REGULARITY OF GEODESIC RAYS
AND MONGE-AMPERE EQUATIONS

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Abstract. It is shown that the geodesic rays constructed as limits of Bergman geodesics from a test configuration are always of class $C^{1,\alpha}$, $0 < \alpha < 1$. An essential step is to establish that the rays can be extended as solutions of a Dirichlet problem for a Monge-Ampère equation on a Kähler manifold which is compact.

1. Introduction

The purpose of this paper is to establish the $C^{1,\alpha}$ regularity, $0 < \alpha < 1$, of the geodesic rays constructed in [PS07] from a test configuration by Bergman geodesic approximations. With the notation given in section 2 below, our main result can be stated as follows:

Theorem 1.1. Let $L \to X$ be a positive holomorphic line bundle over a compact complex manifold $X$ of dimension $n$. Let $\rho$ be a test configuration for the polarization $(X, L)$. Let $D^\times = \{0 < |w| \leq 1\}$ be the punctured unit disk, and let $\pi_X$ be the natural projection $X \times D^\times \to X$. For each metric $h_0$ on $L$ with positive curvature $\omega_0 \equiv -\frac{i}{2} \partial \bar{\partial} \log h_0 > 0$, let $\Phi(z, w)$ be the $\pi_X^*(\omega_0)$-plurisubharmonic function on $X \times D^\times$ defined by

$$\Phi(z, w) = \lim_{k \to \infty} \left[ \sup_{\ell \geq k} \Phi_\ell(z, w) \right]^*, \quad (z, w) \in X \times D^\times,$$

(1.1)

where $\Phi_\ell(z, w)$ are the functions defined by (2.10) below. Then for any $0 < \alpha < 1$, $\Phi(z, w)$ is a $C^{1,\alpha}$ generalized solution of the Dirichlet problem

$$\left(\pi_X^*(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi\right)^{n+1} = 0 \text{ on } X \times D^\times, \quad \Phi(z, w) = 0 \text{ when } |w| = 1.$$  

(1.2)

The fact that $\Phi(z, w)$ is locally bounded and a solution of the Dirichlet problem was established in [PS07], so the new part of the theorem is the $C^{1,\alpha}$ regularity. In the case of toric varieties, the $C^{1,\alpha}$ regularity of geodesic rays was previously established by Song and Zelditch [SZ08], using an explicit analysis of orthonormal bases for $H^0(X, L^k)$ and the theory of large deviations. They also pointed out that, already for toric varieties, geodesic rays from test configurations can be at best $C^{1,1}$.

The interpretation of the completely degenerate Monge-Ampère equation in (1.2)
as the equation for geodesics in the space of Kähler potentials of class $c_1(L)$ on $X$ is well-known and due to Donaldson [D99], Semmes [S], and Mabuchi [M].

In [PS09], $C^{1,\alpha}$ geodesic rays were constructed in all generality from test configurations by a different approach, namely viscosity methods for the degenerate complex Monge-Ampère equation on a compactification $\bar{X}_D$ of $X \times D^\times$. Thus our theorem can be established by showing that the above solution, more precisely $\Phi(z, w) - \Phi_1(z, w)$, can also be extended to $\bar{X}_D$ and that such solutions must be unique. For this, it is essential to show that $\Phi(z, w) - \Phi_1(z, w)$ is uniformly bounded on $X \times D^\times$. We accomplish that with the help of a “lower-triangular” property of Donaldson’s equivariant imbeddings, relating $k$-th powers of sections of $H^0(X, L)$ to sections of $H^0(X, L^k)$, which may be of independent interest (see Lemma 2.2 below).

The uniqueness follows from a comparison theorem for Monge-Ampère equations on Kähler manifolds with boundary, using the approximation theorems for plurisubharmonic functions obtained recently by Blocki and Kolodziej [BK] (see also Demailly and Paun [DP] for other approximation theorems). It is well-known that such approximation theorems would imply comparison theorems by a straightforward adaptation to Kähler manifolds of the classic comparison theorem of Bedford and Taylor [BTS2] for domains in $\mathbb{C}^m$.

The proof of Theorem 1.1 also shows that the function $\Phi_\ell - \Phi_1$ extends for all $\ell$ to a smooth function on $X_{\text{red}}$, the variety underlying the test configuration $X$. Thus its limsup envelope $\Phi - \Phi_1$ is an upper semi-continuous function on $X_{\text{red}}$ whose restriction to the central fiber of $X_{\text{red}}$ is a well-defined bounded upper semi-continuous function.

As has been stressed in [PS06], each test configuration defines a generalized vector field on the space of Kähler potentials, with the vector at each potential $h_0$ given by the tangent vector $\dot{\phi}$ to the geodesic at the initial time. This observation can now be given a precise formulation using the measures recently introduced by Berndtsson [B09a]: for each generalized $C^{1,\alpha}$ geodesic $(-\infty, 0] \to \phi(z, t) \equiv \Phi(z, e^t)$, the functional $\mu_\phi : C_0^0(\mathbb{R}) \ni f \to \int_X f(\phi)\omega_\phi^n(\cdot, t)$ defines a Borel measure on $\mathbb{R}$ which is independent of $t$. Taking $t = 0$, we can think of this measure as a way of characterizing $\dot{\phi}(0)$ by its moments. If $\Phi$ is the geodesic constructed in Theorem 1.1, the corresponding assignment $h_0 \to \mu_\phi$ can be viewed as a precise realization of the generalized vector field defined by the test configuration $\rho$.

We note that Theorem 1.1 gives the regularity of the limiting function $\Phi(z, w)$, but it does not provide information on the precise rate of convergence of $\Phi_k$. For toric varieties, very precise rates of convergence have been provided by Song and Zelditch [SZ06, SZ08]. For general manifolds, in the case of geodesic segments, the precise rate of $C^0$ convergence was obtained a few years ago by Berndtsson [B09a] with an additional twisting by $\frac{1}{2}K_X$ and very recently in [B09b] for the $\Phi_k$ themselves.

Finally, we would like to mention that geodesics have been constructed by Arezzo and Tian [AT], Chen [C00, C08], Chen and Tang [CT], Chen and Sun [CS], Blocki [B09] and others in various geometric situations. For geodesic segments, the $C^{1,\alpha}$ regularity has been established by Chen [C00]. Their construction by Bergman approximations is in [PS06]. This construction has also been extended by Rubinstein and Zelditch [RZ] to the construction of harmonic maps in the space of Kähler potentials in the case of toric varieties.
2. The extension to a compact Kähler manifold

In this section, we show how the generalized geodesic rays constructed in [PS07], originally defined on \( X \times \{ 0 < |w| \leq 1 \} \), actually extend as bounded solutions of a complex Monge-Ampère equation over a compact Kähler manifold \( \tilde{X}_D \supset X \times \{ 0 < |w| \leq 1 \} \). We begin by introducing the notation and recalling the results of [PS07].

2.1. Test configurations. Let \( L \to X \) be a positive line bundle over a compact complex manifold \( X \) of dimension \( n \). A test configuration \( \rho \) for \( L \to X \) [D02] is a homomorphism \( \rho : \mathbb{C}^\times \to \text{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C}) \), where \( \mathcal{L} \) is a \( \mathbb{C}^\times \) equivariant line bundle with ample fibers over a scheme \( \mathcal{X} \), and \( \pi : \mathcal{X} \to \mathbb{C} \) is a flat \( \mathbb{C}^\times \) equivariant map of schemes, with \( (\pi^{-1}(1), \mathcal{L}_{|_{\pi^{-1}(1)}}) \) isomorphic to \((X, L')\) for some fixed \( r > 0 \). Replacing \( L \) by \( L' \), we may assume that \( r = 1 \).

It is convenient to denote \( (\pi^{-1}(w), \mathcal{L}_{|_{\pi^{-1}(w)}}) \) by \((X_w, L_w)\). In particular, for each \( \tau \neq 0 \), \( \rho(\tau) \) is an isomorphism between \((X_w, L_w)\) and \((X_{\tau w}, L_{\tau w})\).

The central fiber \((X_0, L_0)\) is fixed under the action of \( \rho \). Thus, for each \( k \), \( \rho \) induces a one-parameter subgroup of automorphisms

\[
\rho_k(\tau) : H^0(X_0, L_0^k) \to H^0(X_0, L_0^k), \quad \tau \in \mathbb{C}^\times.
\]

Since \( \rho_k(\tau) \) is an algebraic one-parameter subgroup, there is a basis of \( H^0(X_0, L_0^k) \) in which \( \rho(\tau) \) is represented by a diagonal matrix with entries \( \tau^{\eta^{(k)}_\alpha} \), where \( \eta^{(k)}_\alpha \) are integers, \( 0 \leq \alpha \leq N_k \equiv \dim H^0(X_0, L_0^k) - 1 \). Set

\[
\lambda^{(k)}_\alpha = \eta^{(k)}_\alpha - \frac{1}{N_k + 1} \sum_{\beta=0}^{N_k} \eta^{(k)}_\beta,
\]

so that \( (\lambda^{(k)}_\alpha) \) is the traceless component of \( (\eta^{(k)}_\alpha) \). For a fixed \( k \), we shall refer to \( \eta^{(k)}_\alpha \) and \( \lambda^{(k)}_\alpha \) respectively as the weights and the traceless weights of the test configuration \( \rho \).

It is convenient to introduce an \( (N_k + 1) \times (N_k + 1) \) diagonal matrix \( B_k \) whose diagonal entries are given by the weights \( \eta^{(k)}_\alpha \). Such a matrix is determined up to a permutation of the diagonal entries \( \eta^{(k)}_\alpha \), and we fix one choice once and for all. Then the traceless weights \( \lambda^{(k)}_\alpha \) are the diagonal entries of the matrix \( A_k \) defined by \( A_k = B_k - (N_k + 1)^{-1}(\text{Tr} B_k)I \), and

\[
\text{Tr} B_k = \sum_{\alpha=0}^{N_k} \eta^{(k)}_\alpha, \quad \text{Tr} A_k = 0.
\]

For sufficiently large \( k \), the functions \( k(N_k + 1) \) and \( \text{Tr} B_k \) are polynomials in \( k \) of degree \( n + 1 \), so we have an asymptotic expansion

\[
\frac{\text{Tr} B_k}{k(N_k + 1)} \equiv F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots.
\]

The Donaldson-Futaki invariant of \( \rho \) is defined to be the coefficient \( F_1 \).

2.2. Equivariant imbeddings of test configurations. An essential property of test configurations, due to Donaldson [D05], is that the entire configuration can be imbedded equivariantly in \( \mathbb{C}P^{N_k} \times \mathbb{C} \) in a way which respects a given \( L^2 \) metric on \( H^0(X, L^k) \). The following formulation [PS07, Lemmas 2.1-2.3] is most convenient for our purposes.
Let $s(k) = \{s^{(k)}(z)\}_{\alpha=0}^{N_k}$ be a basis for $H^0(X, L^k)$. For all $k$ sufficiently large, it defines a Kodaira imbedding
\begin{equation}
\iota_s(k) : X \ni z \rightarrow [s^{(k)}_0(z) : s^{(k)}_1(z) : \cdots : s^{(k)}_{N_k}(z)] \in \mathbb{CP}^{N_k}
\end{equation}
of $X$ into $\mathbb{CP}^{N_k}$, with $O(1)$ pulled back to $L^k$. If $h_0$ is a fixed metric on $L$ with $\omega_0 \equiv -\frac{1}{2} \partial \bar{\partial} \log h_0 > 0$, then $H^0(X, L^k)$ can be equipped with the $L^2$ metric defined by the metric $h_0^k$ on sections of $L^k$ and the volume form $\omega_0^k/n!$. For simplicity, we shall refer to this $L^2$ metric on $H^0(X, L^k)$ as just the “$L^2$ metric defined by $h_0$". Of particular importance then are the bases $s(k)$ which are orthonormal with respect to this $L^2$ metric.

**Lemma 2.1.** Let $\rho : \mathbb{C}^\times \rightarrow \text{Aut}(L \rightarrow X \rightarrow \mathbb{C})$ be a test configuration, and fix a metric $h_0$ on $L$ with positive curvature $\omega_0$ and a corresponding $L^2$ metric on $H^0(X, L^k)$. Then there is an orthonormal basis $s(k)$ of $H^0(X, L^k) = H^0(X_1, L_1^k)$ with respect to the $L^2$ metric defined by $h_0$ and an imbedding
\begin{equation}
I_s : (L \rightarrow X \rightarrow \mathbb{C}) \rightarrow (O(1) \times \mathbb{C} \rightarrow \mathbb{CP}^{N_k} \times \mathbb{C} \rightarrow \mathbb{C})
\end{equation}
satisfying
\begin{enumerate}
\item $I_s|_X = \iota_s(k)$;
\item $I_s$ intertwines $\rho(\tau)$ and $B_k$,
\end{enumerate}
\begin{equation}
I_s(\rho(\tau)\ell_w) = (\tau B_s I_s(\ell_w), \tau w), \quad \ell_w \in L_w, \quad \tau \in \mathbb{C}^\times.
\end{equation}

Let $E_k = \pi_*(L^k)$ be the direct images of the bundles $L^k$. Thus $E_k \rightarrow \mathbb{C}$ is a vector bundle over $\mathbb{C}$ of rank $N_k + 1$, and its sections $S(w)$ are holomorphic sections of $L_w$ for each $w \in \mathbb{C}$. The action of $\mathbb{C}^\times$ on the sections $S$ is given by
\begin{equation}
S^\tau(w) = \rho(\tau)^{-1}S(w\tau).
\end{equation}

Then a third key statement in the equivariant imbedding lemma is:
\begin{enumerate}
\item The functions $S_\alpha(w) = w^{n_\alpha}\rho(w)s_\alpha, \quad w \in \mathbb{C}^\times$, extend to a basis for the free $\mathbb{C}[w]$ module of all sections of $E_k \rightarrow \mathbb{C}$ and they have the property $S_\alpha(1) = s_\alpha$. This extension still satisfies the relation
\begin{equation}
\rho(\tau)^{-1}S_\alpha(w) = \tau^{n_{\alpha}(k)}S_\alpha(w), \quad w \in \mathbb{C}.
\end{equation}

In the language of [PS07], the Hermitian generator $\Theta$ of the test configuration is the isomorphism $H^0(X_0, L_0^k) \rightarrow H^0(X_1, L_1^k)$ sending the basis $S_\alpha(0)$ to the basis $S_\alpha(1)$.

**2.3. The construction of geodesics.** We come now to the construction of geodesics by Bergman approximations. Let $\rho : \mathbb{C}^\times \rightarrow \text{Aut}(L \rightarrow X \rightarrow \mathbb{C})$ be a test configuration for $L \rightarrow X$, and fix a metric $h_0$ on $L$ with positive curvature $\omega_0$. Let $s(k) = \{s^{(k)}(z)\}$ be an orthonormal basis for $H^0(X, L^k)$ with respect to the $L^2$ metric defined by $h_0$ as in Lemma 2.1. Let $D^\times = \{w \in \mathbb{C}; 0 < |w| \leq 1\}$ be the punctured disk. Define the functions $\Phi_k : X \times D^\times \rightarrow \mathbb{R}$ by
\begin{equation}
\Phi_k(z, w) = \frac{1}{k} \log \sum_{\alpha=0}^{N_k} |w|^{2n_{\alpha}(k)} |s^{(k)}_\alpha(z)|^2_{h_0^k} - \frac{n}{k} \log k
\end{equation}
and $\Phi(z, w)$ by
\begin{equation}
\Phi(z, w) = \lim_{k \to \infty} \sup_{r \geq k} \Phi_k(z, w)^r.
\end{equation}
where \( \eta^{(k)}_\alpha \) are the weights of the test configuration \( \rho \), \( \ast \) denotes the upper semi-continuous envelope, i.e.

\[
f^\ast(z) = \lim_{\varepsilon \to 0^+} \sup_{|w-z|<\varepsilon} f(w),
\]

and \( |s_\alpha(z)|^2 \equiv s_\alpha(z)\overline{s_\alpha(z)}h_0(z)^k \) denotes the norm-squared of \( s_\alpha(z) \) with respect to the metric \( h_0^k \). Then it is shown in [PS07] that \( \Phi(z, w) \) is a generalized geodesic ray in the sense that

(a) \( \pi_X^\ast(\omega_0) + \frac{i}{2} \partial \overline{\partial} \Phi \geq 0 \) on \( X \times D^\times \), where \( \pi_X \) is the projection \( X \times D^\times \to X \);

(b) for each finite \( T > 0 \), we have

\[
(2.13) \quad \sup_k|\Phi^k(z, w)|, |\Phi(z, w)| \leq C_T \text{ for } (z, w) \in X \times \{e^{-T} < |w| \leq 1\}
\]

with \( C_T \) a constant independent of \( z, w \) and \( k \), but possibly depending on \( T \);

(c) \( \Phi(z, w) \) is continuous when \( |w| = 1 \) and is a solution in the sense of pluripo-tential theory of the following Dirichlet problem:

\[
(2.14) \quad (\pi_X^\ast(\omega_0) + \frac{i}{2} \partial \overline{\partial} \Phi)^{n+1} = 0 \text{ on } X \times D^\times ; \quad \Phi(z, w) = 0 \text{ when } |w| = 1.
\]

The geodesic \( \Phi(z, w) \) is non-constant if the test configuration is non-trivial, that is, not holomorphically equivalent to a product test configuration. We note that in the boundary value problem \( (2.14) \), the behavior of \( \Phi(z, w) \) near \( w = 0 \) is not specifically assigned.

2.4. **Formulation in terms of equivariant imbeddings.** We come now to the main task in this chapter, which is to identify the solution \( (2.14) \) with the restriction to \( X \times D^\times \) of the solution of a standard Dirichlet problem on a compact Kähler manifold \( X_D \) with boundary.

Let \( \pi_{\text{red}} : X_{\text{red}} \to \mathbb{C} \) be the projection map, and let \( D = \{ w \in \mathbb{C} : |w| \leq 1 \} \). Here \( X_{\text{red}} \) is the variety underlying the scheme \( X \). Let \( X^\times_D = \pi_{\text{red}}^{-1}(D) \), \( X^\times_D = \pi_{\text{red}}^{-1}(D^\times) \).

The space \( X^\times_D \) is isomorphic to \( X \times D^\times \) under the correspondence

\[
(2.15) \quad X \times D^\times \ni (z, w) \to \rho(w)(z) \in X_w,
\]

where \( z \in X \) is viewed as a point in \( X_1 \). This correspondence lifts to a correspondence between \( L \times D^\times \) and the restriction \( L^\times_D \) of \( L \) over \( X^\times_D \).

Let \( p : \hat{X} \to X_{\text{red}} \to \mathbb{C} \) be an \( S^1 \) equivariant smooth resolution and \( \hat{L} = p^*L \).

The first step is to show that the functions \( \Phi_k(z, w) - \Phi_1(z, w) \) of \( (2.10) \), which are defined on \( X \times D^\times \), may be extended to plurisubharmonic functions on all of \( X^\times_D = p^{-1}(X_D) \).

Let us fix a metric \( h_0 \) on \( L \) with positive curvature \( \omega_0 \). Let \( s(k) \) be the orthonormal basis for \( H^0(X, L^k) \) with respect to \( h_0 \) provided by Lemma 2.1 and let \( I_{s(k)} \)

[1] Actually, in [PS07], the weights \( \eta^{(k)}_\alpha \) in the definition of \( \Phi_k(z, w) \) were replaced by the traceless weights \( \lambda^{(k)}_\alpha \). If we denote by \( \Phi_k^{(\ast)}(z, w) \) the functions obtained in this manner with the traceless weights, then we have

\[
(2.12) \quad \Phi_k(z, w) = \Phi_k^{(\ast)}(z, w) + \frac{T \beta_k}{k(N_k + 1)} \log |w|^2.
\]

It follows that the complex Hessians of \( \Phi_k(z, w) \) and \( \Phi_k^{(\ast)}(z, w) \) are identical. However, the behaviors near \( |w| = 0 \) of \( \Phi_k(z, w) \) and \( \Phi_k^{(\ast)}(z, w) \) are different, and for our purposes, it is important to work with \( \Phi_k(z, w) \).
be a corresponding equivariant imbedding of the test configuration. Let \( \Phi_k(z, w) \) be defined by (2.10). Define a closed \((1,1)\)-form \( \Omega_k \) on \( \tilde{X}_D \) by

\[
(2.16) \quad \Omega_k = \frac{1}{k} (I_{z(k)} \circ p)^* \omega_{FS}
\]

where \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{CP}^{N_k} \). Define as well a Hermitian metric \( H_k \) on \( \tilde{L} \) by \( H_k = (I_{z(k)} \circ p)^*(h_{FS})^{1/k} \), where \( h_{FS} \) is the Fubini-Study metric on the hyperplane bundle \( O(1) \) over \( \mathbb{CP}^{N_k} \). Thus \( \Omega_k \) is the curvature of \( H_k \). The restriction of \( \omega_k \) to \( \tilde{X}_D \) can be readily worked out explicitly in terms of the coordinates \((z, w)\).

Using the intertwining property of the equivariant imbedding,

\[
(2.17) \quad X \times D^x \ni (z, w) \to \rho(w)z \to I_{z(k)}(\rho(w)z) = (w^{I_{z(k)}} I_{z(k)}(z), w),
\]

we find that \( I_{z(k)} \) is given by

\[
(2.18) \quad X \times D^x \ni (z, w) \to ([|w| s_0(z) : w| s_1(z) : \cdots : w| s_n(z)], w) .
\]

Since the Fubini-Study metric \( h_{FS} \) on \( O(1) \) at \([s_0 : s_1 : \cdots : s_n] \in \mathbb{CP}^{N_k}\) is given by \( h_{FS} = (|s_0|^2 + \cdots + |s_n|^2)^{-1} \), we obtain the following expression for \( \Omega_k \):

\[
(2.19) \quad \Omega_k|_{X \times D^x} = \frac{i}{k} \frac{2}{2} \partial \overline{\partial} \log \sum_{\alpha=0}^{N_k} |w|^{2\alpha} |s_\alpha(z)|^2.
\]

Recalling that the norm with respect to \( h_0^k \) of a section \( s(z) \) of \( L^k \) is given by \(|s(z)|_{h^0}^2 = |s(z)|^2 h_0^k \), we find the following key relation between the \((1,1)\)-forms \( \Omega_k \) and the potentials \( \Phi_k(z, w) \) defined earlier in (2.10):

\[
(2.20) \quad \Omega_k|_{X \times D^x} = \pi^*(\omega_0) + \frac{i}{2} \partial \overline{\partial} \Phi_k(z, w).
\]

2.5. The extension of \( \Psi_k \) to the total space \( \tilde{X}_D \). The relation (2.20) that we have just obtained shows that the form \( \pi^*(\omega_0) + \frac{i}{2} \partial \overline{\partial} \Phi_k(z, w) \), defined originally on \( \tilde{X}_D \), admits the natural extension \( \Omega_k \) to the whole of \( \tilde{X}_D \).

Since the form \( \pi^*(\omega_0) \) does not extend by itself to \( \tilde{X} \), we rewrite \( \Omega_k \) as

\[
(2.21) \quad \Omega_k = \Omega_1 + \frac{i}{2} \partial \overline{\partial} (\Phi_k - \Phi_1) \equiv \Omega_1 + \frac{i}{2} \partial \overline{\partial} \Psi_k.
\]

The function \( \Psi_k = \Phi_k - \Phi_1 \) has a simple interpretation that shows that it extends as a smooth function to the whole of \( X_D \): as we saw earlier in §2.2, under the maps \( I_{z(k)} \) and \( I_{z(1)} \) of the test configuration \( \rho \), the Fubini-Study metric \( h_{FS} \) pulls back respectively to \( H_k^1 = (\sum_\alpha |w|^{2\alpha} |s_\alpha(z)|^2)^{-1} \) and \( H_1 = (\sum_\alpha |w|^{2\alpha} |s_1(z)|^2)^{-1} \) on \( L \times D^x \). Thus

\[
(2.22) \quad \Psi_k = \log \frac{H_1}{H_k} - n \frac{k}{k} \log k.
\]

The right hand side is a well-defined, smooth scalar function over the whole of \( \tilde{X}_D \), since it is the logarithm of the ratio of two smooth metrics on the same line bundle \( \tilde{L} \to \tilde{X}_D \).
Furthermore, the coefficients

\[ \alpha \]

if

\[ \omega \]

Now let

\[ S \]

be made uniform in

Lemma 2.2.

Lemma 2.2. Fix a test configuration \( \rho \) and a metric \( h_0 \) on \( L \) with positive curvature \( \omega_0 \). For each \( k \), let \( s(k) = \{ s_\alpha^{(k)} \}_\alpha \) be an orthonormal basis for \( H^0(X, L^k) \) as in Lemma 2.1. Then for any \( s_\beta^{(1)} \) in \( s(1) \), we can write

\[ (s_\beta^{(1)})^k = \sum_{\eta_\beta^{(k)} \leq k} a_{\beta \alpha} s_\alpha^{(k)}, \]

where \( a_{\beta \alpha} \in \mathbb{C} \) and the subindex indicates the range of indices \( \alpha \) which are allowed. Furthermore, the coefficients \( a_{\beta \alpha} \) satisfy the bound

\[ |a_{\beta \alpha}| \leq V^{\frac{1}{2}} M^k, \]

where we have set \( M = \sup_{0 \leq \beta \leq N_1} \sup_X |s_\beta^{(1)}| h_0 \) and \( V = \int_X \omega_0^n \).

Proof of Lemma 2.2. For each \( k \), let \( E_k = \pi_*(L^k) \rightarrow \mathbb{C} \), and let \( S_0(w), \ldots, S_{N_k}(w) \) be a basis for the free \( \mathbb{C}[w] \) module of sections of \( E_k \rightarrow \mathbb{C} \), as provided in Lemma 2.1. Now let \( S_\beta \) be an element of this basis for \( E_1 \rightarrow \mathbb{C} \) and some \( \beta \) with \( 0 \leq \beta \leq N_1 \). Then \( \rho(\tau)^{-1} S_\beta(w) = \tau^{\eta_\beta^{(1)}} S_\beta(w) \) which implies

\[ \rho(\tau)^{-1} S_\beta(w) = \tau^{k\eta_\beta^{(1)}} S_\beta(w). \]

On the other hand, \( S_\beta^k \) is a section of \( E_k \), so we may write

\[ S_\beta^k(w) = \sum_{\alpha=0}^{N_k} a_\alpha(w) S_\alpha(w) \]

for certain uniquely defined polynomials \( a_\alpha(w) \in \mathbb{C}[w] \). Applying the \( \mathbb{C}^\times \) action to both sides of (2.27) we obtain

\[ \sum_{\alpha=0}^{N_k} \tau^{k\eta_\alpha^{(1)}} a_\alpha(w) S_\alpha(w) = \tau^{k\eta_\beta^{(1)}} S_\beta^k(w) = \rho(\tau)^{-1} S_\beta^k(w \tau) = \sum_{\alpha=0}^{N_k} a_\alpha(w \tau) \tau^{\eta_\alpha^{(k)}} S_\alpha(w). \]

Comparing coefficients we obtain

\[ \tau^{k\eta_\alpha^{(1)}} a_\alpha(w) = a_\alpha(w \tau) \tau^{\eta_\alpha^{(k)}}. \]

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In general, given a non-negative smooth, closed (1,1)-form \( \Omega \) on a complex manifold \( X \), we say that a scalar function \( \Phi \) is \( \Omega \)-plurisubharmonic if \( f_\alpha + \Phi \) is plurisubharmonic on \( U_\alpha \) for each \( \alpha \) if \( X = \bigcup_\alpha U_\alpha \) is a covering of \( X \) by coordinate charts \( U_\alpha \) with \( \Omega = \frac{1}{2} \partial \bar{\partial} f_\alpha \) on \( U_\alpha \).
Setting \( w = 1 \) we see that \( a_\alpha(\tau) = a_\beta \tau^r \) for some integer \( r_\alpha \) and some \( a_\beta \in \mathbb{C} \). But \( a_\alpha(w) \) is a polynomial. Thus \( r_\alpha \geq 0 \) and \( a_\alpha(w) = a_\beta w^r \) for all \( w \in \mathbb{C} \).

The equation (2.29) implies that if \( a_\beta \neq 0 \), we have \( k_\beta^{(1)} = r_\alpha + \eta_\alpha(k) \) and thus \( \eta_\alpha(k) \leq k_\beta^{(1)} \). Evaluating (2.27) at \( w = 1 \) we obtain the first part of the lemma.

Finally, the orthonormality of the sections \( s_\alpha^{(k)} \) implies

\[
|a_\beta| = |\int \langle (s_\beta^{(1)})^k, s_\alpha^{(k)} \rangle_{h_0^*} \omega^n_0 | \leq \int |s_\beta^{(1)}|_{h_0} \cdot |s_\alpha^{(k)}|_{h_0^*} \omega^n_0 \leq M^k V^\frac{1}{k}.
\]

Remark. It may happen that \( \eta_\alpha^{(k)} < k_\beta^{(1)} \) for all \( \alpha \) with \( a_\beta \neq 0 \); that is, it may happen that \( a_\alpha(w) \) vanishes at \( w = 0 \) for all \( \alpha \). This would mean that \( S_\beta(0) \) is a non-zero section of \( H^0(X_0, L_0) \) but that \( S_\beta^k(0) = 0 \in H^0(X_0, L_0^k) \); in other words, the section \( S_\beta(0) \) is nilpotent (which is possible if \( X_0 \) is a non-reduced scheme, that is, if \( X_0 \) has nilpotent elements in its structure sheaf).

\[\square\]

**Lemma 2.3.** The complex manifold \( \tilde{X}_D \) always admits a Kähler metric.

**Proof of Lemma 2.3.** This lemma is proved in [PS07a]. In fact, it is proved there that there exists a line bundle \( \mathcal{M} \) on \( X_D \) which is trivial on \( X^\times \) and such that \( L^m \otimes \mathcal{M} \) is positive for some fixed positive power \( m \). The desired Kähler metric on \( \tilde{X}_D \) can then be taken to be the ratio of the curvature of \( L^m \otimes \mathcal{M} \) by \( m \). \( \square \)

**Lemma 2.4.** There exists a finite constant \( C \) so that

\[
\sup_{k \geq 1} \sup_{\tilde{X}_D} |\Psi_k| \leq C < \infty.
\]

In particular,

\[
\sup_{\tilde{X}_D} |\Psi| \leq C < \infty.
\]

**Proof of Lemma 2.4.** Let \( H \) be a Kähler metric on \( \tilde{X}_D \), which exists by Lemma 2.3. Since \( \Psi_k \) is \( \Omega_1 \)-plurisubharmonic, it follows that \( \Delta_H \Psi_k \geq -C_1 \), where \( \Delta_H \) is the Laplacian with respect to \( H \), and \( C_1 \) is an upper bound for the trace of \( \Omega_1 \) with respect to the metric \( H \). On the other hand, \( \Psi_k |_{\partial \tilde{X}_D} \to -\Phi_1 \) uniformly as \( k \to \infty \), and thus \( \Psi_k |_{\partial \tilde{X}_D} \leq C_2 \). Let \( u \) be the smooth function on \( \tilde{X}_D \) which is the solution of the Dirichlet problem

\[
\Delta_H u = -C_1 \text{ on } \tilde{X}_D, \quad u = C_2 \text{ on } \partial \tilde{X}_D.
\]

By the maximum principle, we have \( \Psi_k \leq u \) for all \( k \), and this gives the upper bound.

To establish the lower bound, it suffices to prove that \( \Psi_k \geq -C \) on \( X_D^\times \), where \( C \) is a constant independent of \( k \), since each function \( \Psi_k \) is smooth on \( \tilde{X}_D \). On \( X^\times \), we can use the explicit expressions for \( X \times D^\times \) and write

\[
\Psi_k = \log \left( \frac{\sum_{\alpha=0}^{N_k} |w|^{2\eta_\alpha^{(k)}} |s_\alpha^{(1)}|_{h_0^*}^2}{\sum_{\beta=0}^{N_1} |w|^{2\eta_\beta^{(1)}} |s_\beta^{(1)}|_{h_0}^2} \right) - \frac{n}{k} \log k.
\]

Now fix \( w \) with \( 0 < |w| \leq 1 \), fix \( z \in X \), and choose \( \beta_0 \) so that

\[
|w|^{2\eta_\beta^{(1)}} |s_\beta^{(1)}(z)|_{h_0}^2 = \sup_{0 \leq \beta \leq N_1} |w|^{2\eta_\beta^{(1)}} |s_\beta^{(1)}(z)|_{h_0}^2.
\]
In view of Lemma 2.2, we can write
\begin{equation}
\int (b) \Psi \text{ satisfies the same completely degenerate equation on } \tilde{\Omega}, \text{ by Lemma 2.6.} \tag{2.36}
\end{equation}

|w|^{2k\eta^{(i)}_{\alpha_0}} |s^{(k)}_{\alpha_0}(z)|^{2}_{h_0} \leq M^{2k} V \left( \sum_{k\eta^{(i)}_{\alpha_0} \geq \eta^{(k)}_{\alpha}} |w|^{\alpha_{\nu}} |s^{(k)}_{\alpha}(z)|^{2}_{h_0} \right)^{\frac{1}{\alpha}} \\tag{2.37}
\end{equation}

Since \(|w| \leq 1\), we have then
\begin{equation}
|w|^{2k\eta^{(i)}_{\alpha_0}} |s^{(k)}_{\alpha_0}(z)|^{2}_{h_0} \leq M^{2k} V \left( \sum_{k\eta^{(i)}_{\alpha_0} \geq \eta^{(k)}_{\alpha}} |w|^{\alpha_{\nu}} |s^{(k)}_{\alpha}(z)|^{2}_{h_0} \right)^{\frac{1}{\alpha}} \\tag{2.38}
\end{equation}

Returning to \(\Psi_k\), we can now write
\begin{equation}
\Psi_k(z, w) \geq \log \left( \frac{\sum_{\alpha=0}^{N_k} |w|^{2\eta^{(1)}_{\alpha}} |s^{(1)}_{\alpha}(z)|^{2}_{h_0}}{n} \right) - \frac{n}{k} \log k \\tag{2.39}
\end{equation}

in view of the preceding inequality. This establishes Lemma 2.3 since \(N_k \leq C k^n\).

\section{The Monge-Ampère equation on the whole of \(\tilde{\Omega}_D\).}

We can now prove the main theorem of this section:

\textbf{Theorem 2.5.} Let \(L \to X\) be a positive line bundle over a compact complex manifold, let \(\rho\) be a test configuration, and let \(h_0\) be a metric on \(L\) with positive curvature \(\omega_0\). Let \(\tilde{X}\) be an \(S^1\) invariant resolution \(\pi : \tilde{X} \to X_{\text{red}} \to \mathbb{C}\) of \(X\), and let \(\tilde{X}_D = (\pi \circ p)^{-1}(D)\). Let \(\Phi_k, \Phi\) be defined as in (2.10) and (2.11). Set
\begin{equation}
\Psi = \Phi - \Phi_1 \text{ on } X \times D^k. \\tag{2.40}
\end{equation}

Then the function \(\Psi\) extends to a bounded, \(\Omega_1\)-plurisubharmonic function on \(\tilde{X}_D\), which is a generalized solution of the following Dirichlet problem on \(\tilde{X}_D\),
\begin{equation}
(\Omega_1 + \frac{i}{2} \partial \bar{\partial} \Psi)^{n+1} = 0 \text{ on } \tilde{X}_D, \quad \Psi = -\Phi_1 \text{ on } \partial \tilde{X}_D. \tag{2.41}
\end{equation}

\textbf{Proof of Theorem 2.5.} The function \(\Psi\) satisfies the completely degenerate Monge-Ampère equation on \(X^{\nu}_{\alpha}\). Since the singular set \(X_0\) is an analytic subvariety and since Lemma 2.2 implies that the function \(\Psi\) defined by (2.23) is a bounded, \(\Omega_1\)-plurisubharmonic function on \(\tilde{X}_D\), it follows from general pluripotential theory that \(\Psi\) satisfies the same completely degenerate equation on \(\tilde{X}_D\). Alternatively, a direct proof of this fact can also be given, since we already have at hand all the necessary ingredients. It suffices to observe that \(\Psi_k\) satisfies the following properties and apply Theorem 3 of [PS06]:

\textbf{Lemma 2.6.} The functions \(\Psi_k\) satisfy
\begin{itemize}
  \item[(a)] \(\sup_k \sup_{\tilde{X}_D} |\Psi_k| \leq C < \infty\).
  \item[(b)] \(\int_{\tilde{X}_D} (\Omega_1 + \frac{i}{2} \partial \bar{\partial} \Psi_k)^{n+1} \leq C \frac{1}{k}\).
\end{itemize}
(c) Let $T$ be the vector field $T = \frac{\partial}{\partial w}$ defined in a neighborhood of the boundary $|w| = 1$ on $\tilde{X}_D$, where $t = \log |w|$. Then $\sup_U |T \Psi_k| \leq C$, where $C$ is a constant and $U$ is a neighborhood of the boundary $|w| = 1$, independent of $k$.

(d) \[ \sup_{\partial \tilde{X}_D} |\Psi_k + \Phi_1| \leq a_k, \]
with $a_k$ decreasing to 0 and $\sum_{k=1}^{\infty} a_k < \infty$.

**Proof of Lemma 2.4.** Part (a) is just the statement of Lemma 2.3. Part (b) follows from the fact that the form $\Omega_1 + \frac{i}{2} \partial \bar{\partial} \Psi_k$ is smooth on $\tilde{X}_D$ and that its Monge-Ampère mass on $\tilde{X}_D^\times$ coincides with the Monge-Ampère mass of $\omega_k = (\pi^*_{\tilde{X}}(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi_k)$ on $X \times D^\times$. As we already observed in footnote 1, $\Phi_k$ and $\Phi^p_k$ have the same complex Hessian. So the desired estimate follows from the analogous estimate for the Monge-Ampère mass of $(\pi^*_{\tilde{X}}(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi^p_k)$ established in Lemma 4.3 of [PS07]. Part (c) follows from the bound $|\hat{\eta}_k| \leq C\eta$, established in Lemma 3.1 of [PS07]. Finally, part (d), with $a_k = C k^{-2}$, follows from the Tian-Yau-Zelditch theorem [TY03, Y93, Z] (see also Catlin [Ca] and Lu [L]) as shown in the case of geodesic segments in [PS06]. \qed

2.8. **Positivity of the background form away from $p^{-1}(X_0)$**. The equation (2.40) provides an extension of the degenerate complex Monge-Ampère equation to the compact manifold with boundary $\tilde{X}_D$. It is however written with respect to a background $(1,1)$-form $\Omega_1$ which may be degenerate. In preparation for uniqueness theorems for the complex Monge-Ampère equation, we rewrite it now with a background $(1,1)$-form which is non-negative everywhere and strictly positive away from $p^{-1}(X_0)$.

For this, we make use of Lemma 1 of [PS09], which asserts the existence of a $S^1$ invariant metric $H_0$ on $\hat{L}$ with $H_0|_{\partial \tilde{X}_D} = h_0$ and

\[ \Omega_0 = -\frac{i}{2} \partial \bar{\partial} \log H_0 \geq 0 \quad \text{on} \quad \tilde{X}_D, \quad \Omega_0 > 0 \quad \text{on} \quad \tilde{X}_D^\times. \]

Let $\Psi_0$ be defined by

\[ \Psi_0 = \log \frac{H_0}{(\varphi_1 \circ p)^*(H_{FS})} = \log \frac{H_0}{H_1}, \]

which is a smooth function on $\tilde{X}_D$, since $H_0$, $H_1$ are two smooth metrics on the same line bundle $\hat{L}$. Restricted to $\partial \tilde{X}_D$,

\[ \Psi_0|_{\partial \tilde{X}_D} = \log \frac{h_0}{(\sum_{\alpha=0}^{N_1} |s^{(1)}_\alpha|^2)^{-1}} = \log \sum_{\alpha=0}^{N_1} |s^{(1)}_\alpha|^2 h_0 = \Phi_1|_{\partial \tilde{X}_D}. \]

Let $\Psi$ be the solution on $\tilde{X}_D$ of the completely degenerate Monge-Ampère equation with background form $\Omega_1$ as given in Theorem 2.5. Define the function $\Phi$ on $\tilde{X}_D$ by

\[ \Phi = \Psi + \Psi_0. \]

Clearly $(\Omega_0 + \frac{i}{2} \partial \bar{\partial} \Phi)^{n+1} = 0$ on $\tilde{X}_D$. Furthermore, restricted to the boundary $\partial \tilde{X}_D$,

\[ \Phi|_{\partial \tilde{X}_D} = \Psi|_{\partial \tilde{X}_D} + \Psi_0|_{\partial \tilde{X}_D} = -\Phi_1|_{\partial \tilde{X}_D} + \Phi_1|_{\partial \tilde{X}_D} = 0. \]

In summary, we have obtained the following alternative formulation of Theorem 2.5.
Theorem 2.7. Let the setting be the same as in Theorem 2.5 and let \( H_0 \) be a metric on \( \mathcal{L} \) as in \((2.41)\), \( \Psi_0 = \log \frac{H_0}{H_1} \), and \( \hat{\Psi} \equiv \Phi - \Phi_1 + \Psi_0 \). Then the function \( \hat{\Psi} \) is a bounded, \( \Omega_0 \)-plurisubharmonic generalized solution of the following Dirichlet problem:

\[
(\Omega_0 + \frac{i}{2} \partial \bar{\partial} \hat{\Phi})^{n+1} = 0 \text{ on } \tilde{\mathcal{X}}_\Omega, \quad \hat{\Phi}|_{\partial \tilde{\mathcal{X}}_\Omega} = 0.
\]

(2.45)  

3. A uniqueness theorem for completely degenerate complex Monge-Ampère equations

There has been considerable progress recently on uniqueness theorems for the complex Monge-Ampère equation and in particular for certain broad classes of possibly unbounded solutions (see e.g. Blocki [B03], Blocki and Kolodziej [BK], Dinew [D], and references therein). For our purposes we need a version of the comparison principle of Bedford and Taylor which can be formulated as follows:

Theorem 3.1. Let \((M, \Omega)\) be a compact Kähler manifold with smooth boundary \( \partial M \) and dimension \( m \), and let \( \Omega_0 \) be a smooth, non-negative, closed \((1, 1)\)-form. Then we have

\[
\int_{\{u < v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m \leq \int_{\{u < v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} v)^m
\]

for all \( u, v \in L^\infty \), \( \Omega_0 \)-plurisubharmonic, and satisfying \( \liminf_{z \to \partial M} (u(z) - v(z)) \geq 0 \).

The proof is a straightforward adaption of the original proof of Bedford-Taylor [BT82] in \( \mathbb{C}^m \) to our setting using the approximation theorem for plurisubharmonic functions of Blocki and Kolodziej [BK].

Theorem 3.1 implies the following uniqueness theorem for \( \Omega_0 \)-plurisubharmonic solutions of completely degenerate Monge-Ampère equations, where the form \( \Omega_0 \) is allowed to be degenerate along an analytic subvariety:

Theorem 3.2. Let \((M, \Omega)\) be a Kähler manifold with smooth boundary \( \partial M \) and dimension \( m \), and let \( u, v \in L^\infty \) be \( \Omega_0 \)-plurisubharmonic functions satisfying

\[
(\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m = (\Omega_0 + \frac{i}{2} \partial \bar{\partial} v)^m = 0, \quad \limsup_{z \to \partial M} (u(z) - v(z)) = 0.
\]

If \( \Omega_0 \) is \( \geq 0 \) everywhere and \( > 0 \) away from an analytic subvariety of strictly positive codimension which does not intersect \( \partial M \), then we must have \( u = v \) on \( M \).

Proof. By adding the same large constant to both \( u \) and \( v \), we may assume that \( u, v > 0 \). Arguing by contradiction we begin by assuming that \( S = \{u < v\} \neq \emptyset \).

Since \( u, v \) are \( \Omega_0 \)-plurisubharmonic, the set \( S \) must have strictly positive measure (it suffices to work in local coordinates and apply the corresponding well-known property of plurisubharmonic functions on \( \mathbb{C}^m \)). Furthermore, since we can write

\[
S = \bigcup_{\varepsilon > 0} \{u < (1 - \varepsilon) v\} = \bigcup_{\varepsilon > 0} S_{\varepsilon},
\]

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it follows that $S_\varepsilon$ must have strictly positive measure for some $\varepsilon > 0$. Fix one such value of $\varepsilon$. Since $u \geq v \geq (1 - \varepsilon)v$ on $\partial M$, we may apply the comparison principle for Kähler manifolds and obtain

$$0 \geq \int_{S_\varepsilon} (\Omega_0 + \frac{i}{2} \partial \overline{\partial} u)^m \geq \int_{S_\varepsilon} (\Omega_0 + (1 - \varepsilon) \frac{i}{2} \partial \overline{\partial} v)^m$$

$$= \int_{S_\varepsilon} \{(1 - \varepsilon)(\Omega_0 + \frac{i}{2} \partial \overline{\partial} v) + \varepsilon \Omega_0\}^m \geq \varepsilon^m \int_{S_\varepsilon} \Omega_0^m$$

(3.4)

since the form $\Omega_0 + \frac{i}{2} \partial \overline{\partial} v$ is non-negative. Let $V_\delta$ be the complement of a neighborhood of the divisor $D$, with $\Omega_0^m \geq \delta \Omega^m$ for each $\delta > 0$ small enough. Clearly for each $\delta > 0$

$$\int_{S_\varepsilon} \Omega_0^m \geq \int_{S_\varepsilon \cap V_\delta} \Omega_0^m \geq \delta \int_{S_\varepsilon \cap V_\delta} \Omega^m.$$  

(3.5)

Since $M \setminus D = \bigcup_{\delta > 0} V_\delta$ and $D$ has measure 0 with respect to the volume form $\Omega^m$, we have

$$0 < \int_{S_\varepsilon} \Omega^m = \lim_{\delta \to 0} \int_{S_\varepsilon \cap V_\delta} \Omega^m,$$  

(3.6)

which implies that $\int_{S_\varepsilon \cap V_\delta} \Omega^m > 0$ for some $\delta > 0$. Altogether, we obtain a contradiction. Thus $\{u < v\}$ must be empty. Interchanging the roles of $u$ and $v$ completes the proof.

4. Proof of Theorem 1.1

We now prove Theorem 1.1. In Theorem 2.7, we have shown that the function $\hat{\Phi}$ is a bounded, $\Omega_0$-plurisubharmonic solution of the Dirichlet problem (2.45) on $\tilde{X}_D$. On the other hand, in [PS09] (Theorem 3), it was shown that the same Dirichlet problem admits a bounded, $\Omega_0$-plurisubharmonic solution which is $C^{1,\alpha}$ for any $0 < \alpha < 1$ on $\mathcal{X}_D^\times$. By Theorem 3.2, it follows that the two solutions must coincide. Thus $\hat{\Phi}$ is $C^{1,\alpha}$ on $\mathcal{X}_D^\times$. Since $\hat{\Phi} = \Phi - \Phi_1 + \Psi_0$ and both $\Phi_1$ and $\Psi_0$ are smooth on $\mathcal{X}_D^\times$, it follows that $\Phi$ is $C^{1,\alpha}$ on $\mathcal{X}_D^\times = X \times D^\times$.  

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