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FIXED POINTS AND PERIODIC POINTS OF ORIENTATION-REVERSING PLANAR HOMEOMORPHISMS

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(Communicated by Bryna Kra)

Dedicated to the memory of Professor Andrzej Lasota (1932–2006)

Abstract. Two results concerning orientation-reversing homeomorphisms of the plane are proved. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-reversing planar homeomorphism with a continuum X invariant (i.e. $h(X) = X$). First, suppose there are at least n bounded components of $\mathbb{R}^2 \setminus X$ that are invariant under h. Then there are at least $n + 1$ components of the fixed point set of h in X. This provides an affirmative answer to a question posed by K. Kuperberg. Second, suppose there is a k-periodic orbit in X with $k > 2$. Then there is a 2-periodic orbit in X, or there is a 2-periodic component of $\mathbb{R}^2 \setminus X$. The second result is based on a recent result of M. Bonino concerning linked periodic orbits of orientation-reversing homeomorphisms of the 2-sphere $\mathbb{S}^2$. These results generalize to orientation-reversing homeomorphisms of $\mathbb{S}^2$.

1. Introduction

Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-reversing homeomorphism of the plane onto itself with a continuum X invariant (i.e. $h(X) = X$). Suppose there are at least n bounded components of $\mathbb{R}^2 \setminus X$ that are invariant under h. In 1989 Krystyna Kuperberg [10] asked whether h must always have $n+1$ fixed points in X. Earlier, in 1978, Harold Bell [1] showed that this is true for $n = 0$. Kuperberg [9] proved this result for $n = 1$. Subsequently, she also showed [10] that h must have at least $k + 2$ fixed points in X, whenever $n \geq 2^k$. Drawing on ideas from [9] and [10] we will present an affirmative answer to the above question. More precisely, we will prove the following stronger result.

Theorem 1.1. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-reversing homeomorphism of the plane onto itself with a continuum X invariant, and suppose there are at least n bounded components of $\mathbb{R}^2 \setminus X$ that are invariant under h. Then $Fix(X, h)$, the set of fixed points of h in X, has at least $n + 1$ components.*

In the present paper we also discuss another problem concerning periodic points of orientation-reversing homeomorphisms. Recently, Marc Bonino [2] showed that if $h : \mathbb{S}^2 \to \mathbb{S}^2$ is an orientation-reversing homeomorphism of $\mathbb{S}^2$ onto itself with an orbit $\mathcal{O}$ of period $k > 2$, then h must also have an orbit $\mathcal{O}'$ of period 2. Using

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Nielsen's theory he strengthened his result in [3], showing that if h has a k-periodic orbit $\mathcal{O}$ with $k > 2$, then there is a 2-periodic orbit $\mathcal{O}'$ such that $\mathcal{O}$ and $\mathcal{O}'$ are linked. Two orbits $\mathcal{O}$ and $\mathcal{O}'$ are linked in the sense of Bonino if one cannot find a Jordan curve $C \subseteq \mathbb{S}^2$ separating $\mathcal{O}$ and $\mathcal{O}'$ which is freely isotopic to $h(C)$ in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$. C and $h(C)$ are freely isotopic in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$ if there is an isotopy $\{i_t : \mathbb{S}^1 \to \mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}') : 0 \leq t \leq 1\}$ from $i_0(\mathbb{S}^1) = C$ to $i_1(\mathbb{S}^1) = h(C)$; i.e. $i_t(\mathbb{S}^1)$ is a Jordan curve for any t ($\mathbb{S}^1$ denotes the unit circle). Exploiting heavily results from the second paper we will show the following.

Theorem 1.2. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-reversing homeomorphism with a continuum X invariant (i.e. $h(X) = X$). Suppose h has a k-periodic orbit in X with $k > 2$.*

(i) *If X does not separate the plane, then h has a 2-periodic orbit in X.*
(ii) *If X separates the plane, then h has a 2-periodic orbit in X, or there is a 2-periodic component U of $\mathbb{R}^2 \setminus X$.*

The above result seems to be related to a special case of the Sarkovskii Theorem [11], which asserts that a self-map of the arc that has a point of period $k > 2$ must also have a point of period 2.

2. Preliminaries

Given a set D, by $\operatorname{Int} D$ and ∂D we will denote respectively the interior and the boundary of D. Throughout this paper h is an orientation-reversing homeomorphism of the plane $\mathbb{R}^2$ onto itself and X is a continuum (i.e. connected and compact subset of the plane) invariant under h; that is, $h(X) = X$. Denote by $Fix(X, h)$ the set of fixed points of h in X; i.e. $Fix(X, h) = \{x \in X : h(x) = x\}$. Components of $\mathbb{R}^2 \setminus X$ are called complementary domains of X. A point x (a complementary domain U of X) is k-periodic if $h^k(x) = x$ but $h^p(x) \neq x$ ($h^k(U) = U$ but $h^p(U) \neq U$) for any positive integer $p < k$. $\mathcal{O}$ is a k-periodic orbit if $\mathcal{O} = \{x, h^1(x), \ldots, h^{k-1}(x)\}$ for a k-periodic point x. Let us recall the methods of [9] and [10] that we will rely on in order to prove Theorem 1.1. Let U be a bounded complementary domain of $\mathbb{R}^2 \setminus X$ that is invariant under h. With modification of h outside of X one can ensure that there is an annulus $\mathbb{A}$ invariant under h such that $X \subseteq \mathbb{A}$. $\mathbb{A}$ is topologically a geometric annulus $\{(r, \theta) \in \mathbb{R}^2 : 1 \leq r \leq 2, 0 \leq \theta < 2\pi\}$, given in polar coordinates, with two boundary components $\mathbb{A}^+ = \{(r, \theta) \in \mathbb{R}^2 : r = 2, 0 \leq \theta < 2\pi\}$ and $\mathbb{A}^- = \{(r, \theta) \in \mathbb{R}^2 : r = 1, 0 \leq \theta < 2\pi\}$. The continuum X is essentially inscribed into $\mathbb{A}$; i.e. $\mathbb{A}^- \subseteq U$. Now, one can consider the universal covering space of $\mathbb{A}$ given by $\tilde{\mathbb{A}} = \{(x, y) \in \mathbb{R}^2 : 1 \leq y \leq 2\}$, with the covering map $\tau : \tilde{\mathbb{A}} \to \mathbb{A}$ determined by $\tau(x, y) = (y, 2\pi x (\operatorname{mod} 2\pi))$. Let $\tilde{h} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift homeomorphism of $h|\mathbb{A}$ (i.e. $\tau \circ h = \tilde{h} \circ \tau$). Note that for any $p = (r, \theta)$ in $\mathbb{A}$ its fiber is the set $\tau^{-1}(p) = \{(\frac{\theta}{2\pi} + n, r) : n \in \mathbb{Z}\}$, and p is a fixed point of h iff $\tau^{-1}(p)$ is invariant under $\tilde{h}$. The main ingredients from [9] and [10] that we will need are the following facts.

(1) Given a fixed point $p = (r, \theta) \in \mathbb{A}$ and a lift $\tilde{h}$ of h there is an integer $m[\tilde{h}, p]$ such that $\tilde{h}(\frac{\theta}{2\pi} + n, r) = (\frac{\theta}{2\pi} - n + m[\tilde{h}, p], r)$ for every $(\frac{\theta}{2\pi} + n, r) \in \tau^{-1}(p)$.
(2) $\tilde{h}$ has a fixed point in $\tau^{-1}(p)$ iff $m[\tilde{h}, p]$ is even.
(3) If $m[\tilde{h}, p]$ is even, then $\tilde{h}(x + 1, y)$ is a lift homeomorphism of h that does not have a fixed point in $\tau^{-1}(p)$.

(4) $\tilde{\mathbb{A}}$ can be compactified by two points, say b_1, b_2, so that $\tilde{X} = \tau^{-1}(X) \cup \{b_1, b_2\}$ is a continuum invariant under $\tilde{h}$, and the latter can be extended to an orientation-reversing homeomorphism of the entire plane onto itself.

Let $\tilde{h_1} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$ be a lift of h and $\tilde{h}_2(x, y) = \tilde{h}_1(x + 1, y)$ be another lift, fixed once and for all. For simplicity we will use the same symbols $\tilde{h}_1, \tilde{h}_2$ to denote the extensions of these two lifts to the entire plane.

Proposition 2.1. *If Y is a subcontinuum of the set of fixed points of h, then Y does not separate the plane.*

Proof. If F is the fixed point set of a homeomorphism f of a connected topological manifold M, then either each component of $M \setminus F$ is invariant under f or there are exactly two components of $M \setminus F$ and f interchanges them [6]. Since in the case of planar homeomorphisms the unbounded complementary domain of F is always invariant under h, therefore the above implies that all components of $\mathbb{R}^2 \setminus F$ must be invariant under h. Consequently if Y were a continuum of fixed points of h separating the plane, then Y could be essentially inscribed into the annulus $\mathbb{A}$ with $\mathbb{A}^-$ and $\mathbb{A}^+$ invariant under h, and h would induce the identity on the homology group $\mathrm{H}_1(\mathbb{A}, \mathbb{Z})$. Therefore any lift $\tilde{h}$ of h to the universal cover $\tilde{\mathbb{A}}$ would preserve the orientation on the two boundary components of $\tilde{\mathbb{A}}$, at the same time keeping them invariant. Consequently $\tilde{h}$ would be orientation-preserving on $\tilde{\mathbb{A}}$, contradicting the fact that any lift of h to $\tilde{\mathbb{A}}$ must be orientation-reversing. □

Lemma 2.2. *Suppose p is a fixed point of h and let Y be the component of p in $Fix(X, h)$. Then*

$$m[\tilde{h}_1, p] = m[\tilde{h}_1, q](\mathrm{mod} 2)$$

for every $q \in Y$.

Proof. First, Y does not separate the plane. Suppose $m[\tilde{h}_1, p]$ is even. Let α be the fixed point of $\tilde{h}_1$ in $\tau^{-1}(p)$ and let K be the component of $\tau^{-1}(Y)$ containing α. To the contrary, suppose the above claim is false and let $q \in Y$ be such that $m[\tilde{h}_1, q]$ is odd. For every $\beta \in \tau^{-1}(q)$ we have $\beta \neq \tilde{h}_1(\beta)$. Let γ be in $K \cap \tau^{-1}(q)$. Then $\tilde{h}_1(\gamma) \neq \gamma$ and $\tilde{h}_1(\gamma) \in \tilde{h_1}(K)$. Since $\tilde{h}_1(K)$ is also a component of $\tau^{-1}(Y)$ and $\alpha \in \tilde{h}_1(K)$, then $K = \tilde{h}_1(K)$. Consequently, K contains two elements from the same fiber $\tau^{-1}(q) \in \tau^{-1}(Y)$. But this contradicts the following observation indicated in [5], which in turn will complete the proof.

Since $Y \subseteq \mathbb{A}$ does not separate the plane, one can choose a disk $D \subseteq \mathbb{A}$ around Y; i.e., $Y \subseteq \operatorname{Int} D$, and $\operatorname{Int} D$ being simply connected lifts to disjoint homeomorphic copies of $\operatorname{Int} D$ in $\tilde{\mathbb{A}}$. Consequently Y lifts to disjoint homeomorphic copies in $\tilde{\mathbb{A}}$. Since K is one of them, it cannot contain two points from the same fiber $\tau^{-1}(q)$. □

As a consequence of the above, for a given component Y of $Fix(X, h)$ one can choose any $p \in Y$ and say that $m[\tilde{h}_1, Y]$ is even (or odd) if $m[\tilde{h}_1, p]$ is of the same parity.

3. Proof of Theorem 1.1

Proof of Theorem 1.1. We will prove this theorem by induction. First, observe that the case when $n = 0$ is the theorem of Bell [1]. Indeed, if X is a nonseparating plane continuum, then by Bell's theorem h must have a fixed point in X, and therefore there is at least one component of $Fix(X, h)$.

For the sake of induction suppose the theorem is true for $n = k - 1$. Now we will show that the theorem holds true for $n = k$.

Assume $U_1, \dots, U_k$ are bounded complementary domains of X invariant under h, and that $\mathbb{A}^-$ is inscribed into U_k. We may assume that there is a fixed point u_i of h in each U_i. Without loss of generality assume that $u_1, \dots, u_p$ are all fixed points of h such that there is a fixed point of $\tilde{h}_1$ in the fiber $\tau^{-1}(u_i)$, for all $i = 1, \dots, p$. In other words, each set from $U_1, \dots, U_p$ contains in its lift $\tau^{-1}(U_i)$ a bounded complementary domain of $\tilde{X}$ that is invariant under $\tilde{h}_1$. Equivalently, $m[\tilde{h}_1, u_i]$ is even for $i = 1, \dots, p$ and $m[\tilde{h}_1, u_i]$ is odd for $i = p+1, \dots, k-1$.

Let $q = k - 1 - p$. Note that p, q are nonnegative integers (possibly with p or q equal to 0). Since $\tilde{X}$ is a continuum with p bounded complementary domains invariant under $\tilde{h}_1$ and $p \leq k - 1$, by the induction hypothesis there are $p + 1$ components of $Fix(\tilde{h}_1, \tilde{X})$. Let $A_1, \dots, A_{p+1}$ be those components.

For every $i = 1, \dots, p + 1$ there is a component X_i of $Fix(h, X)$ such that $X_i = \tau(A_i)$. Note that $\tau(A_i)$ and $\tau(A_t)$ are disjoint for $i \neq t$ since any fiber of a fixed point of h contains no more that one fixed point of $\tilde{h}_1$. Therefore $\{X_i : i = 1, \dots, p + 1\}$ consists of $p + 1$ distinct components of $Fix(X, h)$.

Now, $\tau^{-1}(X_i)$ is invariant under $\tilde{h}_1$, and $m[\tilde{h}_1, X_i]$ is even for every $i = 1, \dots, p + 1$. $\tau^{-1}(X_i)$ is also invariant under $\tilde{h}_2$ but contains no fixed point of $\tilde{h}_2$, since $m[\tilde{h}_2, X_i]$ is odd for every $i = 1, \dots, p + 1$. For $i = 1, \dots, p$, no $\tau^{-1}(u_i)$ contains a fixed point of $\tilde{h}_2$, since $m[\tilde{h}_2, u_i]$ is odd. For $i = p + 1, \dots, k - 1$, every $\tau^{-1}(u_i)$ contains a fixed point of $\tilde{h}_2$, since $m[\tilde{h}_2, u_i]$ is even. Therefore, there are $q = (k-1)-p$ bounded complementary domains of $\tilde{X}$ that are invariant under $\tilde{h}_2$. Again, by the induction hypothesis, there must be $q+1$ components of $Fix(\tilde{h}_2, \tilde{X})$. Denote them by $C_1, \dots, C_q$. For every $j = 1, \dots, q+1$, $\tau(C_j)$ is a component of $Fix(h, X)$. Note that $\tau(C_j)$ and $\tau(C_t)$ are disjoint for $j \neq t$ since any fiber of a fixed point of h contains no more that one fixed point of $\tilde{h}_2$. Therefore $\{\tau(C_j) : j = 1, \dots, q + 1\}$ consists of $q + 1$ distinct components of $Fix(X, h)$. Since each $\tau^{-1}(X_i)$ contains no fixed point of $\tilde{h}_2$, no $\tau(C_j)$ can coincide with any X_i. Therefore there are $p + 1 + q + 1 = k + 1$ components of $Fix(h, X)$. This completes the proof. $\square$

Note that Theorem 1.1 generalizes to orientation-reversing homeomorphisms of $\mathbb{S}^2$. More precisely, we get the following as a corollary.

Theorem 3.1. *Let $g : \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation-reversing homeomorphism of $\mathbb{S}^2$ onto itself with a continuum X invariant, and suppose there are at least n components of $\mathbb{S}^2 \setminus X$ that are invariant under g. Then $Fix(X, g)$ has at least n components.*

Proof. First suppose that $\mathbb{S}^2 \setminus X$ has exactly one component U invariant under g. We can assume that there is a fixed point u of g in U. Notice that $\mathbb{S}^2 \setminus \{u\}$ is topologically the plane, and $G = g|(\mathbb{S}^2 \setminus \{u\})$, obtained by the restriction of g to $\mathbb{S}^2 \setminus \{u\}$, is an orientation-reversing homeomorphism of the plane onto itself with the continuum X invariant. Now, since X has no bounded complementary domains invariant under G, by a theorem of Bell there is at least one component of $Fix(X, G) = Fix(X, g)$. Bell's theorem applies to nonseparating plane continua, but in the above case if X separates the plane and none of the bounded complementary domains is invariant under G, then these domains can be added to X to form a nonseparating plane continuum Y with $Fix(X, G) = Fix(Y, G)$.

Second, suppose that $\mathbb{S}^2 \setminus X$ has at least two components U_1 and U_2 invariant under g. Then there is an annulus $\mathbb{A}$ such that $X \subseteq \mathbb{A}$, $\mathbb{A}^- \subseteq U_1$ and $\mathbb{A}^+ \subseteq U_2$. Since U_1 and U_2 are invariant under g, then h does not interchange $\mathbb{A}^-$ and $\mathbb{A}^+$, and one can repeat the proof of Theorem 1.1. □

4. Proof of Theorem 1.2

Theorem 1.2 seems to fit well in the following context. The Cartwright-Littlewood-Bell theorem (see [7] and [1]) states that any planar homeomorphism fixes a point in an invariant nonseparating continuum. Morton Brown [5] and O.H. Hamilton [8] exhibited that, in the case of orientation-preserving homeomorphisms, this theorem can be deduced directly from a theorem of Brouwer [4]. Brouwer showed that any orientation-preserving homeomorphism with at least one bounded orbit must have a fixed point. Briefly, the idea behind these short proofs of the fixed point theorem was to separate the invariant continuum from the fixed-point set F, and then for an open invariant component U in $\mathbb{R}^2 \setminus F$ containing X argue that U contains no fixed point, thus contradicting the theorem of Brouwer. The inspiration for the proof of Theorem 1.2 comes from these very papers, but since the set of 2-periodic points does not need to be closed (in contrast with the fixed-point set), one cannot just replace the theorem of Brouwer with a theorem of Bonino from [2] and use the same arguments. Instead, we will use Bonino's result from [3] and show that no 2-periodic orbit in an invariant component of $\mathbb{R}^2 \setminus X$ can be linked to a k-periodic ($k > 2$) orbit in X.

Proof of Theorem 1.2. Compactify $\mathbb{R}^2$ by a point ∞ to obtain $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ and extend the given homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ to a homeomorphism $\tilde{h} : \mathbb{S}^2 \to \mathbb{S}^2$ by setting $\tilde{h}|\mathbb{R}^2 = h$ and $\tilde{h}(\infty) = \infty$. h and $\tilde{h}$ have exactly the same k-periodic points for any $k > 1$.

By Bonino's result there is an orbit $\mathcal{O}' \subseteq \mathbb{S}^2$ of $\tilde{h}$ of period exactly 2. We will show that any such 2-periodic orbit that lies in an invariant complementary domain of X is not linked to $\mathcal{O}$.

Suppose $\mathcal{O}' \cap X = \emptyset$ and $\mathcal{O}' \subseteq U$ for a complementary domain U of X invariant under h. Since $\mathcal{O}'$ and X are closed, there is a Jordan curve $S \subseteq \mathbb{S}^2$ separating $\mathcal{O}'$ from X. Let D be one of the two disks in $\mathbb{S}^2$ bounded by S, such that $X \subseteq \operatorname{Int} D$. Then $D \cap \mathcal{O}' = \emptyset$. Since X is invariant under $\tilde{h}$, by continuity of $\tilde{h}$, there is a disk C such that $C \subseteq \operatorname{Int} D$ and $\tilde{h}(C) \subseteq \operatorname{Int} D$. Since both C and $\tilde{h}(C)$ contain X in its interior, there is a disk $B \subseteq C \cap \tilde{h}(C)$ that contains X in its interior. Therefore C and $\tilde{h}(C)$ are freely isotopic in the annulus $D \setminus \operatorname{Int} B$, thus freely isotopic in $\mathbb{S}^2 \setminus (\mathcal{O} \cup \mathcal{O}')$. This shows that if $\mathcal{O}' \subseteq \mathbb{S}^2 \setminus X$ is a 2-periodic orbit, then $\mathcal{O}'$ and $\mathcal{O}$ are not linked. Therefore the 2-periodic orbit $\mathcal{O}'$ linked to $\mathcal{O}$, guaranteed by a theorem of Bonino in [3], must be in X or in a 2-periodic component of $\mathbb{S}^2 \setminus X$. □

Corollary 4.1. *Suppose there is a k-periodic component of $\mathbb{R}^2 \setminus X$, for $k > 2$. Then either there is a 2-periodic orbit in X or there is a 2-periodic component of $\mathbb{R}^2 \setminus X$.*

Proof. Let W be a k-periodic complementary domain of X ($k > 2$). Without loss of generality one may assume that there is a k-periodic point w in W (w is a fixed point of h^k). Consider $Y = X \cup W \cup h(W) \cup \ldots \cup h^{k-1}(W)$. Clearly Y is a continuum invariant under h. Now apply Theorem 1.2. □

Remark. It is clear from the proof of Theorem 1.2 that this theorem holds also for any orientation-reversing homeomorphism of $\mathbb{S}^2$. On the other hand, it is not apparent to the present author if one can improve Theorem 1.2 and get rid of the 2-periodic component of $\mathbb{R}^2 \setminus X$ to guarantee that, under the assumptions, there will be a 2-periodic point in X. Nonetheless, the following example shows that one cannot do it for $\mathbb{S}^2$.

Example. Let $\mathbb{S}^2$ be given in spherical coordinates by $\mathbb{S}^2 = \{(r, \theta, \phi) : r = 1, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi\}$. Consider the Jordan curve $S \subseteq \mathbb{S}^2$ determined by $S = \{(r, \theta, \phi) : r = 1, \phi = \frac{\pi}{2}\}$. Let U^+, U^- be the two disks in $\mathbb{S}^2 \setminus S$ bounded by S. Fix $k > 2$ and consider the orientation-reversing homeomorphism $g : \mathbb{S}^2 \to \mathbb{S}^2$ determined by

$$g(r, \theta, \phi) = (r, \theta + \frac{2\pi}{k}, \pi - \phi).$$

g interchanges U^+ and U^-, reflecting $\mathbb{S}^2$ about S and then rotating $\mathbb{S}^2$ by $\frac{2\pi}{k}$. Notice that $g^2(r, \theta, \phi) = (r, \theta + \frac{4\pi}{k}, \phi)$ and $g^k(r, \theta, \phi) = (r, \theta, \phi) = id_{\mathbb{S}^2}(r, \theta, \phi)$. Clearly, g is an orientation-reversing homeomorphism of $\mathbb{S}^2$ with the continuum S invariant, and any point in S is of period exactly k, but the only points of period 2 are the two poles, which are not in S.

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