. 260]) that any dense linear subspace of $L^4(X_1 \times X_2, \mu_1 \otimes \mu_2)$ is a core of T_h. Therefore, to prove the result it is sufficient to show that $\operatorname{dom}(T_{h_1}) \otimes \operatorname{dom}(T_{h_2})$ is dense in $L^4(X_1 \times X_2, \mu_1 \otimes \mu_2)$. Since $h_j \in L^p(X_j, \mu_j)$, for all $p \geq 1$ it follows that all simple functions (linear combinations of characteristic functions of measurable sets) are in $\operatorname{dom}(T_{h_j})$. It follows that the linear span $\mathfrak{S}$ of characteristic functions of rectangles with measurable sets as edges is contained in $\operatorname{dom}(T_{h_1}) \otimes \operatorname{dom}(T_{h_2})$. But it is well known that $\mathfrak{S}$ is dense in $L^p(X_1 \times X_2, \mu_1 \otimes \mu_2)$ for each $p > 0$.

2° The same method of proof can be used to prove a slightly stronger statement: *If for $j = 1, 2$, the linear space $D_j \subset \mathsf{H}_j$ is a core of A_j, then $D_1 \otimes D_2$ is a core of A.* For details see [14, Theorem VIII.3].

3° Denote by $\operatorname{ess.\,ran}(f)$ the essential range of a function f (cf. §2.2). Then for our functions h_1, h_2, h, it is easily verified that

$$\operatorname{ess.\,ran}(h) = \overline{\operatorname{ess.\,ran}(h_1) + \operatorname{ess.\,ran}(h_2)},$$

where the bar denotes closure in the topology of $\mathbb{R}$. It follows that the spectra of A_1, A_2 and A are related by

$$\operatorname{spec}(A) = \overline{\operatorname{spec}(A_1) + \operatorname{spec}(A_2)}.$$

We note here that the set $\operatorname{spec}(A_1) + \operatorname{spec}(A_2)$ need not be closed. Indeed, it is easy to construct selfadjoint operators A_1 and A_2 (both e.g. on the space ℓ^2 of square-summable sequences) such that $\operatorname{spec}(A_1) = \mathbb{N}_+$, the set of positive integers, and $\operatorname{spec}(A_2) = \left\{-\nu - \frac{1}{\nu} \mid \nu \in \mathbb{N}_+\right\}$.

3. The complex Laplacian

3.1. **Definition and basic properties.** We now recall the definition and basic properties of the complex Laplacian $\Box$. Details may be found in the texts [11, 6].

Let Ω be a Hermitian manifold, i.e., a complex manifold with a Hermitian metric. We let $L^2_{p,q}(\Omega)$ be the Hilbert space of square integrable differential forms of bidegree (p,q) and let $L^2_*(\Omega)$ be the orthogonal Hilbert space sum of the $L^2_{p,q}(\Omega)$, so that $L^2_*(\Omega)$ is the Hilbert space of all square integrable forms on Ω.

We can define a realization of the $\overline{\partial}$-operator as a densely defined closed Hilbert space operator from the space $L^2_*(\Omega)$ to itself. This realization has the domain

$$\mathrm{dom}(\overline{\partial}) = \{f \in L^2_*(\Omega) \mid \overline{\partial} f \in L^2_*(\Omega)\},$$

where $\overline{\partial} f$ is taken in the sense of distributions. We denote by $\overline{\partial}^*$ the Hilbert space adjoint of $\overline{\partial}$. This is again a densely defined closed operator on $L^2_*(\Omega)$, whose domain $\mathrm{dom}(\overline{\partial}^*)$ is very different from that of $\overline{\partial}$. We define the *complex Laplacian* on Ω to be the operator

$$\Box = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.$$

Then, it can be shown that $\Box$ is a densely defined closed and unbounded operator, which is *selfadjoint* and *nonnegative.* Note that by the definition of domains of compositions and sums of unbounded operators, we have that

$$\mathrm{dom}(\Box) = \left\{ f \in L^2_*(\Omega) \mid f \in \mathrm{dom}(\overline{\partial}) \cap \mathrm{dom}(\overline{\partial}^*), \overline{\partial} f \in \mathrm{dom}(\overline{\partial}^*) \text{ and } \overline{\partial}^* f \in \mathrm{dom}(\overline{\partial}) \right\}.$$

A very important special case is when Ω is realized as a relatively compact and smoothly bounded domain in a larger Hermitian manifold $\mathcal{M}$, and given as $\Omega = \{z \in \mathcal{M} \mid \rho(z) < 0\}$, where the gradient $\nabla\rho$ is normalized to unit length on $\mathsf{b}\Omega$. In this case, if $f \in \mathcal{C}^2_*(\overline{\Omega})$ is a form smooth up to the boundary, the condition that $f \in \mathrm{dom}(\Box)$ is equivalent to f satisfying on $\mathsf{b}\Omega$ the $\overline{\partial}$-*Neumann boundary conditions*

$$\begin{cases} f \rfloor \nabla\rho & = 0 \text{ and} \\ \overline{\partial} f \rfloor \nabla\rho & = 0, \end{cases} \tag{9}$$

where $\rfloor$ denotes the contraction of a form by a vector field.

3.2. **Differential forms on product manifolds.** We now generalize equations (5) and (6) to spaces of differential forms on manifolds. Let H_1 and H_2 be vector spaces of differential forms on the manifolds Ω_1 and Ω_2 respectively. Let π_j denote the projection from the product $\Omega = \Omega_1 \times \Omega_2$ to the factor Ω_j. It is easy to see that the identification

$$f \otimes g = \pi_1^* f \wedge \pi_2^* g$$

linearly extends to an injective map of $\mathsf{H}_1 \otimes \mathsf{H}_2$ into the space of differential forms on Ω. In particular, if we take $\mathsf{H}_j = L^2_*(\Omega_j)$, the Hilbert space of forms square integrable with respect to the Hermitian metric (see [6, Chapter 5] for detailed definitions), we get an injective map $L^2_*(\Omega_1) \otimes L^2_*(\Omega_2) \hookrightarrow L^2_*(\Omega)$ which can be extended by closure to obtain a natural identification

$$L^2_*(\Omega_1) \widehat{\otimes} L^2_*(\Omega_2) \cong L^2_*(\Omega).$$

Note that for (p,q) forms, reading off degrees on each side, this construction gives a representation of $L^2_{p,q}(\Omega)$ as an orthogonal direct sum of tensor products:

$$L^2_{p,q}(\Omega) = \bigoplus_{\substack{p_1+p_2=p\\ q_1+q_2=q}} L^2_{p_1,q_1}(\Omega_1)\widehat{\otimes} L^2_{p_2,q_2}(\Omega_2). \tag{10}$$

It is clear how this construction extends to more than two factors.

3.3. **Proof of main theorem.** We introduce some more notation. Denote by $\Box^1$ and $\Box^2$ the complex Laplacians on Ω_1 and Ω_2 respectively (this is unambiguous since we never consider powers of the complex Laplacian). Then for $j = 1, 2$, the operator $\Box^j$ is a densely defined, selfadjoint, nonnegative operator on $L^2_*(\Omega_j)$, and we will denote its restriction to $L^2_{p,q}(\Omega_j)$ by $\Box^j_{p,q}$. Let $\Omega = \Omega_1 \times \Omega_2$ be the product Hermitian manifold (with product metric). Let D be the operator on $L^2_*(\Omega)$ with domain $\mathrm{dom}(\Box^1) \otimes \mathrm{dom}(\Box^2)$, defined as

$$D = \Box^1 \otimes I_2 + I_1 \otimes \Box^2, \tag{11}$$

where I_j is the identity operator on $L^2_*(\Omega_j)$. Note that D is densely defined on $L^2_*(\Omega)$, but it is not clear a priori whether D is closable or not.

Denote by $\Box$ the complex Laplacian on the product Ω. We claim that D is closable, and its closure is $\Box$. The first step of the proof is the following:

Lemma 3.1. $\mathrm{dom}(D) \subset \mathrm{dom}(\Box)$, *and the restriction of* $\Box$ *to* $\mathrm{dom}(D) = \mathrm{dom}(\Box^1)\otimes \mathrm{dom}(\Box^2)$ *coincides with* D.

Proof. Denote by $\overline{\partial}_j$ the L^2 Cauchy-Riemann operator on Ω_j and by $\overline{\partial}^*_j$ its Hilbert adjoint, and let $\overline{\partial}, \overline{\partial}^*$ denote the corresponding objects on the product $\Omega = \Omega_1 \times \Omega_2$. It is easy to see from the definitions of the domains of these operators that $\mathrm{dom}(\overline{\partial}) \supset \mathrm{dom}(\overline{\partial}_1) \otimes \mathrm{dom}(\overline{\partial}_2)$ and $\mathrm{dom}(\overline{\partial}^*) \supset \mathrm{dom}(\overline{\partial}^*_1) \otimes \mathrm{dom}(\overline{\partial}^*_2)$. Further, on $\mathrm{dom}(\overline{\partial}_1) \otimes \mathrm{dom}(\overline{\partial}_2)$ we have the Leibniz formula

$$\overline{\partial} = \overline{\partial}_1 \otimes I_1 + \sigma_1 \otimes \overline{\partial}_2, \tag{12}$$

where σ_1 is the operator on $L^2_*(\Omega_1)$, which when restricted to $L^2_{p,q}(\Omega_1)$ is multiplication by $(-1)^{p+q}$. Note that $\sigma_1^2 = I_1$ and for any operator S of degree d on $L^2_*(\Omega_1)$ (i.e., $\deg(Sf) - \deg(f) = d$ for every homogeneous form f in $\mathrm{dom}(S)$) we have $\sigma_1 S = (-1)^d S\sigma_1$.

Similarly for $\overline{\partial}^*$ we have on $\mathrm{dom}(\overline{\partial}^*_1) \otimes \mathrm{dom}(\overline{\partial}^*_2)$ that

$$\overline{\partial}^* = \overline{\partial}^*_1 \otimes I_1 + \sigma_1 \otimes \overline{\partial}^*_2. \tag{13}$$

Now let $f_j \in \mathrm{dom}(\Box^j)$ and set $f = f_1 \otimes f_2$. We verify that $f \in \mathrm{dom}(\Box)$. Indeed, since $f_j \in \mathrm{dom}(\overline{\partial}_j) \cap \mathrm{dom}(\overline{\partial}^*_j)$, it follows that

$$f \in \left(\mathrm{dom}(\overline{\partial}_1) \otimes \mathrm{dom}(\overline{\partial}_2)\right) \cap \left(\mathrm{dom}(\overline{\partial}^*_1) \otimes \mathrm{dom}(\overline{\partial}^*_2)\right).$$

Now, using also the facts that $\overline{\partial} f_j \in \mathrm{dom}(\overline{\partial}^*)$ and (12), it follows that $\overline{\partial} f \in \mathrm{dom}(\overline{\partial}^*)$. Similarly, using $\overline{\partial}^* f_j \in \mathrm{dom}(\overline{\partial})$ and (13) we obtain that $\overline{\partial}^* f \in \mathrm{dom}(\overline{\partial})$. It follows that $f \in \mathrm{dom}(\Box)$. So $\mathrm{dom}(D) \subset \mathrm{dom}(\Box)$.

Now we compute

$$\begin{aligned}\overline{\partial}\overline{\partial}^* f &= (\overline{\partial}_1 \otimes I_2 + \sigma_1 \otimes \overline{\partial}_2)(\overline{\partial}^*_1 f_1 \otimes f_2 + \sigma_1 f_1 \otimes \overline{\partial}^*_2 f_2)\\ &= \overline{\partial}_1\overline{\partial}^*_1 f_1 \otimes f_2 + \overline{\partial}_1\sigma_1 f_1 \otimes \overline{\partial}^*_2 f_2 + \sigma_1\overline{\partial}^*_1 f_1 \otimes \overline{\partial}_2 f_2 + f_1 \otimes \overline{\partial}_2\overline{\partial}^*_2 f_2\end{aligned}$$

and

$$\begin{aligned}\overline{\partial}^*\overline{\partial}f &= (\overline{\partial}_1^* \otimes I_2 + \sigma_1 \otimes \overline{\partial}_2^*)(\overline{\partial}_1 f_1 \otimes f_2 + \sigma_1 f_1 \otimes \overline{\partial} f_2)\\ &= \overline{\partial}_1^*\overline{\partial}_1 f_1 \otimes f_2 + \sigma_1\overline{\partial}_1 f_1 \otimes \overline{\partial}_2^* f_2 + \overline{\partial}_1^*\sigma_1 f_1 \otimes \overline{\partial}_2 f_2 + f_1 \otimes \overline{\partial}_2^*\overline{\partial}_2 f_2.\end{aligned}$$

Combining, we obtain

$$\begin{aligned}\Box f &= \Box^1 f_1 \otimes f_2 + (\overline{\partial}_1\sigma_1 + \sigma_1\overline{\partial}_1) f_1 \otimes \overline{\partial}_2^* f_2 + (\sigma_1\overline{\partial}_1^* + \overline{\partial}_1^*\sigma_1) f_1 \otimes \overline{\partial}_2 f_2 + f_1 \otimes \Box^2 f_2\\ &= \Box^1 f_1 \otimes f_2 + f_1 \otimes \Box^2 f_2\\ &= Df,\end{aligned}$$

where we have made use of the fact that $\overline{\partial}$ and $\overline{\partial}^*$ are of degree ± 1 respectively. By linear extension, it follows that $\Box = D$ on $\mathrm{dom}(D)$. □

The proof of Theorem 1.1 will also require the following simple observation:

Lemma 3.2. *Let E and F be closed subsets of the set of nonnegative reals. Then $E + F$ is also a closed set.*

Proof. Let z a point in the closure $\overline{E+F}$, and let z_ν be a sequence of points in $E + F$ converging to z. Writing $z_\nu = x_\nu + y_\nu$, where $x_\nu \in E$ and $y_\nu \in F$, we see that x_ν, y_ν are bounded sequences in the closed sets E and F respectively. By the Bolzano-Weierstrass theorem, after passing to a subsequence, we can assume that $x_\nu \to x \in E$ and $y_\nu \to y \in F$. It follows that $z = x + y \in E + F$. □

Proof of equation (1). We first consider the case when $N = 2$. Since $\Box^1$ and $\Box^2$ are selfadjoint, using the results of §2.4, we see that D is an essentially selfadjoint operator on $L^2_*(\Omega)$. By Lemma 3.1, $\Box$ is an extension of D. But since $\Box$ is selfadjoint, this means that $\Box$ is the closure of D. Thanks again to the results of §2.4, it follows that

$$\mathrm{spec}(\Box) = \overline{\mathrm{spec}(\Box^1) + \mathrm{spec}(\Box^2)}.$$

Now, using Lemma 3.2 and the fact that $\Box^j$ is nonnegative so its spectrum consists of nonnegative numbers, it follows that

$$\sigma(\Omega) = \sigma(\Omega_1) + \sigma(\Omega_2).$$

The case of general $N > 2$ now follows by a straightforward induction argument. □

Proof of equation (2). First we assume that $N = 2$. Let $f_1 \in L^2_{p_1,q_1}(\Omega_1)$, and $f_2 \in L^2_{p_2,q_2}(\Omega_2)$, so that $f_1 \otimes f_2 \in L^2_{p_1,q_1}(\Omega_1)\widehat{\otimes}L^2_{p_2,q_2}(\Omega_2)$. Note that $\Box^1$ (resp. $\Box^2$) maps $L^2_{p_1,q_1}(\Omega_1)$ (resp. $L^2_{p_2,q_2}(\Omega_2)$) into itself. Therefore, the formula (11) defining D shows that D maps $L^2_{p_1,q_1}(\Omega_1)\widehat{\otimes}L^2_{p_2,q_2}(\Omega_2)$ into itself, and it follows that so does $\Box$, the closure of D. Therefore, the restriction of $\Box$ defines a selfadjoint operator on each space $L^2_{p_1,q_1}(\Omega_1)\widehat{\otimes}L^2_{p_2,q_2}(\Omega_2)$. Denote this restriction by the admittedly barbarous notation $\Box_{\substack{p_1,q_1\\p_2,q_2}}$. Then by the results of §2.4, we have that $\Box_{\substack{p_1,q_1\\p_2,q_2}}$ is the

unique selfadjoint extension of the operator $\Box^1_{p_1,q_1} \otimes I_2 + I_1 \otimes \Box^2_{p_2,q_2}$ (where now I_j denotes the identity map on the Hilbert space $L^2_{p_j,q_j}(\Omega_j)$), and we have for the spectra

$$\begin{aligned} \operatorname{spec}\left(\Box_{\substack{p_1,q_1\\p_2,q_2}}\right) &= \overline{\operatorname{spec}(\Box^1_{p_1,q_1}) + \operatorname{spec}(\Box^2_{p_2,q_2})} \\ &= \overline{\sigma_{p_1,q_1}(\Omega_1) + \sigma_{p_2,q_2}(\Omega_2)} \\ &= \sigma_{p_1,q_1}(\Omega_1) + \sigma_{p_2,q_2}(\Omega_2), \end{aligned} \tag{14}$$

where we have used Lemma 3.2.

Now consider a Hilbert space H represented as an orthogonal direct sum

$$\mathsf{H} = \bigoplus_{k=1}^{n} \mathsf{H}_k,$$

and let A be an operator on H which maps each H_k to itself. Denoting by A_k the restriction of A to H_k (interpreted as an operator on H_k), we find there is a direct sum decomposition $A = \bigoplus_{k=1}^{n} A_k$. Then we have (see [17, Theorem 2.23]):

$$\operatorname{spec}(A) = \bigcup_{k=1}^{n} \operatorname{spec}(A_k). \tag{15}$$

Now, by (10), the space $L^2_{p,q}(\Omega)$ is represented as an orthogonal direct sum of subspaces $L^2_{p_1,q_1}(\Omega_1)\widehat{\otimes}L^2_{p_2,q_2}(\Omega_2)$ (with $p_1 + p_2 = p$ and $q_1 + q_2 = q$), and the operator $\Box_{p,q}$ maps each of these subspaces to itself. Therefore, using (15), we have

$$\begin{aligned} \operatorname{spec}(\Box_{p,q}) &= \bigcup_{\substack{p_1+p_2=p\\q_1+q_2=q}} \operatorname{spec}\left(\Box_{\substack{p_1,q_1\\p_2,q_2}}\right) \\ &= \bigcup_{\substack{p_1+p_2=p\\q_1+q_2=q}} \left(\operatorname{spec}(\Box^1_{p_1,q_1}) + \operatorname{spec}(\Box^2_{p_2,q_2})\right) \qquad \text{by (14)}, \end{aligned}$$

In the notation used in the statement of Theorem 1.1, this reads

$$\sigma_{p,q}(\Omega) = \bigcup_{\substack{p_1+p_2=p\\q_1+q_2=q}} \left(\sigma_{p_1,q_1}(\Omega_1) + \sigma_{p_2.q_2}(\Omega_2)\right),$$

which proves (2) in the case $N = 2$.

Now we extend this result by induction to general $N > 2$. The induction will require the use of the following formula regarding Minkowski sums:

$$E + \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} (E + F_i), \tag{16}$$

which follows directly from the definition. We assume that the result has been established for $N-1$ factors, and we consider N factors $\Omega_1, \ldots, \Omega_N$. We set $\Omega'_j = \Omega_j$

for $1 \leq j \leq N-2$ and let $\Omega'_{N-1} = \Omega_{N-1} \times \Omega_N$. Then, if $\Omega = \Omega_1 \times \cdots \times \Omega_N = \Omega'_1 \times \cdots \times \Omega'_{N-1}$, we have by the induction hypothesis

$$\begin{aligned}
\sigma_{\boldsymbol{p},\boldsymbol{q}}(\Omega) &= \bigcup_{\substack{\sum_{j=1}^{N-1} p_j = \boldsymbol{p} \\ \sum_{j=1}^{N-1} q_j = \boldsymbol{q}}} \left(\sigma_{p_1,q_1}(\Omega'_1) + \cdots + \sigma_{p_{N-1},q_{N-1}}(\Omega'_{N-1})\right) \\
&= \bigcup_{\substack{\sum_{j=1}^{N-1} p_j = \boldsymbol{p} \\ \sum_{j=1}^{N-1} q_j = \boldsymbol{q}}} \Bigg(\sigma_{p_1,q_1}(\Omega_1) + \cdots + \sigma_{p_{N-2},q_{N-2}}(\Omega_{N-2}) \\
&\qquad + \left(\bigcup_{\substack{P_1+P_2=p_{N-1} \\ Q_1+Q_2=q_{N-1}}} \left(\sigma_{P_1,Q_1}(\Omega_{N-1}) + \sigma_{P_2,Q_2}(\Omega_N)\right) \right) \Bigg) \\
&\quad \text{(using the result for } N=2) \\
&= \bigcup_{\substack{\sum_{j=1}^{N-1} p_j = \boldsymbol{p} \\ \sum_{j=1}^{N-1} q_j = \boldsymbol{q}}} \Bigg(\bigcup_{\substack{P_1+P_2=p_{N-1} \\ Q_1+Q_2=q_{N-1}}} \Big(\sigma_{p_1,q_1}(\Omega_1) + \cdots + \sigma_{p_{N-2},q_{N-2}}(\Omega_{N-2}) \\
&\qquad + \sigma_{P_1,Q_1}(\Omega_{N-1}) + \sigma_{P_2,Q_2}(\Omega_N)\Big) \Bigg) \\
&\quad \text{(using (16))} \\
&= \bigcup_{\substack{\sum_{j=1}^{N} p_j = \boldsymbol{p} \\ \sum_{j=1}^{N} q_j = \boldsymbol{q}}} \left(\sigma_{p_1,q_1}(\Omega_1) + \cdots + \sigma_{p_N,q_N}(\Omega_N)\right) \\
&\quad \text{(renaming the indices).}
\end{aligned}$$

This completes the proof. □

4. Proof of Theorem 1.2

We first establish a couple of lemmas.

Lemma 4.1. *Let A be a nonnegative selfadjoint operator on a Hilbert space H. Then the following are equivalent:*

1. *The range of A is closed.*
2. *There is a $C > 0$ such that for each $x \in \operatorname{dom}(A) \cap \ker(A)^{\perp}$,*

$$\|Ax\| \geq C \|x\| . \tag{17}$$

3. *There is a $c > 0$ such that the intersection of the open interval $(0, c)$ with the set $\operatorname{spec}(A)$ is empty.*

Proof. The equivalence of (1) and (2) is a standard fact in functional analysis (see [6, Lemma 4.1.1]). We show that (2) and (3) are equivalent. Using the spectral

theorem as stated in §2.2, we can assume that H is the space $L^2(X,\mu)$ for some measure space (X,μ), and $Af = hf$ for a nonnegative function h on X.

First assume that (3) is true. Recall from §2.1 that the spectrum of A coincides with the essential range of the function f. This means that on the complement of the set $\{h=0\} \subset X$, the function h satisfies $h \geq c$ a.e. Then, we have for any $f \in \operatorname{dom}(A) \subset L^2(X,\mu)$,

$$\begin{aligned}\|Af\|^2 &= \int |hf|^2 \, d\mu \\ &= \int_{\{h=0\}} |hf|^2 \, d\mu + \int_{\{h>0\}} |hf|^2 \, d\mu \\ &\geq 0 + c^2 \int |f|^2 \, d\mu \\ &= c^2 \|f\|^2 ,\end{aligned}$$

so that (3) implies (2). Now assume that (3) is violated. For a positive integer ν, let

$$E_\nu = \left\{ x \in X \mid \frac{1}{2^{\nu+1}} \leq h(x) < \frac{1}{2^\nu} \right\}$$

and let $\mu_\nu = \mu(E_\nu)$. Since (3) is not true, it is possible to find a sequence of integers $\nu_k \uparrow \infty$ such that $\mu_{\nu_k} > 0$. For each k define

$$f_k = \begin{cases} \dfrac{1}{\sqrt{\mu_{\nu_k}}} & \text{on } E_{\nu_k}, \\ 0 & \text{elsewhere.} \end{cases}$$

We claim that f_k is orthogonal to $\ker(A)$. Indeed, $g \in \ker(A)$ if and only if g has support in the set $\{h=0\}$ (cf. equation (4)). Since the support of f_k is by construction disjoint from that of g, it follows that $\int f_k \overline{g} d\mu = 0$. Also,

$$\begin{aligned}\|f_k\|^2 &= \int_{E_{\nu_k}} \frac{1}{\mu_{\nu_k}} d\mu \\ &= 1,\end{aligned}$$

but on the other hand we have

$$\begin{aligned}\|Af_k\|^2 &= \int |hf_k|^2 \, d\mu \\ &= \int_{E_{\nu_k}} \frac{h^2}{\mu_{\nu_k}} d\mu \\ &\leq \frac{1}{4^{\nu_k}}.\end{aligned}$$

It follows that $f_k \in \operatorname{dom}(A) \cap \ker(A)^\perp$, but no constant such as C in (17) exists. This completes the proof. □

Lemma 4.2. *Let A_1, A_2 be nonnegative selfadjoint operators on Hilbert spaces $\mathsf{H}_1, \mathsf{H}_2$, and let A be the closure of $A_1 \otimes I_2 + I_1 \otimes A_2$ as an operator on $\mathsf{H}_1 \widehat{\otimes} \mathsf{H}_2$. (Recall that A was shown in §2.4 to be selfadjoint.) Then we have*

$$\ker(A) = \ker(A_1) \widehat{\otimes} \ker(A_2). \tag{18}$$

Proof. We use the representation of A_1, A_2, A by multiplication operators developed in §2.4. After using the unitary isomorphism U, proving (18) is reduced to proving that $\ker(T_h) = \ker(T_{h_1})\widehat{\otimes}\ker(T_{h_2})$. The functions h_1 and h_2 on X_1 and X_2, respectively, which represent A_1 and A_2 by multiplication are now nonnegative a.e., and the subset $h^{-1}(0)$ of $X_1 \times X_2$ is identical to $h_1^{-1}(0) \times h_2^{-1}(0)$. Indeed, since $h_1 \geq 0$ and $h_2 \geq 0$ a.e., the only way $h(x_1, x_2) = h_1(x_1) + h_2(x_2)$ can vanish a.e. is by the vanishing a.e. of both h_1 and h_2. Using the representation (4) of the kernel, we obtain

$$\begin{aligned}\ker(T_h) &= L^2(h^{-1}(0), \mu_1 \otimes \mu_2)\\ &= L^2(h_1^{-1}(0) \times h_2^{-1}(0), \mu_1 \otimes \mu_2)\\ &= L^2(h_1^{-1}(0), \mu_1)\widehat{\otimes}L^2(h_2^{-1}(0), \mu_2)\\ &= \ker(T_{h_1})\widehat{\otimes}\ker(T_{h_2}).\end{aligned}$$

□

Proof of Theorem 1.2. It is sufficient to prove the result for $N = 2$, the general case following by a simple induction argument. We recall the following standard fact from Hodge theory: on a Hermitian manifold the following are equivalent: (a) $\Box$ has closed range, (b) $\overline{\partial}$ has closed range, (c) $\overline{\partial}^*$ has closed range, (d) (Kodaira; see [7, p. 165], or [3, Lemma 2.2]) every L^2-Dolbeault cohomology class has a unique harmonic representative: more precisely, the inclusion $\ker(\Box) \subset \ker(\overline{\partial})$ induces an isomorphism on the cohomology level.

We use the notation used in the proof of Lemma 3.1. Therefore, assume that the operators $\overline{\partial}_1$ and $\overline{\partial}_2$ have closed range in $L^2_*(\Omega_1)$ and $L^2_*(\Omega_2)$ respectively. Therefore for $j = 1, 2$, the operator $\Box^j$ also has closed range in $L^2_*(\Omega_j)$. It follows from Lemma 4.1 that there exist $c_j > 0$ such that $\sigma(\Omega_j) \cap (0, c_j) = \emptyset$. But $\sigma(\Omega) = \sigma(\Omega_1) + \sigma(\Omega_2)$ by Theorem 1.1, so if $c = \min(c_1, c_2)$, then clearly $\sigma(\Omega) \cap (0, c) = \emptyset$. It follows now from Lemma 4.1 that $\Box$ has closed range in $L^2_*(\Omega)$. We conclude that $\overline{\partial}$ has closed range in $L^2_*(\Omega)$.

Now, $\Box^j$ is a nonnegative operator, $j = 1, 2$, and $\Box$ is the unique selfadjoint extension of $\Box^1 \otimes I_2 + I_1 \otimes \Box^2$, so by Lemma 4.2 we have

$$\ker(\Box) = \ker(\Box^1)\widehat{\otimes}\ker(\Box^2).$$

Since, by part (d) of the result quoted in the first paragraph of this proof, $\ker(\Box) \cong H^*_{L^2}(\Omega)$ and $\ker(\Box^j) \cong H^*_{L^2}(\Omega_j)$ for $j = 1, 2$, the Künneth formula (3) now follows in the case $N = 2$. □

5. Point spectra and eigenvectors

In this section we consider certain simple special cases of Theorem 1.1. All these could have been deduced directly from the representation of $\Box$ constructed in §3.3. However, we give direct elementary arguments wherever possible in view of the importance of the special cases considered.

5.1. **Expansions in eigenforms.** We use the notation of Theorem 1.1 and §3.3.

Proposition 5.1. (a) *For $j = 1, \ldots, N$, let $\alpha_j \in \sigma(\Omega_j)$ be an eigenvalue, and let $E^j_{\alpha_j} \subset \mathrm{dom}(\Omega_j) \subset L^2_*(\Omega_j)$ be the corresponding eigenspace. Then $\sum_{j=1}^N \alpha_j \in \sigma(\Omega)$ is an eigenvalue and the corresponding eigenspace is $E^1_{\alpha_1}\widehat{\otimes}\ldots\widehat{\otimes}E^N_{\alpha_N}$.*

(b) *If each $\sigma(\Omega_j)$ consists only of eigenvalues, so does $\sigma(\Omega)$.*

Proof. By a simple induction argument, it suffices to consider the case $N = 2$ for both (a) and (b). For part (a), we let $f_j \in E^j_{\alpha_j} \subset \mathrm{dom}(\Box_j)$ so that $f_1 \otimes f_2 \in \mathrm{dom}(\Box)$ by Lemma 3.1. A computation using (11) and Lemma 3.1 now shows that $\alpha_1 + \alpha_2$ is an eigenvalue of $\Box$ with eigenvector $f_1 \otimes f_2$. Part (a) follows, since the algebraic tensor product is dense in the Hilbert tensor product (cf. §2.3).

For part (b), continuing to assume $N = 2$, we note that the hypothesis implies that for $j = 1, 2$,

$$L^2_*(\Omega_j) = \bigoplus_{\lambda \in \sigma(\Omega_j)} E^j_\lambda,$$

where E^j_λ denotes the eigenspace of $\Box_j$ corresponding to the eigenvalue λ. Therefore, we have

$$L^2_*(\Omega) = \bigoplus_{\substack{\lambda \in \sigma(\Omega_1) \\ \mu \in \sigma(\Omega_2)}} E^1_\lambda \widehat{\otimes} E^2_\mu.$$

Therefore, the span of the eigenspaces corresponding to the points of $\sigma(\Omega_1) + \sigma(\Omega_2)$ is dense in the Hilbert space $L^2_*(\Omega)$. It follows that the full spectral decomposition of $\Box$ on Ω is given by projection on the eigenspaces corresponding to $\sigma(\Omega_1) + \sigma(\Omega_2)$. Part (b) now follows. $\Box$

We now consider Ω_j, $j = 1, \ldots, N$ such that each $\sigma(\Omega_j)$ consists of eigenvalues only. This happens in many important cases, for example, when each Ω_j is a smoothly bounded pseudoconvex domain of finite type in some $\mathbb{C}^{n_j}$. For $\lambda \in \sigma(\Omega_j)$ denote by π^j_λ the orthogonal projection from $L^2_*(\Omega_j)$ to the subspace E^j_λ. Then we have the following representation:

$$\Box^j = \sum_{\lambda \in \sigma(\Omega_j)} \lambda \pi^j_\lambda,$$

where the series on the right converges in the strong operator topology; i.e. for every $f \in \mathrm{dom}(\Box^j)$, $\sum_{\lambda \in \sigma(\Omega_j)} \lambda \pi^j_\lambda f$ converges to $\Box^j f$ in the norm topology of $L^2_*(\Omega_j)$. Our computations show that:

Corollary 5.2. *On Ω:*

$$\Box = \sum_{\substack{\lambda_j \in \sigma(\Omega_j) \\ j=1,\ldots,N}} (\lambda_1 + \cdots + \lambda_N)\, \pi^1_{\lambda_1} \widehat{\otimes} \ldots \widehat{\otimes} \pi^N_{\lambda_N}.$$

Here $\pi^1_{\lambda_1} \widehat{\otimes} \ldots \widehat{\otimes} \pi^N_{\lambda_N}$ is the projection operator on $L^2_*(\Omega)$ obtained as the closure of the bounded operator $\pi^1_{\lambda_1} \otimes \ldots \otimes \pi^N_{\lambda_N}$ on the algebraic tensor product subspace $L^2_*(\Omega_1) \otimes \ldots \otimes L^2_*(\Omega_N)$. This is clearly a projection onto a subspace of the eigenspace of $\Box$ corresponding to the eigenvalue $\lambda = \sum_j \lambda_j$ of $\Box$ (it is not necessarily the *full* projection corresponding to λ, since there may be more than one way of representing λ as a sum of eigenvalues of the complex Laplacian on the factor domains).

Denote by $\pi^{j,(p,q)}_\lambda$ the projection from $L^2_{p,q}(\Omega_j)$ onto the eigenspace of $\Box_{p,q}$ corresponding to the eigenvalue $\lambda \in \sigma_{p,q}(\Omega_j)$. An argument similar to the one above shows also that

Corollary 5.3. *We also have on Ω:*

$$\Box_{\boldsymbol{p},\boldsymbol{q}} = \bigoplus_{\substack{\sum_{j=1}^N p_j = \boldsymbol{p} \\ \sum_{j=1}^N q_j = \boldsymbol{q}}} \sum_{\lambda_j \in \sigma_{p_j,q_j}(\Omega_j)} (\lambda_1 + \cdots + \lambda_N)\, \pi_{\lambda_1}^{1,(p_1,q_1)} \widehat{\otimes} \ldots \widehat{\otimes} \pi_{\lambda_N}^{N,(p_N,q_N)}.$$

5.2. **Example: Polydomains.** We now consider the special case in which each Ω_j is a bounded domain in the complex plane $\mathbb{C}$ with smooth boundary, so that the pseudoconvex domain $\Omega = \Omega_1 \times \cdots \times \Omega_N \subset \mathbb{C}^N$ is a so-called *polydomain*. We consider the spectrum of $\Box_{0,q}$ on Ω. For convenience we will write $\Box_q$ for $\Box_{0,q}$, and $\sigma_q(\Omega_j)$ for $\sigma_{0,q}(\Omega_j)$.

Note that the spectrum $\sigma(\Omega_j)$ of $\Box^j$ consists of eigenvalues only. Indeed $\Box_1^j$ is the same as the usual Laplacian with Dirichlet boundary conditions, which is well-known to have a compact inverse, the Green operator $\mathcal{G}$. We can write the eigenvalues in $\sigma_1(\Omega_j)$ as an increasing sequence

$$0 < \mu_1^j \leq \mu_2^j \leq \ldots,$$

where we repeat each eigenvalue according to its (finite) multiplicity, and let $Y_k^j(z_j)d\overline{z_j}$ denote an eigenform of $\Box_1^j$ corresponding to the eigenvalue μ_k^j, where z_j denotes the natural coordinate on Ω_j, and the eigenforms are so chosen for each eigenvalue with multiplicity that the collection $\{Y_k^j\}_{k\in\mathbb{N}}$ is a complete orthogonal set in $L^2(\Omega_j)$.

Now $u \in \Box_0^j \cap \mathcal{C}^1(\overline{\Omega_j})$ means that $\overline{\partial} u = 0$ on $\mathsf{b}\Omega_j$, by (9). For smooth f therefore, the equation $\Box_0^j u = f$ takes the form

$$\begin{cases} \Delta u = -4f & \text{on } \Omega_j, \\ \dfrac{\partial u}{\partial \overline{z}} = 0 & \text{on } \mathsf{b}\Omega_j. \end{cases}$$

If $v = \frac{\partial u}{\partial \overline{z}}$, this can be rewritten as the Dirichlet problem

$$\begin{cases} \Delta v = -4f_{\overline{z}} & \text{on } \Omega_j, \\ v = 0 & \text{on } \mathsf{b}\Omega_j, \end{cases}$$

where Δ is the usual Laplacian on $\mathbb{R}^2$. It is now easy to see that

$$u = \frac{1}{\pi z} * \left(\mathcal{G}(-4f_{\overline{z}})\right)^0 + h,$$

where h is an L^2 holomorphic function, $\mathcal{G}$ is the (compact) solution operator of the Dirichlet Laplacian, $g \mapsto g^0$ is the extension-by-zero of a function on Ω_j to $\mathbb{C}$, and $\frac{1}{\pi z}$ is the fundamental solution of the $\overline{\partial}$-equation on $\mathbb{C}$. It easily follows that the inverse modulo the kernel of $\Box_0^j$ is compact; i.e. the restriction of $\Box_0^j$ to $\ker(\Box_0^j)^\perp$ has compact inverse. $\sigma_0(\Omega_j)$, the spectrum of $\Box_0^j$, consists of eigenvalues only, which can be written in ascending order as

$$0 = \lambda_0^j < \lambda_1^j \leq \lambda_2^j \leq \ldots,$$

where the positive eigenvalues are of finite multiplicity and they are repeated according to their multiplicity. As noted above, the eigenspace corresponding to the eigenvalue $\lambda_0^j = 0$ is the Bergman space of L^2-holomorphic functions on Ω_j, and we let $\{H_k^j\}_{k\in\mathbb{N}}$ be a complete orthogonal set in the Bergman space $L^2(\Omega_j) \cap \mathcal{O}(\Omega_j)$. For $k \geq 1$, let Z_k^j be an eigenfunction of $\Box_0^j$ corresponding to the eigenvalue λ_k^j.

We can again assume that these have been chosen such that the family $\{Z_k^j\}_{k\in\mathbb{N}}$ is a complete orthogonal set in $\ker(\Box_0^j)^\perp$.

For a subset J of $\{1,\dots,n\}$ of cardinality q, where $J=\{j_1,\dots,j_q\}$ with $j_1<\cdots<j_q$, we write $d\overline{z}^J = d\overline{z}_{j_1}\wedge\cdots\wedge d\overline{z}_{j_q}$, with the understanding that $d\overline{z}^\emptyset = 1$. We also use the standard convention that a sum over an empty set is 0. With these notational preliminaries, Corollary 5.3 gives rise to the following description of the eigenstructure of the operator $\Box_q$ on Ω:

Proposition 5.4. *Let J be a subset of $\{1,\dots,n\}$ of cardinality q, and let $\mathbf{k}=(k_1,\dots,k_n)\in\mathbb{N}_+^n$ be an n-tuple of positive integers. Then*

$$\mu(J,\mathbf{k}) = \sum_{j\in J}\mu_{k_j}^j$$

is an eigenvalue of $\Box_q$, and

$$W(J,\mathbf{k}) = \left(\prod_{j\in J} Y_{k_j}^j(z_j)\prod_{j\notin J} H_{k_j}^j(z_j)\right) d\overline{z}^J$$

is an eigenform corresponding to this eigenvalue. Further, if $q<n$, then

$$\lambda(J,\mathbf{k}) = \sum_{j\in J}\mu_{k_j}^j + \sum_{j\notin J}\lambda_{k_j}^j$$

is also an eigenvalue of $\Box_q$, with eigenform

$$V(J,\mathbf{k}) = \left(\prod_{j\in J} Y_{k_j}^j(z_j)\prod_{j\notin J} Z_{k_j}^j(z_j)\right) d\overline{z}^J.$$

Moreover, this is the complete list of eigenvalues and eigenforms of $\Box_q$ as J ranges over all subsets of $\{1,\dots,n\}$ of size q and $\mathbf{k}$ ranges over $\mathbb{N}_+^n$ and gives the full spectral decomposition of $\Box_q$.

If $q<n$, the eigenvalue $\mu(J,\mathbf{k})$ has infinite multiplicity, since there are infinitely many $\mathbf{k}$ corresponding to the same eigenvalue, and for distinct $\mathbf{k}$ we have distinct eigenforms $W(J,\mathbf{k})$. If $q=n$ on the other hand, all the eigenvalues $\mu(J,\mathbf{k})$ are of finite multiplicity, as one would expect from the Dirichlet problem in a bounded domain. Since $\Box_q$ has eigenvalues of infinite multiplicity for $q<n$, it immediately follows that for $0<q<n$, the inverse of $\Box_q$, the $\overline{\partial}$-Neumann operator N_q, is noncompact.

The special case of Proposition 5.4 when $\Omega_j=\{z\in\mathbb{C}\mid |z|<a_j\}$ for some $a_j>0$ (so that Ω is a polydisc) was obtained in the paper [12]. In this case, the functions Y_k^j and Z_k^j have explicit representations in terms of Bessel functions.

Acknowledgements

The author wishes to express his gratitude to Professor Mei-Chi Shaw for her support and advice, and to the referee for helpful suggestions.

References

[1] Bertrams, Julia: Randregularität von Lösungen der $\overline{\partial}$-Gleichung auf dem Polyzylinder und zweidimensionalen analytischen Polyedern. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1986. MR867085 (88b:32044)

[2] Brüning, J. and Lesch, M.: Hilbert complexes. *J. Funct. Anal.* **108** (1992), no. 1, 88–132. MR1174159 (93k:58208)

[3] Chakrabarti, Debraj and Shaw, Mei-Chi: The Cauchy-Riemann equations on a product domain. Preprint. Available at `arXiv:0911.0103`

[4] Cheeger, Jeff: On the Hodge theory of Riemannian pseudomanifolds. *Proc. Sympos. Pure Math.* **36**, Amer. Math. Soc, Providence, RI, 1980, 91-145. MR573430 (83a:58081)

[5] Cheeger, Jeff: Spectral geometry of singular Riemannian spaces. *J. Differential Geom.* **18**, no. 4 (1983), 575-657. MR730920 (85d:58083)

[6] Chen, So-Chin and Shaw, Mei-Chi: *Partial differential equations in several complex variables.* AMS/IP Studies in Advanced Mathematics, 19. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001. MR1800297 (2001m:32071)

[7] de Rham, Georges: *Variétés différentiables. Formes, courants, formes harmoniques.* Hermann et Cie, Paris, 1955. MR0068889 (16:957b)

[8] Ehsani, Dariush: Solution of the $\overline{\partial}$-Neumann problem on a non-smooth domain. *Indiana Univ. Math. J.* **52** (2003), no. 3, 629–666. MR1986891 (2004f:32051)

[9] Ehsani, Dariush: Solution of the $\overline{\partial}$-Neumann problem on a bi-disc. *Math. Res. Lett.* **10** (2003), no. 4, 523–533. MR1995791 (2004f:32052)

[10] Ehsani, Dariush: The $\overline{\partial}$-Neumann problem on product domains in $\mathbb{C}^n$. *Math. Ann.* **337** (2007), 797–816. MR2285738 (2007j:32041)

[11] Folland, Gerald B. and Kohn, Joseph J.: *The Neumann problem for the Cauchy-Riemann complex.* Annals of Mathematics Studies, No. 75. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1972. MR0461588 (57:1573)

[12] Fu, Siqi: Spectrum of the $\overline{\partial}$-Neumann Laplacian on polydiscs. *Proc. Amer. Math. Soc.* **135** (2007), no. 3, 725–730. MR2262868 (2007h:32056)

[13] Halmos, P. R: What does the spectral theorem say? *Amer. Math. Monthly* **70** (1963), 241–247. MR0150600 (27:595)

[14] Reed, Michael and Simon, Barry: *Methods of modern mathematical physics, Vol. I: Functional analysis.* Second edition. Academic Press, Inc., New York, 1980. MR751959 (85e:46002)

[15] Shaw, Mei-Chi: Global solvability and regularity for $\overline{\partial}$ on an annulus between two weakly pseudoconvex domains. *Trans. Amer. Math. Soc.* **291** (1985), 255–267. MR797058 (86m:32030)

[16] Shaw, Mei-Chi: The closed range property for $\overline{\partial}$, on domains with pseudoconcave boundary. *Proceedings of the Complex Analysis Conference, Fribourg, Switzerland* (2008).

[17] Teschl, Gerald: *Mathematical methods in quantum mechanics. With applications to Schrödinger operators.* Graduate Studies in Mathematics, 99, American Mathematical Society, Providence, RI, 2009. MR2499016

[18] Zucker, Steven: L_2 cohomology of warped products and arithmetic groups. *Invent. Math.* **70** (1982), no. 2, 169–218. MR684171 (86j:32063)

Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556

E-mail address: `dchakrab@nd.edu`