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ARITHMETICAL RANK OF TORIC IDEALS ASSOCIATED TO GRAPHS

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Abstract. Let $I_G \subset K[x_1, \ldots, x_m]$ be the toric ideal associated to a finite graph G. In this paper we study the binomial arithmetical rank and the G-homogeneous arithmetical rank of I_G in 2 cases:

(1) G is bipartite,
(2) I_G is generated by quadratic binomials.

In both cases we prove that the binomial arithmetical rank and the G-homogeneous arithmeticaal rank coincide with the minimal number of generators of I_G.

1. Introduction

Let G be a finite, connected and undirected graph having no loops and no multiple edges on the vertex set $V(G) = \{v_1, \ldots, v_n\}$, and let $E(G) = \{e_1, \ldots, e_m\}$ be the set of edges of G. The *incidence matrix* of G is the $n \times m$ matrix $M_G = (a_{i,j})$ defined by

$$a_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ is one of the vertices in } e_j, \\ 0, & \text{otherwise.} \end{cases}$$

Let K be an algebraically closed field, and let $A_G = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$ be the set of vectors in $\mathbb{Z}^n$, where $\mathbf{a}_i = (a_{1,i}, \ldots, a_{n,i})$ for $1 \leq i \leq m$. The *toric ideal* I_G associated to G is the kernel of the K-algebra homomorphism

$$\phi : K[x_1, \ldots, x_m] \to K[t_1, \ldots, t_n]$$

given by

$$\phi(x_i) = t_1^{a_{1,i}} \cdots t_n^{a_{n,i}} \qquad \text{for all } i = 1, \ldots, m.$$

The ideal I_G is prime, and therefore $rad(I_G) = I_G$. The *toric variety* $\mathbb{V}(I_G)$ associated to G is the set

$$\{(u_1, \ldots, u_m) \in K^m | F(u_1, \ldots, u_m) = 0, \forall F \in I_G\}$$

of zeroes of I_G. For every graph G the variety $\mathbb{V}(I_G)$ is an *extremal toric variety*, i.e. the vector configuration A_G is extremal; see Remark 2.1.

The polynomial ring $K[x_1, \ldots, x_m]$ has a natural G-graded structure given by setting $\deg_G(x_i) = \mathbf{a}_i$ for $i = 1, \ldots, m$. For $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the

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G-degree of the monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$\deg_G(\mathbf{x}^{\mathbf{u}}) := u_1\mathbf{a}_1 + \cdots + u_m\mathbf{a}_m \in \mathbb{N}A_G,$$

where $\mathbb{N}A_G$ is the semigroup generated by A_G. Remark that $\mathbb{N}A_G$ is *pointed*, i.e. zero is the only invertible element. A polynomial $F \in K[x_1, \ldots, x_m]$ is called *G-homogeneous* if the monomials in nonzero terms of F have the same G-degree. An ideal I is G-homogeneous if it is generated by G-homogeneous polynomials.

The toric ideal I_G is generated by all the binomials $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ such that

$$\deg_G(\mathbf{x}^{\mathbf{u}_+}) = \deg_G(\mathbf{x}^{\mathbf{u}_-}),$$

where $\mathbf{u}_+ \in \mathbb{N}^m$ and $\mathbf{u}_- \in \mathbb{N}^m$ denote the positive and negative part of $\mathbf{u} = \mathbf{u}_+ - \mathbf{u}_-$, respectively (see [11]).

A basic problem in commutative algebra asks one to compute the smallest integer s for which there exist polynomials $F_1, \ldots, F_s$ in I_G such that $I_G = rad(F_1, \ldots, F_s)$. This integer is called the *arithmetical rank* of I_G and will be denoted by $\mathrm{ara}(I_G)$. An usual approach to this problem is to restrict to a certain class of polynomials and ask how many polynomials from this class can generate the toric ideal up to radical. Restricting the polynomials to the class of binomials we arrive at the notion of the *binomial arithmetical rank* of I_G, denoted by $\mathrm{bar}(I_G)$. Also, if all of the polynomials $F_1, \ldots, F_s$ satisfying $I_G = rad(F_1, \ldots, F_s)$ are G-homogeneous, the smallest integer s is called *the G-homogeneous arithmetical rank* of I_G and will be denoted by $\mathrm{ara}_G(I_G)$. From the definitions, the generalized Krull's principal ideal theorem and the graded version of Nakayama's Lemma, we deduce the following inequalities for a toric ideal I_G:

$$\mathrm{ht}(I_G) \leq \mathrm{ara}(I_G) \leq \mathrm{ara}_G(I_G) \leq \mathrm{bar}(I_G) \leq \mu(I_G).$$

Here $\mathrm{ht}(I_G)$ denotes the height and $\mu(I_G)$ denotes the minimal number of generators of I_G. When $\mathrm{ht}(I_G) = \mu(I_G)$ the ideal I_G is called a *complete intersection*.

A case of particular interest is when $\mathrm{bar}(I_G) = \mathrm{ht}(I_G)$. When K is a field of characteristic zero, this is equivalent to saying that I_G is complete intersection; see [1]. Complete intersection bipartite graphs have been characterized in [4], [8]. In most cases, when G is bipartite, the equality $\mathrm{bar}(I_G) = \mathrm{ht}(I_G)$ does not hold. In section 3 we prove that $\mathrm{bar}(I_G) = \mathrm{ara}_G(I_G) = \mu(I_G)$ for any bipartite graph G. In addition we show that the equality $\mathrm{bar}(I_G) = \mathrm{ara}_G(I_G) = \mu(I_G)$ also holds for any graph G such that the toric ideal I_G is generated by quadratic binomials.

2. Basics on toric ideals associated to graphs

Let G be a graph. A *walk* of length q of G is a finite sequence of the form

$$\Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{q-1}, v_q\}, \{v_q, v_{q+1}\});$$

this walk is *closed* if $v_{q+1} = v_1$. An *even* closed walk is a closed walk of even length. A *cycle* of G is a closed walk

$$\Gamma = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\})$$

with $v_i \neq v_j$ for all $1 \leq i < j \leq q$. Notice that if $e_i = \{v_{i_k}, v_{i_l}\}$ is an edge of G, then $\phi(x_i) = t_{i_k} t_{i_l}$. Given an even closed walk $\Gamma = (e_{i_1}, \ldots, e_{i_{2q}})$ of G with each $e_k \in E(G)$, we have that

$$\phi(\prod_{k=1}^{q} x_{i_{2k-1}}) = \phi(\prod_{k=1}^{q} x_{i_{2k}}),$$

and therefore the binomial

$$f_\Gamma := \prod_{k=1}^{q} x_{i_{2k-1}} - \prod_{k=1}^{q} x_{i_{2k}}$$

belongs to I_G. Remark that if Γ is an even cycle of G, then the monomials $M = \prod_{k=1}^{q} x_{i_{2k-1}}$ and $N = \prod_{k=1}^{q} x_{i_{2k}}$ are squarefree. From Proposition 3.1 in [12] we have that

$$I_G = (\{f_\Gamma | \Gamma \text{ is an even closed walk of } G\}).$$

Let $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{Z}^m$ be a vector. Then the *support* of $\mathbf{u}$, denoted by $\mathrm{supp}(\mathbf{u})$, is the set $\{i \in \{1, \ldots, m\} | u_i \neq 0\}$. For a monomial $\mathbf{x}^{\mathbf{u}}$ we define $\mathrm{supp}(\mathbf{x}^{\mathbf{u}}) := \mathrm{supp}(\mathbf{u})$. A non-zero vector $\mathbf{u} = (u_1, \ldots, u_m) \in \ker_{\mathbb{Z}}(M_G)$ is called a *circuit* of A_G if its support is minimal with respect to inclusion and all the coordinates of $\mathbf{u}$ are relatively prime, where $\ker_{\mathbb{Z}}(M_G) = \{\mathbf{v} \in \mathbb{Z}^m | M_G \mathbf{v}^t = \mathbf{0}^t\}$. The binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ associated to a vector $\mathbf{u} \in \ker_{\mathbb{Z}}(M_G)$ is also called a circuit. A binomial $B = \mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ is called *primitive* if there exists no other binomial $\mathbf{x}^{\mathbf{v}_+} - \mathbf{x}^{\mathbf{v}_-} \in I_G$ such that $\mathbf{x}^{\mathbf{v}_+}$ divides $\mathbf{x}^{\mathbf{u}_+}$ and $\mathbf{x}^{\mathbf{v}_-}$ divides $\mathbf{x}^{\mathbf{u}_-}$. For a circuit $B = \mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ we have, from Corollary 8.1.4 in [13], that $B = f_\Gamma$ for an even closed walk Γ of G, since every circuit is also primitive.

If G is a bipartite graph, then G has no odd cycles, so, from Proposition 4.2 in [12], a binomial f_Γ, where Γ is an even closed walk of G, is a circuit if and only if Γ is an even cycle.

For the rest of this section we recall some fundamental material from [7].

We shall denote by $\mathcal{C}_G$ the set of circuits of A_G. Let

$$\mathcal{C} := \{E \subset \{1, \ldots, m\} \mid \mathrm{supp}(\mathbf{u}_+) = E \text{ or } \mathrm{supp}(\mathbf{u}_-) = E \text{ where } \mathbf{u} \in \mathcal{C}_G\},$$

and let $\mathcal{C}_{min}$ be the set of minimal elements of $\mathcal{C}$.

To every toric ideal I_G we associate the rational polyhedral cone

$$\sigma = pos_{\mathbb{Q}}(A_G) := \{\lambda_1 \mathbf{a}_1 + \cdots + \lambda_m \mathbf{a}_m \mid \lambda_i \in \mathbb{Q}_{\geq 0}\}.$$

A *face* $\mathcal{F}$ of σ is any set of the form

$$\mathcal{F} = \sigma \cap \{\mathbf{x} \in \mathbb{Q}^n : \mathbf{c}\mathbf{x} = 0\},$$

where $\mathbf{c} \in \mathbb{Q}^n$ and $\mathbf{c}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \sigma$. Given an edge $e_i = \{v_{i_1}, v_{i_2}\}$ of G, we have that $pos_{\mathbb{Q}}(\mathbf{a}_i)$ is a face of σ with defining vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ having coordinates

$$c_j = \begin{cases} 0, & \text{if } j = i_1, i_2, \\ 1, & \text{otherwise.} \end{cases}$$

Remark 2.1. For every graph G the vector configuration A_G is extremal, i.e. for any $B \subsetneqq A_G$ we have $pos_{\mathbb{Q}}(B) \subsetneqq pos_{\mathbb{Q}}(A_G)$.

To see this consider a set $B = \{\mathbf{a}_{i_1}, \ldots, \mathbf{a}_{i_k}\} \subsetneqq A_G$ and assume that the vector $\mathbf{a}_j$ is not in B. Let $\mathbf{c}$ be the defining vector of the face $pos_{\mathbb{Q}}(\mathbf{a}_j)$. If $\mathbf{a}_j$ belongs to $pos_{\mathbb{Q}}(B)$, then $\mathbf{a}_j = \lambda_1 \mathbf{a}_{i_1} + \cdots + \lambda_k \mathbf{a}_{i_k}$, where $\lambda_1, \ldots, \lambda_k$ are nonnegative rationals and there is at least one λ_r different from zero. Thus $\mathbf{c}\mathbf{a}_j = \lambda_1(\mathbf{c}\mathbf{a}_{i_1}) + \cdots + \lambda_k(\mathbf{c}\mathbf{a}_{i_k})$, and therefore $0 = \lambda_1(\mathbf{c}\mathbf{a}_{i_1}) + \cdots + \lambda_k(\mathbf{c}\mathbf{a}_{i_k})$. But

$$\lambda_1(\mathbf{c}\mathbf{a}_{i_1}) + \cdots + \lambda_k(\mathbf{c}\mathbf{a}_{i_k}) > 0,$$

a contradiction. Consequently $\mathbf{a}_j$ does not belong to $pos_{\mathbb{Q}}(B)$, so $pos_{\mathbb{Q}}(B)$ is a proper subset of $pos_{\mathbb{Q}}(A_G)$.

For a subset E of $\{1,\dots,m\}$ we denote by σ_E the subcone $pos_{\mathbb{Q}}(\mathbf{a}_i | i \in E)$ of σ. We adopt the convention that $\sigma_\emptyset = \{\mathbf{0}\}$. The *relative interior* of σ_E, denoted by $relint_{\mathbb{Q}}(\sigma_E)$, is the set of all strictly positive rational linear combinations of $\mathbf{a}_i$, $i \in E$.

Definition 2.2 ([7])**.** We associate to G the simplicial complex Δ_G with vertices the elements of $\mathcal{C}_{min}$. Let $T \subset \mathcal{C}_{min}$; then $T \in \Delta_G$ if

$$\bigcap_{E \in T} relint_{\mathbb{Q}}\left(\sigma_E\right) \neq \emptyset.$$

In particular, $\{E, E'\} \in \Delta_G$ if and only if there exists a circuit $\mathbf{u} \in \mathcal{C}_G$ such that $\mathrm{supp}(\mathbf{u}_+) = E$ and $\mathrm{supp}(\mathbf{u}_-) = E'$.

Let J be a subset of $\Omega := \{0, 1, \dots, \dim(\Delta_G)\}$. A set $\mathcal{M} = \{T_1, \dots, T_s\}$ of simplices of Δ_G is called a *J-matching* in Δ_G if $T_k \cap T_l = \emptyset$ for every $1 \leq k, l \leq s$ and $\dim(T_k) \in J$ for every $1 \leq k \leq s$; see also Defnition 2.1 in [7]. Let $\mathrm{supp}(\mathcal{M}) = \bigcup_{i=1}^{s} T_i$, which is a subset of the vertices $\mathcal{C}_{min}$. A J-matching $\mathcal{M}$ in Δ_G is called a *maximal* J-matching if $\mathrm{supp}(\mathcal{M})$ has the maximum possible cardinality among all J-matchings.

Given a maximal J-matching $\mathcal{M} = \{T_1, \dots, T_s\}$ in Δ_G, we shall denote by $card(\mathcal{M})$ the cardinality s of the set $\mathcal{M}$. In addition, by $\delta(\Delta_G)_J$ we denote the minimum of the set

$$\{card(\mathcal{M}) | \mathcal{M} \text{ is a maximal } J\text{-matching in } \Delta_G\}.$$

It follows from the definitions that if $\Delta_G = \bigcup_{i=1}^{t} \Delta_G^i$, then

$$\delta(\Delta_G)_J = \sum_{i=1}^{t} \delta(\Delta_G^i)_J,$$

where Δ_G^i are the connected components of Δ_G.

Example 2.3. Consider the complete graph $\mathcal{K}_4$ on the vertex set $\{v_1, \dots, v_4\}$. We consider one variable x_{ij}, $1 \leq i < j \leq 4$, for each edge $\{v_i, v_j\}$ of $\mathcal{K}_4$ and form the polynomial ring $K[x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}]$. From Proposition 4.2 in [12] we have that the toric ideal $I_{\mathcal{K}_4}$ has 3 circuits, namely $f_{\Gamma_1} = x_{12}x_{34} - x_{14}x_{23}$, $f_{\Gamma_2} = x_{12}x_{34} - x_{13}x_{24}$ and $f_{\Gamma_3} = x_{13}x_{24} - x_{14}x_{23}$, corresponding to the 3 even cycles Γ_1, Γ_2 and Γ_3, respectively, of $\mathcal{K}_4$. In fact $I_{\mathcal{K}_4}$ is minimally generated by two of the above binomials, so it is a complete intersection of height 2. The simplicial complex $\Delta_{\mathcal{K}_4}$ has three vertices, namely $E_1 = \{12, 34\}$, $E_2 = \{14, 23\}$ and $E_3 = \{13, 24\}$. It consists of all subsets of the set $\{E_1, E_2, E_3\}$. There are four maximal $\{0,1\}$-matchings in $\Delta_{\mathcal{K}_4}$, namely $\{\{E_1, E_2\}, \{E_3\}\}$, $\{\{E_1, E_3\}, \{E_2\}\}$, $\{\{E_2, E_3\}, \{E_1\}\}$ and $\{\{E_1\}, \{E_2\}, \{E_3\}\}$. We have that $\delta(\Delta_{\mathcal{K}_4})_{\{0,1\}} = 2$, which is attained for example by the maximal $\{0,1\}$-matching $\{\{E_1, E_2\}, \{E_3\}\}$. In addition, $\delta(\Delta_{\mathcal{K}_4})_{\{0,1,2\}} = 1$, which is attained by the maximal $\{0,1,2\}$-matching $\{\{E_1, E_2, E_3\}\}$.

Using the fact that A_G is an extremal vector configuration and also using two results from [7], namely Theorem 4.6 and Theorem 3.5, we get the following theorem:

Theorem 2.4. *For a toric ideal I_G we have $\delta(\Delta_G)_{\{0,1\}} \leq \mathrm{bar}(I_G)$ and $\delta(\Delta_G)_\Omega \leq \mathrm{ara}_G(I_G)$.*

3. Arithmetical rank

Let $I_G \subset K[x_1, \ldots, x_m]$ be the toric ideal associated to a graph G. A binomial $B \in I_G$ is called *indispensable* if every system of binomial generators of I_G contains B or $-B$, while a monomial M is called *indispensable* if every system of binomial generators of I_G contains a binomial B such that M is a monomial of B. Let $\mathcal{N}_G$ be the monomial ideal generated by all $\mathbf{x}^{\mathbf{u}}$ for which there exists a nonzero $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{w}} \in I_G$. From Proposition 3.1 in [2] we have that the set of indispensable monomials is the unique minimal generating set of $\mathcal{N}_G$. The following lemma will be useful in the proof of Theorem 3.2 and Proposition 3.4.

Lemma 3.1. *Assume that either G is a bipartite graph or I_G is generated by quadratic binomials. Let $T_G = \{M_1, \ldots, M_r\}$ be the set of indispensable monomials; then $\mathcal{C}_{min} = \{\mathrm{supp}(M_1), \ldots, \mathrm{supp}(M_r)\}$.*

Proof. First consider the case where G is a bipartite graph. From Theorem 3.2 in [8] we have that I_G is minimally generated by all binomials of the form f_Γ, where Γ is an even cycle of G with no chord. Combining the above theorem and Theorem 2.3 in [10], we obtain that a binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ is indispensable if and only if it is of the form f_Γ, for an even cycle of G with no chord. Notice that in some cases there are circuits of I_G of the form f_Γ, for an even cycle Γ of G with a chord. If $B_1, \ldots, B_s$ are the indispensable binomials of I_G, then the toric ideal I_G is generated by the indispensable binomials. In addition, the monomials of B_i, $1 \leq i \leq s$, are all indispensable and also they form T_G. We will prove that

$$\mathcal{C}_{min} \subset \{\mathrm{supp}(M_1), \ldots, \mathrm{supp}(M_r)\}.$$

Let $E \in \mathcal{C}_{min}$ and let $\sigma = pos_{\mathbb{Q}}(A_G)$. From Theorem 4.6 in [7] the simplicial complexes Δ_G and $\mathcal{D}_\sigma$ are identical; see [7] for the definition of the last complex. Using the fact that I_G is generated by the binomials $B_1, \ldots, B_s$ and Corollary 5.7 in [5], we have that there is a monomial M_i such that $\mathrm{cone}(M_i) = \sigma_E$. For the definition and results about the cone of a monomial, see [5]. But $\mathrm{cone}(M_i) = \sigma_{\mathrm{supp}(M_i)}$, since in this case all vectors belong to an extreme ray of σ, so $\sigma_E = \sigma_{\mathrm{supp}(M_i)}$ and therefore $E = \mathrm{supp}(M_i)$. Thus

$$\mathcal{C}_{min} \subset \{\mathrm{supp}(M_1), \ldots, \mathrm{supp}(M_r)\}.$$

Now consider a set $E = \mathrm{supp}(M_i)$, $1 \leq i \leq r$, and we will prove that it also belongs to $\mathcal{C}_{min}$. Suppose not; then there is an $E' \subsetneqq E$ such that $E' = \mathrm{supp}(\mathbf{x}^{\mathbf{u}_+})$ or $E' = \mathrm{supp}(\mathbf{x}^{\mathbf{u}_-})$, where $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ is a circuit. Without loss of generality we can assume that $E' = \mathrm{supp}(\mathbf{x}^{\mathbf{u}_+})$. The monomials $\mathbf{x}^{\mathbf{u}_+}, \mathbf{x}^{\mathbf{u}_-}$ are squarefree and also $\mathbf{x}^{\mathbf{u}_+}$ divides M_i, since $E' \subsetneqq E$, a contradiction to the fact that M_i is indispensable.

Assume now that I_G is generated by quadratic binomials. Let $\{B_1, \ldots, B_s\}$ be a quadratic set of generators of I_G, and let S_G be the set of monomials appearing in the binomials $B_1, \ldots, B_s$. We will prove that S_G coincides with T_G; i.e. S_G is the minimal generating set of the ideal $\mathcal{N}_G$. Every monomial N of S_G belongs to the ideal $\mathcal{N}_G$. On the other hand, for a monomial $\mathbf{x}^{\mathbf{u}} \in \mathcal{N}_G$ there exists a monomial $\mathbf{x}^{\mathbf{w}}$ such that $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{w}} \in I_G$. But $I_G = (B_1, \ldots, B_s)$, so there is a monomial $N' \in S_G$ which divides $\mathbf{x}^{\mathbf{u}}$, and therefore S_G is a set of generators for the ideal $\mathcal{N}_G$. In addition S_G is a minimal generating set, since every monomial N of S_G is quadratic. Thus S_G is the set of indispensable monomials.

Using the fact that I_G is generated by the binomials $B_1, \ldots, B_s$ and Corollary 5.7 in [5], we can easily prove that

$$\mathcal{C}_{min} \subset \{\mathrm{supp}(M_1), \ldots, \mathrm{supp}(M_r)\}.$$

It remains to prove that

$$\{\mathrm{supp}(M_1), \ldots, \mathrm{supp}(M_r)\} \subset \mathcal{C}_{min}.$$

Let $E = \mathrm{supp}(M_i)$, $1 \leq i \leq r$, and assume that E does not belong to $\mathcal{C}_{min}$. Then there is an $E' \subsetneqq E$, i.e. $E' = \{i\}$ is a singleton, and a circuit $x_i^{g_i} - N_i \in I_G$ such that $E' = \mathrm{supp}(x_i^{g_i})$ and $E' \cap \mathrm{supp}(N_i) = \emptyset$. Let $R = A_G - \{\mathbf{a}_i\} \subsetneqq A_G$. Then $\deg_G(N_i)$ belongs to $pos_{\mathbb{Q}}(R)$, so $g_i\mathbf{a}_i$ also belongs to $pos_{\mathbb{Q}}(R)$, since $g_i\mathbf{a}_i = \deg_G(N_i)$, and therefore $\mathbf{a}_i \in pos_{\mathbb{Q}}(R)$. Thus $pos_{\mathbb{Q}}(A_G) = pos_{\mathbb{Q}}(R)$, a contradiction to the fact that A_G is an extremal vector configuration. □

Let $\mathcal{F} \subset I_G$ be a set of binomials. We shall denote by $S(\mathbf{b})_{\mathcal{F}}$ the graph with vertices the elements of $\deg_G^{-1}(\mathbf{b}) = \{\mathbf{x}^{\mathbf{u}} \mid \deg_G(\mathbf{x}^{\mathbf{u}}) = \mathbf{b}\}$ and edges the sets $\{\mathbf{x}^{\mathbf{u}}, \mathbf{x}^{\mathbf{v}}\}$ whenever $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is a monomial multiple of a binomial in $\mathcal{F}$. The next theorem computes the binomial arithmetical rank and the G-homogeneous arithmetical rank of I_G for a bipartite graph G.

Theorem 3.2. *Let G be a bipartite graph; then* $\mathrm{bar}(I_G) = \mu(I_G)$ *and* $\mathrm{ara}_G(I_G) = \mu(I_G)$.

Proof. First we will prove that $\{E, E'\}$ is an edge of Δ_G if and only if there is an indispensable binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ with $\mathrm{supp}(\mathbf{u}_+) = E$ and $\mathrm{supp}(\mathbf{u}_-) = E'$. The one implication is easy. Let $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ be an indispensable binomial with $\mathrm{supp}(\mathbf{u}_+) = E$ and $\mathrm{supp}(\mathbf{u}_-) = E'$. Then $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} = f_\Gamma$ for an even cycle Γ of G with no chord. But f_Γ is a circuit, and therefore $\{E, E'\}$ is an edge. Conversely, consider an edge $\{E, E'\}$ of Δ_G; then there is a circuit $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ such that $\mathrm{supp}(\mathbf{u}_+) = E$ and $\mathrm{supp}(\mathbf{u}_-) = E'$. Let $T_G = \{M_1, \ldots, M_r\}$ be the set of indispensable monomials. Then, from Lemma 3.1, there are indispensable monomials M_i, M_j such that $E = \mathrm{supp}(M_i)$ and $E' = \mathrm{supp}(M_j)$. But $\mathbf{x}^{\mathbf{u}_+}$, $\mathbf{x}^{\mathbf{u}_-}$ are squarefree and also M_i, M_j are minimal generators of $\mathcal{N}_G$, so $\mathbf{x}^{\mathbf{u}_+} = M_i$, $\mathbf{x}^{\mathbf{u}_-} = M_j$ and therefore both monomials $\mathbf{x}^{\mathbf{u}_+}$, $\mathbf{x}^{\mathbf{u}_-}$ are indispensable. Let

$$\mathbf{b} = \deg_G(\mathbf{x}^{\mathbf{u}_+}) = \deg_G(\mathbf{x}^{\mathbf{u}_-}).$$

If $B_1, \ldots, B_s$ are the indispensable binomials of I_G, then $\mathcal{F} := \{B_1, \ldots, B_s\}$ is a generating set of I_G, and therefore the graph $S(\mathbf{b})_{\mathcal{F}}$ is connected; see Theorem 3.2 in [3]. Suppose that $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ is not indispensable. Then there exists a path

$$(\{\mathbf{x}^{\mathbf{u}_+} = \mathbf{x}^{\mathbf{u}_0}, \mathbf{x}^{\mathbf{u}_1}\}, \{\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2}\}, \ldots, \{\mathbf{x}^{\mathbf{u}_{t-1}}, \mathbf{x}^{\mathbf{u}_t} = \mathbf{x}^{\mathbf{u}_-}\}),\ t \geq 2,$$

in $S(\mathbf{b})_{\mathcal{F}}$ connecting the vertices $\mathbf{x}^{\mathbf{u}_0}$ and $\mathbf{x}^{\mathbf{u}_t}$. Now consider the binomial $\mathbf{x}^{\mathbf{u}_0} - \mathbf{x}^{\mathbf{u}_1}$. There is a binomial B_i and a monomial P such that $\mathbf{x}^{\mathbf{u}_0} - \mathbf{x}^{\mathbf{u}_1} = PB_i$ since $\{\mathbf{x}^{\mathbf{u}_0}, \mathbf{x}^{\mathbf{u}_1}\}$ is an edge of $S(\mathbf{b})_{\mathcal{F}}$. If $B_i = \mathbf{x}^{\mathbf{w}_+} - \mathbf{x}^{\mathbf{w}_-}$, then $\mathbf{x}^{\mathbf{u}_0} = P\mathbf{x}^{\mathbf{w}_+}$ and therefore $\mathbf{x}^{\mathbf{w}_+}$ divides $\mathbf{x}^{\mathbf{u}_0}$. But $\mathbf{x}^{\mathbf{u}_0}$ is indispensable, so $P = 1$, and therefore the binomial $\mathbf{x}^{\mathbf{u}_0} - \mathbf{x}^{\mathbf{u}_1}$ is indispensable. Thus the monomial $\mathbf{x}^{\mathbf{u}_1}$ is indispensable. Moreover $\mathbf{x}^{\mathbf{u}_1} - \mathbf{x}^{\mathbf{u}_2}$ is indispensable, since $\{\mathbf{x}^{\mathbf{u}_1}, \mathbf{x}^{\mathbf{u}_2}\}$ is an edge of $S(\mathbf{b})_{\mathcal{F}}$ and $\mathbf{x}^{\mathbf{u}_1}$ is indispensable, as well as all the binomials $\mathbf{x}^{\mathbf{u}_{i-1}} - \mathbf{x}^{\mathbf{u}_i}$, $3 \leq i \leq t$. Consequently there are at least two indispensable binomials with the same G-degree, contradicting Theorem 3.4 in [2]. Recall that the G-degree of a binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_G$ is defined to be $\deg_G(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}) := \deg_G(\mathbf{x}^{\mathbf{u}})$.

Now, from Theorem 3.4 in [2], every edge of Δ_G constitutes a connected component. Remark that Δ_G has no connected components which are singletons, since $\mathcal{C}_{min} = \{\text{supp}(M_1), \ldots, \text{supp}(M_r)\}$ and I_G is generated by the indispensable binomials. Thus every connected component of Δ_G is an edge, so Δ_G has s connected components. Let $\Delta_G^i = \{E, E'\}$, $1 \leq i \leq s$, be a connected component of Δ_G. There are two maximal $\{0,1\}$-matchings in Δ_G^i, namely $\{\{E, E'\}\}$ and $\{\{E\}, \{E'\}\}$. We have that $\delta(\Delta_G^i)_{\{0,1\}} = 1$, which is attained by the maximal $\{0,1\}$-matching $\{\{E, E'\}\}$. Consequently,

$$\delta(\Delta_G)_{\{0,1\}} = \sum_{i=1}^{s} \delta(\Delta_G^i)_{\{0,1\}} = s,$$

i.e. $\delta(\Delta_G)_{\{0,1\}} = \mu(I_G)$. From Theorem 2.4 we have that $\text{bar}(I_G) \geq \mu(I_G)$ and therefore $\text{bar}(I_G) = \mu(I_G)$.

In addition, $\delta(\Delta_G)_\Omega = \delta(\Delta_G)_{\{0,1\}}$, since $\dim(\Delta_G) = 1$. So $\delta(\Delta_G)_\Omega = \mu(I_G)$, and therefore, from Theorem 2.4, we have that $\text{ara}_G(I_G) \geq \mu(I_G)$. Thus $\text{ara}_G(I_G) = \mu(I_G)$. □

Example 3.3. Let $G = \mathcal{K}_{3,3}$ be the complete bipartite graph on the vertex set $\{v_1, \ldots, v_6\}$ with 9 edges:

$$\{v_1, v_4\}, \{v_1, v_5\}, \{v_1, v_6\}, \{v_2, v_4\}, \{v_2, v_5\}, \{v_2, v_6\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}.$$

We consider one variable x_{ij}, $1 \leq i < j \leq 6$, for each edge $\{v_i, v_j\}$ of G and form the polynomial ring $K[x_{ij} | 1 \leq i < j \leq 6]$. The toric ideal I_G is minimally generated by 9 binomials:

$$\begin{gathered} x_{14}x_{26} - x_{16}x_{24}, x_{15}x_{36} - x_{16}x_{35}, x_{25}x_{36} - x_{26}x_{35}, \\ x_{24}x_{36} - x_{26}x_{34}, x_{14}x_{25} - x_{15}x_{24}, x_{15}x_{26} - x_{16}x_{25}, \\ x_{24}x_{35} - x_{25}x_{34}, x_{14}x_{36} - x_{16}x_{34}, x_{14}x_{35} - x_{15}x_{34}. \end{gathered}$$

The simplicial complex Δ_G has 18 vertices corresponding to the 18 monomials arising in the above minimal generating set of I_G and 9 edges corresponding to the 9 minimal generators of I_G. From Theorem 3.2 we have that $\text{ara}_G(I_G) = \text{bar}(I_G) = 9$. The height of I_G equals $9-6+1 = 4$; see Proposition 3.2 in [12]. For the arithmetical rank of I_G we have that $4 \leq \text{ara}(I_G) \leq 7$, since I_G equals the radical of the ideal generated by

$$\begin{gathered} x_{14}x_{26} - x_{16}x_{24} + x_{15}x_{36} - x_{16}x_{35}, x_{25}x_{36} - x_{26}x_{35} + x_{14}x_{25} - x_{15}x_{24}, \\ x_{24}x_{36} - x_{26}x_{34}, x_{15}x_{26} - x_{16}x_{25}, x_{24}x_{35} - x_{25}x_{34}, \\ x_{14}x_{36} - x_{16}x_{34}, x_{14}x_{35} - x_{15}x_{34}. \end{gathered}$$

An interesting case occurs when I_G is generated by quadratic binomials. In [9] a combinatorial criterion for the toric ideal I_G to be generated by quadratic binomials is studied. Remark that if $B = x_i x_j - x_k x_l$ is a quadratic binomial in I_G, then $B = f_\Gamma$ for an even cycle Γ of G of length 4. We are going to compute the binomial arithmetical rank and the G-homogeneous arithmetical rank of such an ideal.

We shall denote by $\Delta_{\text{ind}(A_G)}$ the indispensable complex of A_G. For the definition and results about the indispensable complex, see [2].

Proposition 3.4. *Let I_G be a toric ideal generated by quadratic binomials. Then*

(1) *$\{E, E'\}$ is a connected component of Δ_G if and only if there is an indispensable quadratic binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ with* $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_+}) = E$ *and* $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_-}) = E'$.
(2) *Every connected component of Δ_G is either an edge or a 2-simplex .*

Proof. (1) The first goal is to prove that $\{E, E'\}$ is an edge of Δ_G if and only if there is a quadratic binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ with $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_+}) = E$ and $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_-}) = E'$. The one implication follows from the fact that if $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ is a quadratic binomial, then $\operatorname{supp}(\mathbf{u})$ is minimal with respect to inclusion, since A_G is extremal, and therefore the binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ is a circuit. Conversely let $\{E, E'\}$ be an edge of Δ_G and let $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ be a circuit with $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_+}) = E$ and $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_-}) = E'$, where $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{Z}^m$. From Corollary 8.4.16 in [13] we have that $|u_i| \leq 2$. Let $\{B_1, \ldots, B_s\}$ be a quadratic set of generators of I_G, and let $T_G = \{M_1, \ldots, M_r\}$ be the indispensable monomials. We have that both E, E' belong to $\mathcal{C}_{min}$, and therefore, from Lemma 3.1, $E = \operatorname{supp}(M_i)$ and $E' = \operatorname{supp}(M_j)$. But M_i and M_j are quadratic monomials, so E and E' consist of exactly 2 elements. Let $E = \{k, l\}$ and $E' = \{p, q\}$. We will consider three cases:

(i) If the monomials $\mathbf{x}^{\mathbf{u}_+}$ and $\mathbf{x}^{\mathbf{u}_-}$ are squarefree, then $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-}$ is a quadratic binomial with $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_+}) = E$ and $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_-}) = E'$.
(ii) If $\mathbf{x}^{\mathbf{u}_+} = x_k^2 x_l^2$ and $\mathbf{x}^{\mathbf{u}_-} = x_p^2 x_q^2$, then $\deg_G(\mathbf{x}^{\mathbf{u}_+}) = \deg_G(\mathbf{x}^{\mathbf{u}_-})$, and therefore $2\mathbf{a}_k + 2\mathbf{a}_l = 2\mathbf{a}_p + 2\mathbf{a}_q$. Thus $\mathbf{a}_k + \mathbf{a}_l = \mathbf{a}_p + \mathbf{a}_q$, so the binomial $x_k x_l - x_p x_q$ belongs to I_G and also $\operatorname{supp}(x_k x_l) = E$, $\operatorname{supp}(x_p x_q) = E'$.
(iii) If $\mathbf{x}^{\mathbf{u}_+} = x_k^2 x_l$ and $\mathbf{x}^{\mathbf{u}_-} = x_p^2 x_q$, then $\deg_G(\mathbf{x}^{\mathbf{u}_+}) = \deg_G(\mathbf{x}^{\mathbf{u}_-})$, and therefore $2\mathbf{a}_k + \mathbf{a}_l = 2\mathbf{a}_p + \mathbf{a}_q$. Assume that $\mathbf{a}_k - \mathbf{a}_p \neq \mathbf{0}$. Notice that every nonzero coordinate of the above vector equals either 1 or -1. We have that $\mathbf{a}_q - \mathbf{a}_l = 2(\mathbf{a}_k - \mathbf{a}_p)$, so every nonzero coordinate of the vector $\mathbf{a}_q - \mathbf{a}_l$ equals either 2 or -2, a contradiction. Thus $\mathbf{a}_k - \mathbf{a}_p = \mathbf{0}$, so $\mathbf{a}_k = \mathbf{a}_p$, and therefore $k = p$, a contradiction to the fact that $\operatorname{supp}(\mathbf{u}_+) \cap \operatorname{supp}(\mathbf{u}_-) = \emptyset$. Similarly the assumption $\mathbf{x}^{\mathbf{u}_+} = x_k x_l^2$ or $\mathbf{x}^{\mathbf{u}_-} = x_p x_q^2$ again leads to a contradiction.

The second goal is to prove that $\{E, E'\}$ is a connected component of Δ_G if and only if there is an indispensable quadratic binomial $M_i - M_j \in I_G$ with $\operatorname{supp}(M_i) = E$ and $\operatorname{supp}(M_j) = E'$. Suppose that the binomial $M_i - M_j \in I_G$ is indispensable with $\operatorname{supp}(M_i) = E$ and $\operatorname{supp}(M_j) = E'$. From Theorem 3.4 in [2] we have that $\{M_i, M_j\}$ is a facet of the indispensable complex $\Delta_{\operatorname{ind}(A_G)}$. Assume that $\{E, E'\}$ is not a connected component of Δ_G. Let us suppose that $\{E, E''\}$ is an edge of Δ_G. Then there exists a quadratic binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ with $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_+}) = E$ and $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_-}) = E''$. Moreover, $\mathbf{x}^{\mathbf{u}_+} = M_i$ and $\mathbf{x}^{\mathbf{u}_-} = M_k$, since the monomials $\mathbf{x}^{\mathbf{u}_+}$, $\mathbf{x}^{\mathbf{u}_-}$ are quadratic and therefore indispensable. Notice that $\deg_G(M_i) = \deg_G(M_k)$. Thus

$$\deg_G(M_i) = \deg_G(M_j) = \deg_G(M_k),$$

since the binomial $M_i - M_j$ belongs to I_G, and therefore $\deg_G(M_i) = \deg_G(M_j)$. So $\{M_i, M_j, M_k\}$ is a face of $\Delta_{\operatorname{ind}(A_G)}$, a contradiction to the fact that $\{M_i, M_j\}$ is a facet of $\Delta_{\operatorname{ind}(A_G)}$. Consequently $\{E, E'\}$ is a connected component of Δ_G. Conversely assume that $\{E, E'\}$ is a connected component of Δ_G. Then there is a quadratic binomial $\mathbf{x}^{\mathbf{u}_+} - \mathbf{x}^{\mathbf{u}_-} \in I_G$ with $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_+}) = E$ and $\operatorname{supp}(\mathbf{x}^{\mathbf{u}_-}) = E'$. In fact $\mathbf{x}^{\mathbf{u}_+} = M_i$ and $\mathbf{x}^{\mathbf{u}_-} = M_j$ for some indispensable monomials M_i, M_j. Suppose that the above binomial is not indispensable. Then, since M_i is indispensable, there

is an $l \in \{1, \ldots, s\}$ such that $B_l = M_i - M_k$. Set $E'' = \mathrm{supp}(M_k) \in \mathcal{C}_{min}$. We have that $\{E, E''\}$ and $\{E', E''\}$ are edges of Δ_G, since also $M_j - M_k \in I_G$, a contradiction to the fact that $\{E, E'\}$ is a connected component.

(2) Notice that Δ_G has no connected components which are singletons. To see this consider $E = \mathrm{supp}(M_i) \in \mathcal{C}_{min}$. Then there is an $l \in \{1, \ldots, s\}$ such that $B_l = M_i - M_j$. Consequently $\{\mathrm{supp}(M_i), \mathrm{supp}(M_j)\}$ is an edge of Δ_G.

Next we will show that $\{E, E', E''\}$ is a 2-simplex of Δ_G if and only if there are quadratic binomials $M_i - M_j$, $M_j - M_k$, $M_i - M_k$ in I_G with $\mathrm{supp}(M_i) = E$, $\mathrm{supp}(M_j) = E'$ and $\mathrm{supp}(M_k) = E''$. The one implication is easily derived from the fact that if $\{E, E', E''\}$ is a 2-simplex of Δ_G, then every 2-element subset of it is an edge. Conversely we have that

$$\deg_G(M_i) = \deg_G(M_j) = \deg_G(M_k)$$

belongs to the intersection

$$relint_{\mathbb{Q}}\left(\sigma_E\right) \cap relint_{\mathbb{Q}}\left(\sigma_{E'}\right) \cap relint_{\mathbb{Q}}\left(\sigma_{E''}\right).$$

Thus $\{E, E', E''\}$ is a 2-simplex of Δ_G.

Finally, we prove that if $\{E, E', E''\}$ is a 2-simplex of Δ_G, then it is a connected component. There are quadratic binomials $M_i - M_j$, $M_j - M_k$ and $M_i - M_k$ in I_G, where $\mathrm{supp}(M_i) = E$, $\mathrm{supp}(M_j) = E'$ and $\mathrm{supp}(M_k) = E''$. Let us suppose that $M_i = x_{i_1}x_{i_2}$, $M_j = x_{j_1}x_{j_2}$ and $M_k = x_{k_1}x_{k_2}$. In addition there are even cycles Γ_1, Γ_2, Γ_3 of G of length 4 such that $M_i - M_j = f_{\Gamma_1}$, $M_i - M_k = f_{\Gamma_2}$ and $M_j - M_k = f_{\Gamma_3}$. The cycle Γ_1 has 4 edges, namely e_{i_1}, e_{i_2}, e_{j_1} and e_{j_2}, the cycle Γ_2 has 4 edges, namely e_{i_1}, e_{i_2}, e_{k_1} and e_{k_2}, and Γ_3 has 4 edges, namely e_{j_1}, e_{j_2}, e_{k_1} and e_{k_2}. Notice that the edges e_{i_1} and e_{i_2} have no common vertex. The above three cycles have the same vertex set $\mathcal{V}$ consisting of 4 vertices. Moreover these are the only even cycles of length 4 with vertex set $\mathcal{V}$. Let $\mathcal{K}_4$ be the induced subgraph of G on the above vertex set. It is a complete subgraph with 4 vertices and edges

$$E(\mathcal{K}_4) = \{e_{i_1}, e_{i_2}, e_{j_1}, e_{j_2}, e_{k_1}, e_{k_2}\}.$$

If, for example, $\{E, E'''\}$ is an edge of Δ_G, then there is a quadratic binomial $M_i - M_l \in I_G$ with $\mathrm{supp}(M_l) = E''' \in C_{min}$. Furthermore $M_i - M_l = f_{\Gamma_4}$ for an even cycle Γ_4 of G of length 4. The vertex set of Γ_4 is $\mathcal{V}$, since e_{i_1} and e_{i_2} have no common vertex, and therefore Γ_4 coincides with either Γ_1 or Γ_2. Thus M_l equals either M_j or M_k, so $E''' = E'$ or $E''' = E''$. Consequently $\{E, E', E''\}$ is a connected component of Δ_G. □

Remark 3.5. (1) To every connected component $\{E, E'\}$ of Δ_G we can associate an indispensable binomial $f_\Gamma = x_i x_j - x_k x_l \in I_G$ for an even cycle Γ of G of length 4, where $E = \{i, j\}$ and $E' = \{k, l\}$, and also the induced subgraph $\mathcal{H}$ of G on the vertex set of Γ. The subgraph $\mathcal{H}$ is not a complete graph. Moreover the toric ideal $I_{\mathcal{H}}$ is a complete intersection of height 1 and is generated by f_Γ.

(2) $\{E, E', E''\}$ is a 2-simplex of Δ_G if and only if there are quadratic binomials $M_i - M_j$, $M_i - M_k$, $M_j - M_k$ in I_G with $\mathrm{supp}(M_i) = E$, $\mathrm{supp}(M_j) = E'$ and $\mathrm{supp}(M_k) = E''$.

(3) To every connected component of Δ_G which is a 2-simplex we can associate a complete subgraph $\mathcal{K}_4$ of G of order 4. The toric ideal $I_{\mathcal{K}_4}$ is minimally generated by two binomials f_{Γ_1} and f_{Γ_2}, where Γ_1 and Γ_2 are even cycles of length 4 on the vertex set of $\mathcal{K}_4$.

Proposition 3.6. *Let $\Gamma = (e_i, e_p, e_j, e_q)$ be an even cycle of a graph G such that the induced subgraph $\mathcal{H}$ of G on the vertex set of Γ is not a complete graph. If H is a nonzero polynomial in $I_{\mathcal{H}}$, then there exist monomials M, N of H such that x_ix_j divides M and x_px_q divides N.*

Proof. For the toric ideal $I_{\mathcal{H}}$ we have, from Proposition 4.13 in [11], that $I_{\mathcal{H}} = I_G \cap K[x_l | e_l \in E(\mathcal{H})]$. Since $I_{\mathcal{H}} = (f_\Gamma)$, there is a nonzero polynomial $C \in K[x_l | e_l \in E(\mathcal{H})]$ such that $H = Cf_\Gamma$. The polynomial C has a unique representation as a sum of terms $C = C_1 + \cdots + C_s$. Notice that the monomials of two distinguished terms C_k and C_l are different. We have that

$$H = C_1x_ix_j + \cdots + C_sx_ix_j - C_1x_px_q - \cdots - C_sx_px_q. \tag{3.1}$$

Assume that H has no term whose monomial is x_px_q. This implies that in the above expression of H all the terms of the form $C_kx_px_q$ should by cancelled. But these terms cannot cancel by themselves, so terms of the form $C_kx_ix_j$ are used to cancel them. We claim that a term $C_kx_ix_j$ can be used to cancel at most one term of the form $-C_lx_px_q$. Assume that $C_kx_ix_j$ is used to cancel the terms $-C_lx_px_q$ and $-C_rx_px_q$. Let M_1, M_2 and M_3 be the monomials of the terms $C_kx_ix_j$, $-C_lx_px_q$ and $-C_rx_px_q$ respectively. Then $M_1 = M_2$ and $M_1 = M_3$, so $M_2 = M_3$, a contradiction. But x_px_q divides no monomials of H, so there are two cases:

(1) In expression (3.1) every term cancels. Therefore H is equal to zero, a contradiction.
(2) In (3.1) every term of the form $-C_lx_px_q$ cancels, but still there exist terms of the form $C'_kx_ix_j$, where C'_k is different from C_k. Notice that the monomial of such a term coincides with the monomial of a suitable term $-C_lx_px_q$. Thus every term $C'_kx_ix_j$ is divided by x_px_q, a contradiction.

□

The next lemma will be useful in the proof of Theorem 3.8.

Lemma 3.7. *Let G be a graph with edges $E(G) = \{e_1, \ldots, e_m\}$, Γ an even cycle of length 4 and $\mathcal{H}$ the induced subgraph of G on the vertex set of Γ. If $\mathcal{F} \subset I_G$ is a set of G-homogeneous polynomials which generates I_G up to radical, then $\mathcal{F} \cap K[x_i | e_i$ is an edge of $\mathcal{H}]$ that generates $I_{\mathcal{H}}$ up to radical.*

Proof. Let $\{v_{i_1}, \ldots, v_{i_4}\}$ be the vertices of Γ. The rational polyhedral cone $pos_{\mathbb{Q}}(A_{\mathcal{H}})$ is a face of $pos_{\mathbb{Q}}(A_G)$ with defining vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n$ having coordinates

$$c_j = \begin{cases} 0, & \text{if } j = i_1, i_2, i_3, i_4, \\ 1, & \text{otherwise.} \end{cases}$$

Thus, from Proposition 3.2 in [6], we have that $\mathcal{F} \cap K[x_i | e_i \in E(\mathcal{H})]$ generates $I_{\mathcal{H}}$ up to radical. □

The following theorem determines the binomial arithmetical rank and the G-homogeneous arithmetical rank of a toric ideal I_G generated by quadratic binomials.

Theorem 3.8. *Let G be a graph. If I_G is generated by quadratic binomials, then*

(1) $\mathrm{bar}(I_G) = \mu(I_G)$ *and*
(2) $\mathrm{ara}_G(I_G) = \mu(I_G)$.

Proof. (1) Let $g \geq 0$ be the number of indispensable binomials of I_G. Then, from Proposition 3.4 (1), the simplicial complex Δ_G has exactly g connected components which are edges.

We will show that Δ_G has $\frac{s-g}{2}$ connected components which are 2-simplices, where $s = \mu(I_G)$. Let $\mathcal{B} = \{B_1, \ldots, B_s\}$ be a minimal set of quadratic generators of I_G, and let $\{M_1, \ldots, M_r\}$ be the set of indispensable monomials. Notice that $\mathcal{B}$ has $s - g$ binomials which are not indispensable. Given a 2-simplex $\{E, E', E''\}$ of Δ_G, there are quadratic binomials $f_{\Gamma_1} = M_i - M_j$, $f_{\Gamma_2} = M_i - M_k$, $f_{\Gamma_3} = M_j - M_k$ in I_G with $\mathrm{supp}(M_i) = E$, $\mathrm{supp}(M_j) = E'$ and $\mathrm{supp}(M_k) = E''$. Remark that the binomials f_{Γ_1}, f_{Γ_2} and f_{Γ_3}, as well as $-f_{\Gamma_1}$, $-f_{\Gamma_2}$ and $-f_{\Gamma_3}$, are not indispensable. In fact the minimal generating set $\mathcal{B}$ contains exactly two of the binomials f_{Γ_1}, $-f_{\Gamma_1}$, f_{Γ_2}, $-f_{\Gamma_2}$, f_{Γ_3} and $-f_{\Gamma_3}$, since the monomials M_i, M_j, M_k are indispensable and $\{E, E', E''\}$ is a connected component of Δ_G. Let t be the number of connected components which are 2-simplices. Then $\mathcal{B}$ contains at least $2t$ binomials which are not indispensable. So $2t \leq s - g$. On the other hand, if $B_l = M_i - M_j$ is not indispensable, then $\{\mathrm{supp}(M_i), \mathrm{supp}(M_j)\}$ is an edge which is not a connected component of Δ_G. Therefore there is a monomial M_k such that $\{\mathrm{supp}(M_i), \mathrm{supp}(M_j), \mathrm{supp}(M_k)\}$ is a connected component of Δ_G. Moreover, there exists a $p \in \{1, \ldots, s\}$ such that B_p or $-B_p$ equals either $M_i - M_k$ or $M_j - M_k$. But $\mathcal{B}$ is a minimal generating set, so there exist exactly two binomials in $\mathcal{B}$ whose monomials are M_i, M_j and M_k. Thus Δ_G has at least $\frac{s-g}{2}$ connected components which are 2-simplices, so $\frac{s-g}{2} \leq t$. Consequently $t = \frac{s-g}{2}$.

For every connected component Δ_G^i of Δ_G which is an edge, we have $\delta(\Delta_G^i)_{\{0,1\}} = 1$, while for every connected component Δ_G^i of Δ_G which is a 2-simplex, we have $\delta(\Delta_G^i)_{\{0,1\}} = 2$. Consequently,

$$\delta(\Delta_G)_{\{0,1\}} = g + 2\frac{s-g}{2} = s;$$

i.e. $\delta(\Delta_G)_{\{0,1\}} = \mu(I_G)$, and therefore, from Theorem 2.4, $\mathrm{bar}(I_G) = \mu(I_G)$.

(2) Let $\mathcal{F} \subset I_G$ be a set of G-homogeneous polynomials which generate I_G up to radical. Let $\Delta_G^i = \{E, E'\}$, Δ_G^j be two connected components which are edges, and let $\mathcal{H}_i$ and $\mathcal{H}_j$, respectively, be the corresponding induced subgraphs. Let $E = \{k, l\}$ and $E' = \{p, q\}$; then $I_{\mathcal{H}_i} = (f_\Gamma)$, where $f_\Gamma = x_k x_l - x_p x_q$. The cycle Γ has 4 edges, namely e_k, e_l, e_p and e_q. From Proposition 3.6 every nonzero $H \in I_{\mathcal{H}_i}$ is a polynomial in at least 4 variables, namely x_k, x_l, x_p and x_q. We will prove that every nonzero polynomial H, which belongs to $I_{\mathcal{H}_i}$, does not belong to $I_{\mathcal{H}_j}$. Assume that there is a nonzero polynomial $H \in I_{\mathcal{H}_i}$ which belongs to $I_{\mathcal{H}_j}$. From Proposition 4.13 in [11], we have that $I_{\mathcal{H}_j} = I_G \cap K[x_i | e_i \in E(\mathcal{H}_j)]$. But H belongs to $I_{\mathcal{H}_j}$, so H is a polynomial in the ring $K[x_i | e_i \in E(\mathcal{H}_j)]$, and therefore every edge of Γ is also an edge of $\mathcal{H}_j$. Thus the indispensable binomial f_Γ belongs to $I_{\mathcal{H}_j}$, and therefore, from Proposition 3.4 (1), we have that $\{E, E'\}$ is a connected component of Δ_G^j, a contradiction. Given a connected component of Δ_G, which is an edge, and the corresponding induced subgraph $\mathcal{H}$ of G, there exists, from Lemma 3.7, at least one G-homogeneous polynomial $F \in I_{\mathcal{H}}$ in $\mathcal{F}$. The simplicial complex Δ_G has g connected components which are edges, so $\mathcal{F}$ has at least g G-homogeneous polynomials, say $F_1, \ldots, F_g$, belonging to the corresponding toric ideals $I_{\mathcal{H}_i}$, $1 \leq i \leq g$.

Remark that if G has a complete subgraph $\mathcal{K}_4$, then every polynomial F_i, $1 \leq i \leq g$, does not belong to the toric ideal $I_{\mathcal{K}_4}$, since $I_{\mathcal{H}_i} = I_G \cap K[x_r | e_r \in E(\mathcal{H}_i)]$, and every $\mathcal{H}_i$ is not a complete graph.

Let Δ_G^p, Δ_G^q be two connected components which are 2-simplices, and let $\mathcal{K}_{4,p}$ and $\mathcal{K}_{4,q}$, respectively, the corresponding induced subgraphs. We will prove that every nonzero polynomial H, which is in the ideal $I_{\mathcal{K}_{4,p}}$, does not belong to $I_{\mathcal{K}_{4,q}}$. Let $I_{\mathcal{K}_{4,p}} = (f_{\Gamma_1}, f_{\Gamma_2})$, where $f_{\Gamma_1} = x_{i_1}x_{i_2} - x_{i_3}x_{i_4}$ and $f_{\Gamma_2} = x_{i_1}x_{i_2} - x_{i_5}x_{i_6}$. Assume that $H \in I_{\mathcal{K}_{4,p}}$ is a nonzero polynomial which belongs to $I_{\mathcal{K}_{4,q}}$. Since H belongs to $I_{\mathcal{K}_{4,p}}$, every monomial of H is of the form $Cx_{i_1}x_{i_2}$ or $Nx_{i_3}x_{i_4}$ or $Qx_{i_5}x_{i_6}$, for appropriate monomials C, N and Q. But $I_{\mathcal{K}_{4,q}} = I_G \cap K[x_j | e_j \in E(\mathcal{K}_{4,q})]$, so $\mathcal{K}_{4,q}$ has at least 2 edges coming from $\mathcal{K}_{4,p}$. These edges are e_{i_1} and e_{i_2} or e_{i_3} and e_{i_4} or e_{i_5} and e_{i_6}. Notice that the edges e_{i_1} and e_{i_2} have no common vertex. The same holds for e_{i_3} and e_{i_4}, as well as the edges e_{i_5} and e_{i_6}. But $\mathcal{K}_{4,q}$ is a complete graph, so $\mathcal{K}_{4,p} = \mathcal{K}_{4,q}$, a contradiction.

Given a connected component of Δ_G, which is a 2-simplex, and the corresponding induced subgraph $\mathcal{K}_4$ of G, there exist, from Lemma 3.7, at least two G-homogeneous polynomial $H_1, H_2 \in I_{\mathcal{K}_4}$ in $\mathcal{F}$. The simplicial complex Δ_G has $\frac{s-g}{2}$ connected components which are 2-simplices, so $\mathcal{F}$ has also at least $2\frac{s-g}{2} = s-g$ polynomials, which are G-homogeneous, say $H_1, \ldots, H_{s-g}$, belonging to the corresponding toric ideals $I_{\mathcal{K}_{4,i}}$, $1 \leq i \leq s-g$. Thus

$$\operatorname{ara}_G(I_G) \geq g + (s-g) = s;$$

i.e. $\operatorname{ara}_G(I_G) \geq \mu(I_G)$, and therefore $\operatorname{ara}_G(I_G) = \mu(I_G)$. □

Example 3.9. Consider the complete graph $\mathcal{K}_n$, $n \geq 4$, on the vertex set $\{v_1, \ldots, v_n\}$. We consider one variable x_{ij}, $1 \leq i < j \leq n$, for each edge $\{v_i, v_j\}$ of $\mathcal{K}_n$ and form the polynomial ring $K[x_{ij} | 1 \leq i < j \leq n]$. The toric ideal $I_{\mathcal{K}_n}$ is the kernel of the K-algebra homomorphism

$$\phi : K[x_{ij} | 1 \leq i < j \leq n] \to K[t_1, \ldots, t_n]$$

given by

$$\phi(x_{ij}) = t_i t_j.$$

From Proposition 3.2 in [12] the height of $I_{\mathcal{K}_n}$ equals $\binom{n}{2} - n = \frac{n(n-3)}{2}$, i.e. the number of edges minus the number of vertices. It is well known (see for example Proposition 9.2.1 in [13]) that

$$B = \{x_{ij}x_{kl} - x_{il}x_{jk}, x_{ik}x_{jl} - x_{il}x_{jk} | 1 \leq i < j < k < l \leq n\}$$

is a minimal generating set for $I_{\mathcal{K}_n}$. The toric ideal $I_{\mathcal{K}_n}$ has no indispensable binomials, and therefore every connected component of $\Delta_{\mathcal{K}_n}$ is a 2-simplex. Thus $\Delta_{\mathcal{K}_n}$ has $3\binom{n}{4}$ vertices and $\binom{n}{4}$ connected components, which are 2-simplices, corresponding to all complete subgraphs of $\mathcal{K}_n$ of order 4. For the minimal number of generators we have that

$$\mu(I_{\mathcal{K}_n}) = 2\binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{12}.$$

Consequently

$$\operatorname{bar}(I_{\mathcal{K}_n}) = \operatorname{ara}_{\mathcal{K}_n}(I_{\mathcal{K}_n}) = \frac{n(n-1)(n-2)(n-3)}{12}.$$

Using the result of Eisenbud-Evans and Storch that $\operatorname{ara}(I_{\mathcal{K}_n})$ is bounded above by the number of variables of $K[x_{ij}|1 \leq i < j \leq n]$, we get that

$$\frac{n(n-3)}{2} \leq \operatorname{ara}(I_{\mathcal{K}_n}) \leq \frac{n(n-1)}{2}.$$

For the polynomials which minimally generate $I_{\mathcal{K}_n}$ up to radical, we know, from Theorem 5.8 in [5], that there must be at least $3\binom{n}{4}$ monomials in at least $2\binom{n}{4}$ $\mathcal{K}_n$-homogeneous components.

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