FIXED POINTS AND PERIODIC POINTS
OF ORIENTATION-REVERSING PLANAR HOMEOMORPHISMS

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Dedicated to the memory of Professor Andrzej Lasota (1932–2006)

ABSTRACT. Two results concerning orientation-reversing homeomorphisms of the plane are proved. Let \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) be an orientation-reversing planar homeomorphism with a continuum \( X \) invariant (i.e. \( h(X) = X \)). First, suppose there are at least \( n \) bounded components of \( \mathbb{R}^2 \setminus X \) that are invariant under \( h \). Then there are at least \( n + 1 \) components of the fixed point set of \( h \) in \( X \). This provides an affirmative answer to a question posed by K. Kuperberg. Second, suppose there is a \( k \)-periodic orbit in \( X \) with \( k > 2 \). Then there is a 2-periodic orbit in \( X \), or there is a 2-periodic component of \( \mathbb{R}^2 \setminus X \). The second result is based on a recent result of M. Bonino concerning linked periodic orbits of orientation-reversing homeomorphisms of the 2-sphere \( S^2 \). These results generalize to orientation-reversing homeomorphisms of \( S^2 \).

1. INTRODUCTION

Let \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) be an orientation-reversing homeomorphism of the plane onto itself with a continuum \( X \) invariant (i.e. \( h(X) = X \)). Suppose there are at least \( n \) bounded components of \( \mathbb{R}^2 \setminus X \) that are invariant under \( h \). In 1989 Krystyna Kuperberg \cite{10} asked whether \( h \) must always have \( n + 1 \) fixed points in \( X \). Earlier, in 1978, Harold Bell \cite{1} showed that this is true for \( n = 0 \). Kuperberg \cite{9} proved this result for \( n = 1 \). Subsequently, she also showed \cite{10} that \( h \) must have at least \( k + 2 \) fixed points in \( X \), whenever \( n \geq 2^k \). Drawing on ideas from \cite{9} and \cite{10} we will present an affirmative answer to the above question. More precisely, we will prove the following stronger result.

**Theorem 1.1.** Let \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) be an orientation-reversing homeomorphism of the plane onto itself with a continuum \( X \) invariant, and suppose there are at least \( n \) bounded components of \( \mathbb{R}^2 \setminus X \) that are invariant under \( h \). Then \( \text{Fix}(X, h) \), the set of fixed points of \( h \) in \( X \), has at least \( n + 1 \) components.

In the present paper we also discuss another problem concerning periodic points of orientation-reversing homeomorphisms. Recently, Marc Bonino \cite{2} showed that if \( h : S^2 \to S^2 \) is an orientation-reversing homeomorphism of \( S^2 \) onto itself with an orbit \( O \) of period \( k > 2 \), then \( h \) must also have an orbit \( O' \) of period 2. Using

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Nielsen’s theory he strengthened his result in [3], showing that if \( h \) has a \( k \)-periodic orbit \( O \) with \( k > 2 \), then there is a 2-periodic orbit \( O' \) such that \( O \) and \( O' \) are linked. Two orbits \( O \) and \( O' \) are linked in the sense of Bonino if one cannot find a Jordan curve \( C \subseteq S^2 \) separating \( O \) and \( O' \) which is freely isotopic to \( h(C) \) in \( S^2 \setminus (O \cup O') \). \( C \) and \( h(C) \) are freely isotopic in \( S^2 \setminus (O \cup O') \) if there is an isotopy \( \{i_t : S^1 \to S^2 \setminus (O \cup O') : 0 \leq t \leq 1 \} \) from \( i_0(S^1) = C \) to \( i_1(S^1) = h(C) \); i.e. \( \{i_t(S^1)\} \) is a Jordan curve for any \( t \) (\( S^1 \) denotes the unit circle). Exploiting heavily results from the second paper we will show the following.

**Theorem 1.2.** Let \( h : \mathbb{R}^2 \to \mathbb{R}^2 \) be an orientation-reversing homeomorphism with a continuum \( X \) invariant (i.e. \( h(X) = X \)). Suppose \( h \) has a \( k \)-periodic orbit in \( X \) with \( k > 2 \).

(i) If \( X \) does not separate the plane, then \( h \) has a 2-periodic orbit in \( X \).

(ii) If \( X \) separates the plane, then \( h \) has a 2-periodic orbit in \( X \), or there is a 2-periodic component \( U \) of \( \mathbb{R}^2 \setminus X \).

The above result seems to be related to a special case of the Sarkovskii Theorem [11], which asserts that a self-map of the arc that has a point of period \( k > 2 \) must also have a point of period 2.

2. Preliminaries

Given a set \( D \), by \( \text{Int} \) \( D \) and \( \partial D \) we will denote respectively the interior and the boundary of \( D \). Throughout this paper \( h \) is an orientation-reversing homeomorphism of the plane \( \mathbb{R}^2 \) onto itself and \( X \) is a continuum (i.e. connected and compact subset of the plane) invariant under \( h \); that is, \( h(X) = X \). Denote by \( \text{Fix}(X, h) \) the set of fixed points of \( h \) in \( X \); i.e. \( \text{Fix}(X, h) = \{x \in X : h(x) = x \} \). Components of \( \mathbb{R}^2 \setminus X \) are called complementary domains of \( X \). A point \( x \) (a complementary domain \( U \) of \( X \)) is \( k \)-periodic if \( h^k(x) = x \) but \( h^p(x) \neq x \) \((h^k(U) = U \) but \( h^p(U) \neq U)\) for any positive integer \( p < k \). \( O \) is a \( k \)-periodic orbit if \( O = \{x, h^1(x), \ldots, h^{k-1}(x)\} \) for a \( k \)-periodic point \( x \). Let us recall the methods of [9] and [10] that we will rely on in order to prove Theorem [11]. Let \( U \) be a bounded complementary domain of \( \mathbb{R}^2 \setminus X \) that is invariant under \( h \). With modification of \( h \) outside of \( X \) one can ensure that there is an annulus \( \mathcal{A} \) invariant under \( h \) such that \( X \subseteq \mathcal{A} \). \( \mathcal{A} \) is topologically a geometric annulus \( \{(r, \theta) \in \mathbb{R}^2 : 1 \leq r \leq 2, 0 \leq \theta < 2\pi\} \), given in polar coordinates, with two boundary components \( \mathcal{A}^+ = \{(r, \theta) \in \mathbb{R}^2 : r = 2, 0 \leq \theta < 2\pi\} \) and \( \mathcal{A}^- = \{(r, \theta) \in \mathbb{R}^2 : r = 1, 0 \leq \theta < 2\pi\} \). The continuum \( X \) is essentially inscribed into \( \mathcal{A} \); i.e. \( \mathcal{A}^- \subseteq U \). Now, one can consider the universal covering space of \( \mathcal{A} \) given by \( \hat{\mathcal{A}} = \{(x, y) \in \mathbb{R}^2 : 1 \leq y \leq 2\} \), with the covering map \( \tau : \hat{\mathcal{A}} \to \mathcal{A} \) determined by \( \tau(x, y) = (y, 2\pi x (\text{mod} 2\pi)) \). Let \( \hat{h} : \hat{\mathcal{A}} \to \hat{\mathcal{A}} \) be a lift homeomorphism of \( h|_{\mathcal{A}} \) (i.e. \( \tau \circ h = \hat{h} \circ \tau \)). Note that for any \( p = (r, \theta) \in \mathcal{A} \) its fiber is the set \( \tau^{-1}(p) = \{(\frac{r}{2^p} + n, r) : n \in \mathbb{Z}\} \), and \( p \) is a fixed point of \( h \) iff \( \tau^{-1}(p) \) is invariant under \( \hat{h} \). The main ingredients from [9] and [10] that we will need are the following facts.

1. Given a fixed point \( p = (r, \theta) \in \mathcal{A} \) and a lift \( \hat{h} \) of \( h \) there is an integer \( m[\hat{h}, p] \) such that \( \hat{h}(\frac{r}{2^m} + n, r) = (\frac{r}{2^m} - n + m[\hat{h}, p], r) \) for every \( (\frac{r}{2^m} + n, r) \in \tau^{-1}(p) \).
2. \( \hat{h} \) has a fixed point in \( \tau^{-1}(p) \) iff \( m[\hat{h}, p] \) is even.
3. If \( m[\hat{h}, p] \) is even, then \( \hat{h}(x + 1, y) \) is a lift homeomorphism of \( h \) that does not have a fixed point in \( \tau^{-1}(p) \).
(4) \( \tilde{\mathcal{A}} \) can be compactified by two points, say \( b_1, b_2 \), so that \( \tilde{X} = \tau^{-1}(X) \cup \{b_1, b_2\} \) is a continuum invariant under \( \tilde{h} \), and the latter can be extended to an orientation-reversing homeomorphism of the entire plane onto itself.

Let \( \hat{h}_1 : \hat{\mathcal{A}} \to \hat{\mathcal{A}} \) be a lift of \( h \) and \( \hat{h}_2(x, y) = \hat{h}_1(x + 1, y) \) be another lift, fixed once and for all. For simplicity we will use the same symbols \( \hat{h}_1, \hat{h}_2 \) to denote the extensions of these two lifts to the entire plane.

**Proposition 2.1.** If \( Y \) is a subcontinuum of the set of fixed points of \( h \), then \( Y \) does not separate the plane.

**Proof.** If \( F \) is the fixed point set of a homeomorphism \( f \) of a connected topological manifold \( M \), then either each component of \( M \setminus F \) is invariant under \( f \) or there are exactly two components of \( M \setminus F \) and \( f \) interchanges them \([6]\). Since in the case of planar homeomorphisms the unbounded complementary domain of \( F \) is always invariant under \( h \), therefore the above implies that all components of \( \mathbb{R}^2 \setminus F \) must be invariant under \( h \). Consequently if \( Y \) were a continuum of fixed points of \( h \) separating the plane, then \( Y \) could be essentially inscribed into the annulus \( \mathcal{A} \) with \( \mathcal{A}^- \) and \( \mathcal{A}^+ \) invariant under \( h \), and \( h \) would induce the identity on the homology group \( H_1(\mathcal{A}, \mathbb{Z}) \). Therefore any lift \( \hat{h} \) of \( h \) to the universal cover \( \hat{\mathcal{A}} \) would preserve the orientation on the two boundary components of \( \hat{\mathcal{A}} \), at the same time keeping them invariant. Consequently \( \hat{h} \) would be orientation-preserving on \( \hat{\mathcal{A}} \), contradicting the fact that any lift of \( h \) to \( \hat{\mathcal{A}} \) must be orientation-reversing. \( \square \)

**Lemma 2.2.** Suppose \( p \) is a fixed point of \( h \) and let \( Y \) be the component of \( p \) in \( \text{Fix}(X, h) \). Then
\[
m[\hat{h}_1, p] = m[\hat{h}_1, q](\text{mod} 2)
\]
for every \( q \in Y \).

**Proof.** First, \( Y \) does not separate the plane. Suppose \( m[\hat{h}_1, p] \) is even. Let \( \alpha \) be the fixed point of \( \hat{h}_1 \) in \( \tau^{-1}(p) \) and let \( K \) be the component of \( \tau^{-1}(Y) \) containing \( \alpha \). To the contrary, suppose the above claim is false and let \( q \in Y \) be such that \( m[\hat{h}_1, q] \) is odd. For every \( \beta \in \tau^{-1}(q) \) we have \( \beta \neq \hat{h}_1(\beta) \). Let \( \gamma \) be in \( K \cap \tau^{-1}(q) \). Then \( \hat{h}_1(\gamma) \neq \gamma \) and \( \hat{h}_1(\gamma) \in \hat{h}_1(K) \). Since \( \hat{h}_1(K) \) is also a component of \( \tau^{-1}(Y) \) and \( \alpha \in \hat{h}_1(K) \), then \( K = \hat{h}_1(K) \). Consequently, \( K \) contains two elements from the same fiber \( \tau^{-1}(q) \in \tau^{-1}(Y) \). But this contradicts the following observation indicated in \([5]\), which in turn will complete the proof.

Since \( Y \subseteq \mathcal{A} \) does not separate the plane, one can choose a disk \( D \subseteq \mathcal{A} \) around \( Y \); i.e., \( Y \subseteq \text{Int} D \), and \( \text{Int} D \) being simply connected lifts to disjoint homeomorphic copies of \( \text{Int} D \) in \( \hat{\mathcal{A}} \). Consequently \( Y \) lifts to disjoint homeomorphic copies in \( \hat{\mathcal{A}} \). Since \( K \) is one of them, it cannot contain two points from the same fiber \( \tau^{-1}(q) \). \( \square \)

As a consequence of the above, for a given component \( Y \) of \( \text{Fix}(X, h) \) one can choose any \( p \in Y \) and say that \( m[\hat{h}_1, Y] \) is even (or odd) if \( m[\hat{h}_1, p] \) is of the same parity.

3. **Proof of Theorem [11]**

**Proof of Theorem [11].** We will prove this theorem by induction. First, observe that the case when \( n = 0 \) is the theorem of Bell [1]. Indeed, if \( X \) is a nonseparating plane continuum, then by Bell’s theorem \( h \) must have a fixed point in \( X \), and therefore there is at least one component of \( \text{Fix}(X, h) \).
For the sake of induction suppose the theorem is true for \( n = k - 1 \). Now we will show that the theorem holds true for \( n = k \).

Assume \( U_1, \ldots, U_k \) are bounded complementary domains of \( X \) invariant under \( h \) and that \( h_i \) is inserted into \( U_i \). We may assume that there is a fixed point \( u_i \) of \( h \) in each \( U_i \). Without loss of generality assume that \( u_1, \ldots, u_p \) are all fixed points of \( h \) such that there is a fixed point of \( h_1 \) in the fiber \( \tau^{-1}(u_1) \), for all \( i = 1, \ldots, p \). In other words, each set from \( U_1, \ldots, U_p \) contains in its lift \( \tau^{-1}(U_i) \) a bounded complementary domain of \( X \) that is invariant under \( h_1 \). Equivalently, \( m[h_1, u_i] \) is even for \( i = 1, \ldots, p \) and \( m[h_1, u_i] \) is odd for \( i = p + 1, \ldots, k - 1 \).

Let \( q = k - 1 - p \). Note that \( p, q \) are nonnegative integers (possibly with \( p \) or \( q \) equal to 0). Since \( X \) is a continuum with \( p \) bounded complementary domains invariant under \( h_1 \) and \( p \leq k - 1 \), by the induction hypothesis there are \( p + 1 \) components of \( \text{Fix}(h_1, X) \). Let \( A_1, \ldots, A_{p+1} \) be those components.

For every \( i = 1, \ldots, p + 1 \) there is a component \( X_i \) of \( \text{Fix}(h, X) \) such that \( X_i = \tau(A_i) \). Note that \( \tau(A_i) \) and \( \tau(A_t) \) are disjoint for \( i \neq t \) since any fiber of a fixed point of \( h \) contains no more than one fixed point of \( h_1 \). Therefore \( \{X_i : i = 1, \ldots, p + 1\} \) consists of \( p + 1 \) distinct components of \( \text{Fix}(X, h) \).

Now, \( \tau^{-1}(X_i) \) is invariant under \( h_1 \), and \( m[h_1, X_i] \) is even for every \( i = 1, \ldots, p + 1 \). \( \tau^{-1}(X_i) \) is also invariant under \( h_2 \) but contains no fixed point of \( h_2 \), since \( m[h_2, X_i] \) is odd for every \( i = 1, \ldots, p + 1 \). For \( i = 1, \ldots, p \), no \( \tau^{-1}(u_i) \) contains a fixed point of \( h_2 \), since \( m[h_2, u_i] \) is odd. For \( i = p + 1, \ldots, k - 1 \), every \( \tau^{-1}(u_i) \) contains a fixed point of \( h_2 \), since \( m[h_2, u_i] \) is even. Therefore, there are \( q = (k - 1) - p \) bounded complementary domains of \( X \) that are invariant under \( h_2 \). Again, by the induction hypothesis, there must be \( q + 1 \) components of \( \text{Fix}(h_2, X) \). Denote them by \( C_1, \ldots, C_q \). For every \( j = 1, \ldots, q + 1 \), \( \tau(C_j) \) is a component of \( \text{Fix}(h, X) \). Note that \( \tau(C_j) \) and \( \tau(C_t) \) are disjoint for \( j \neq t \) since any fiber of a fixed point of \( h \) contains no more than one fixed point of \( h_2 \). Therefore \( \{\tau(C_j) : j = 1, \ldots, q + 1\} \) consists of \( q + 1 \) distinct components of \( \text{Fix}(X, h) \). Since each \( \tau^{-1}(X_i) \) contains no fixed point of \( h_2 \), no \( \tau(C_j) \) can coincide with any \( X_i \). Therefore there are \( p + 1 + q + 1 = k + 1 \) components of \( \text{Fix}(h, X) \). This completes the proof. \( \square \)

Note that Theorem 3.1 generalizes to orientation-reversing homeomorphisms of \( S^2 \). More precisely, we get the following as a corollary.

**Theorem 3.1.** Let \( g : S^2 \to S^2 \) be an orientation-reversing homeomorphism of \( S^2 \) onto itself with a continuum \( X \) invariant, and suppose there are at least \( n \) components of \( S^2 \setminus X \) that are invariant under \( g \). Then \( \text{Fix}(X, g) \) has at least \( n \) components.

Proof. First suppose that \( S^2 \setminus X \) has exactly one component \( U \) invariant under \( g \). We can assume that there is a fixed point \( u \) of \( g \) in \( U \). Notice that \( S^2 \setminus \{u\} \) is topologically the plane, and \( G = g(S^2 \setminus \{u\}) \), obtained by the restriction of \( g \) to \( S^2 \setminus \{u\} \), is an orientation-reversing homeomorphism of the plane onto itself with the continuum \( X \) invariant. Now, since \( X \) has no bounded complementary domains invariant under \( G \), by a theorem of Bell there is at least one component of \( \text{Fix}(X, G) = \text{Fix}(X, g) \). Bell’s theorem applies to nonseparating plane continua, but in the above case if \( X \) separates the plane and none of the bounded complementary domains is invariant under \( G \), then these domains can be added to \( X \) to form a nonseparating plane continuum \( Y \) with \( \text{Fix}(X, G) = \text{Fix}(Y, G) \).
Second, suppose that $\mathbb{S}^2 \setminus X$ has at least two components $U_1$ and $U_2$ invariant under $g$. Then there is an annulus $A$ such that $X \subseteq A$, $A^- \subseteq U_1$ and $A^+ \subseteq U_2$. Since $U_1$ and $U_2$ are invariant under $g$, then $h$ does not interchange $A^-$ and $A^+$, and one can repeat the proof of Theorem 1.1.

\[\square\]

4. Proof of Theorem 1.2

Theorem 1.2 seems to fit well in the following context. The Cartwright-Littlewood-Bell theorem (see [7] and [1]) states that any planar homeomorphism fixes a point in an invariant nonseparating continuum. Morton Brown [5] and O.H. Hamilton [5] exhibited that, in the case of orientation-preserving homeomorphisms, this theorem can be deduced directly from a theorem of Brouwer [4]. Brouwer showed that any orientation-preserving homeomorphism with at least one bounded orbit must have a fixed point. Briefly, the idea behind these short proofs of the fixed point theorem was to separate the invariant continuum from the fixed-point set, and then for an open invariant component $U$ in $\mathbb{R}^2 \setminus F$ containing $X$ argue that $U$ contains no fixed point, thus contradicting the theorem of Brouwer. The inspiration for the proof of Theorem 1.2 comes from these very papers, but since the set of 2-periodic points does not need to be closed (in contrast with the fixed-point set), one cannot just replace the theorem of Brouwer with a theorem of Bonino from [2] and use the same arguments. Instead, we will use Bonino’s result from [3] and show that no 2-periodic orbit in an invariant component of $\mathbb{R}^2 \setminus X$ can be linked to a $k$-periodic ($k > 2$) orbit in $X$.

\textbf{Proof of Theorem 1.2.} Compactify $\mathbb{R}^2$ by a point $\infty$ to obtain $\mathbb{S}^2 = \mathbb{R}^2 \cup \{\infty\}$ and extend the given homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ to a homeomorphism $\tilde{h} : \mathbb{S}^2 \to \mathbb{S}^2$ by setting $\tilde{h}|_{\mathbb{R}^2} = h$ and $\tilde{h}(\infty) = \infty$. $h$ and $\tilde{h}$ have exactly the same $k$-periodic points for any $k > 1$.

By Bonino’s result there is an orbit $O' \subseteq \mathbb{S}^2$ of $\tilde{h}$ of period exactly 2. We will show that any such 2-periodic orbit that lies in an invariant complementary domain of $X$ is not linked to $O$.

Suppose $O' \cap X = \emptyset$ and $O' \subseteq U$ for a complementary domain $U$ of $X$ invariant under $h$. Since $O'$ and $X$ are closed, there is a Jordan curve $S \subseteq \mathbb{S}^2$ separating $O'$ from $X$. Let $D$ be one of the two disks in $\mathbb{S}^2$ bounded by $S$, such that $X \subseteq \text{Int} D$. Then $D \cap O' = \emptyset$. Since $X$ is invariant under $\tilde{h}$, by continuity of $\tilde{h}$, there is a disk $C$ such that $C \subseteq \text{Int} D$ and $\tilde{h}(C) \subseteq \text{Int} D$. Since both $C$ and $\tilde{h}(C)$ contain $X$ in its interior, there is a disk $B \subseteq C \cap \tilde{h}(C)$ that contains $X$ in its interior. Therefore $C$ and $\tilde{h}(C)$ are freely isotopic in the annulus $D \setminus \text{Int} B$, thus freely isotopic in $\mathbb{S}^2 \setminus (O \cup O')$. This shows that if $O' \subseteq \mathbb{S}^2 \setminus X$ is a 2-periodic orbit, then $O'$ and $O$ are not linked. Therefore the 2-periodic orbit $O'$ linked to $O$, guaranteed by a theorem of Bonino in [3], must be in $X$ or in a 2-periodic component of $\mathbb{S}^2 \setminus X$.

\[\square\]

\textbf{Corollary 4.1.} Suppose there is a $k$-periodic component of $\mathbb{R}^2 \setminus X$, for $k > 2$. Then either there is a 2-periodic orbit in $X$ or there is a 2-periodic component of $\mathbb{R}^2 \setminus X$.

\textbf{Proof.} Let $W$ be a $k$-periodic complementary domain of $X$ ($k > 2$). Without loss of generality one may assume that there is a $k$-periodic point $w$ in $W$ ($w$ is a fixed point of $h^k$). Consider $Y = X \cup W \cup h(W) \cup \ldots \cup h^{k-1}(W)$. Clearly $Y$ is a continuum invariant under $h$. Now apply Theorem 1.2.

\[\square\]
Remark. It is clear from the proof of Theorem 1.2 that this theorem holds also for any orientation-reversing homeomorphism of $S^2$. On the other hand, it is not apparent to the present author if one can improve Theorem 1.2 and get rid of the 2-periodic component of $\mathbb{R}^2 \setminus X$ to guarantee that, under the assumptions, there will be a 2-periodic point in $X$. Nonetheless, the following example shows that one cannot do it for $S^2$.

Example. Let $S^2$ be given in spherical coordinates by $S^2 = \{(r,\theta,\phi) : r = 1, 0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi\}$. Consider the Jordan curve $S \subseteq S^2$ determined by $S = \{(r,\theta,\phi) : r = 1, \phi = \frac{\pi}{2}\}$. Let $U^+, U^-$ be the two disks in $S^2 \setminus S$ bounded by $S$. Fix $k > 2$ and consider the orientation-reversing homeomorphism $g : S^2 \to S^2$ determined by

$$g(r,\theta,\phi) = (r,\theta + \frac{2\pi}{k}, \pi - \phi).$$

$g$ interchanges $U^+$ and $U^-$, reflecting $S^2$ about $S$ and then rotating $S^2$ by $\frac{2\pi}{k}$. Notice that $g^2(r,\theta,\phi) = (r,\theta + \frac{2\pi}{k}, \phi)$ and $g^k(r,\theta,\phi) = (r,\theta,\phi) = \text{id}_{S^2}(r,\theta,\phi)$. Clearly, $g$ is an orientation-reversing homeomorphism of $S^2$ with the continuum $S$ invariant, and any point in $S$ is of period exactly $k$, but the only points of period 2 are the two poles, which are not in $S$.

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