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REGULARITY OF GEODESIC RAYS AND MONGE-AMPÈRE EQUATIONS

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Abstract. It is shown that the geodesic rays constructed as limits of Bergman geodesics from a test configuration are always of class $C^{1,\alpha}$, $0 < \alpha < 1$. An essential step is to establish that the rays can be extended as solutions of a Dirichlet problem for a Monge-Ampère equation on a Kähler manifold which is compact.

1. Introduction

The purpose of this paper is to establish the $C^{1,\alpha}$ regularity, $0 < \alpha < 1$, of the geodesic rays constructed in [PS07] from a test configuration by Bergman geodesic approximations. With the notation given in section 2 below, our main result can be stated as follows:

Theorem 1.1. *Let $L \to X$ be a positive holomorphic line bundle over a compact complex manifold X of dimension n. Let ρ be a test configuration for the polarization (X, L). Let $D^\times = \{0 < |w| \leq 1\}$ be the punctured unit disk, and let π_X be the natural projection $X \times D^\times \to X$. For each metric h_0 on L with positive curvature $\omega_0 \equiv -\frac{i}{2}\partial\bar\partial \log h_0 > 0$, let $\Phi(z,w)$ be the $\pi_X^*(\omega_0)$-plurisubharmonic function on $X \times D^\times$ defined by*

$$\Phi(z,w) = \lim_{k\to\infty}[\sup_{\ell\geq k}\Phi_\ell(z,w)]^*, \quad (z,w) \in X \times D^\times, \tag{1.1}$$

where $\Phi_k(z,w)$ are the functions defined by (2.10) *below. Then for any* $0 < \alpha < 1$, *$\Phi(z,w)$ is a $C^{1,\alpha}$ generalized solution of the Dirichlet problem*

$$(\pi_X^*(\omega_0) + \frac{i}{2}\partial\bar\partial\Phi)^{n+1} = 0 \text{ on } X \times D^\times, \quad \Phi(z,w) = 0 \text{ when } |w| = 1. \tag{1.2}$$

The fact that $\Phi(z,w)$ is locally bounded and a solution of the Dirichlet problem was established in [PS07], so the new part of the theorem is the $C^{1,\alpha}$ regularity. In the case of toric varieties, the $C^{1,\alpha}$ regularity of geodesic rays was previously established by Song and Zelditch [SZ08], using an explicit analysis of orthonormal bases for $H^0(X, L^k)$ and the theory of large deviations. They also pointed out that, already for toric varieties, geodesic rays from test configurations can be at best $C^{1,1}$. The interpretation of the completely degenerate Monge-Ampère equation in (1.2)

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as the equation for geodesics in the space of Kähler potentials of class $c_1(L)$ on X is well-known and due to Donaldson [D99], Semmes [S], and Mabuchi [M].

In [PS09], $C^{1,\alpha}$ geodesic rays were constructed in all generality from test configurations by a different approach, namely viscosity methods for the degenerate complex Monge-Ampère equation on a compactification $\tilde{\mathcal{X}}_D$ of $X \times D^\times$. Thus our theorem can be established by showing that the above solution, more precisely $\Phi(z,w) - \Phi_1(z,w)$, can also be extended to $\tilde{\mathcal{X}}_D$ and that such solutions must be unique. For this, it is essential to show that $\Phi(z,w) - \Phi_1(z,w)$ is uniformly bounded on $X \times D^\times$. We accomplish that with the help of a "lower-triangular" property of Donaldson's equivariant imbeddings, relating k-th powers of sections of $H^0(X,L)$ to sections of $H^0(X,L^k)$, which may be of independent interest (see Lemma 2.2 below).

The uniqueness follows from a comparison theorem for Monge-Ampère equations on Kähler manifolds with boundary, using the approximation theorems for plurisubharmonic functions obtained recently by Blocki and Kolodziej [BK] (see also Demailly and Paun [DP] for other approximation theorems). It is well-known that such approximation theorems would imply comparison theorems by a straightforward adaptation to Kähler manifolds of the classic comparison theorem of Bedford and Taylor [BT82] for domains in $\mathbb{C}^m$.

The proof of Theorem 1.1 also shows that the function $\Phi_\ell - \Phi_1$ extends for all ℓ to a smooth function on $\mathcal{X}_{\rm red}$, the variety underlying the test configuration $\mathcal{X}$. Thus its limsup envelope $\Phi - \Phi_1$ is an upper semi-continuous function on $\mathcal{X}_{\rm red}$ whose restriction to the central fiber of $\mathcal{X}_{\rm red}$ is a well-defined bounded upper semi-continuous function.

As has been stressed in [PS06], each test configuration defines a generalized vector field on the space of Kähler potentials, with the vector at each potential h_0 given by the tangent vector $\dot\phi$ to the geodesic at the initial time. This observation can now be given a precise formulation using the measures recently introduced by Berndtsson [B09b]: for each generalized $C^{1,\alpha}$ geodesic $(-\infty, 0] \in t \to \phi(z,t) \equiv \Phi(z, e^t)$, the functional $\mu_\Phi : C_0^0(\mathbb{R}) \ni f \to \int_X f(\dot\phi)\omega^n_{\phi(\cdot,t)}$ defines a Borel measure on $\mathbb{R}$ which is independent of t. Taking $t = 0$, we can think of this measure as a way of characterizing $\dot\phi(0)$ by its moments. If Φ is the geodesic constructed in Theorem 1.1, the corresponding assignment $h_0 \to \mu_\Phi$ can be viewed as a precise realization of the generalized vector field defined by the test configuration ρ.

We note that Theorem 1.1 gives the regularity of the limiting function $\Phi(z,w)$, but it does not provide information on the precise rate of convergence of Φ_k. For toric varieties, very precise rates of convergence have been provided by Song and Zelditch [SZ06, SZ08]. For general manifolds, in the case of geodesic segments, the precise rate of C^0 convergence was obtained a few years ago by Berndtsson [B09a] with an additional twisting by $\frac{1}{k}K_X$ and very recently in [B09b] for the Φ_k themselves.

Finally, we would like to mention that geodesics have been constructed by Arezzo and Tian [AT], Chen [C00, C08], Chen and Tang [CT], Chen and Sun [CS], Blocki [B09] and others in various geometric situations. For geodesic segments, the $C^{1,\alpha}$ regularity has been established by Chen [C00]. Their construction by Bergman approximations is in [PS06]. This construction has also been extended by Rubinstein and Zelditch [RZ] to the construction of harmonic maps in the space of Kähler potentials in the case of toric varieties.

2. The extension to a compact Kähler manifold

In this section, we show how the generalized geodesic rays constructed in [PS07], originally defined on $X \times \{0 < |w| \leq 1\}$, actually extend as bounded solutions of a complex Monge-Ampère equation over a compact Kähler manifold $\tilde{\mathcal{X}}_D \supset X \times \{0 < |w| \leq 1\}$. We begin by introducing the notation and recalling the results of [PS07].

2.1. **Test configurations.** Let $L \to X$ be a positive line bundle over a compact complex manifold X of dimension n. A test configuration ρ for $L \to X$ [D02] is a homomorphism $\rho : \mathbb{C}^\times \to \mathrm{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C})$, where $\mathcal{L}$ is a $\mathbb{C}^\times$ equivariant line bundle with ample fibers over a scheme $\mathcal{X}$, and $\pi : \mathcal{X} \to \mathbb{C}$ is a flat $\mathbb{C}^\times$ equivariant map of schemes, with $(\pi^{-1}(1), \mathcal{L}_{|_{\pi^{-1}(1)}})$ isomorphic to (X, L^r) for some fixed $r > 0$. Replacing L by L^r, we may assume that $r = 1$.

It is convenient to denote $(\pi^{-1}(w), \mathcal{L}_{|_{\pi^{-1}(w)}})$ by (X_w, L_w). In particular, for each $\tau \neq 0$, $\rho(\tau)$ is an isomorphism between (X_w, L_w) and $(X_{\tau w}, L_{\tau w})$.

The central fiber (X_0, L_0) is fixed under the action of ρ. Thus, for each k, ρ induces a one-parameter subgroup of automorphisms

$$\rho_k(\tau) : H^0(X_0, L_0^k) \to H^0(X_0, L_0^k), \qquad \tau \in \mathbb{C}^\times. \tag{2.1}$$

Since $\rho_k(\tau)$ is an algebraic one-parameter subgroup, there is a basis of $H^0(X_0, L_0^k)$ in which $\rho(\tau)$ is represented by a diagonal matrix with entries $\tau^{\eta_\alpha^{(k)}}$, where $\eta_\alpha^{(k)}$ are integers, $0 \leq \alpha \leq N_k \equiv \dim H^0(X_0, L_0^k) - 1$. Set

$$\lambda_\alpha^{(k)} = \eta_\alpha^{(k)} - \frac{1}{N_k + 1} \sum_{\beta=0}^{N_k} \eta_\beta^{(k)}, \tag{2.2}$$

so that (λ_α^k) is the traceless component of $(\eta_\alpha^{(k)})$. For a fixed k, we shall refer to $\eta_\alpha^{(k)}$ and $\lambda_\alpha^{(k)}$ respectively as the weights and the traceless weights of the test configuration ρ.

It is convenient to introduce an $(N_k + 1) \times (N_k + 1)$ diagonal matrix B_k whose diagonal entries are given by the weights $\eta_\alpha^{(k)}$. Such a matrix is determined up to a permutation of the diagonal entries $\eta_\alpha^{(k)}$, and we fix one choice once and for all. Then the traceless weights $\lambda_\alpha^{(k)}$ are the diagonal entries of the matrix A_k defined by $A_k = B_k - (N_k + 1)^{-1}(\mathrm{Tr}\, B_k)I$, and

$$\mathrm{Tr}\, B_k = \sum_{\alpha=0}^{N_k} \eta_\alpha^{(k)}, \qquad \mathrm{Tr}\, A_k = 0. \tag{2.3}$$

For sufficiently large k, the functions $k(N_k + 1)$ and $\mathrm{Tr}\, B_k$ are polynomials in k of degree $n + 1$, so we have an asymptotic expansion

$$\frac{\mathrm{Tr}\, B_k}{k(N_k + 1)} \equiv F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots . \tag{2.4}$$

The Donaldson-Futaki invariant of ρ is defined to be the coefficient F_1.

2.2. **Equivariant imbeddings of test configurations.** An essential property of test configurations, due to Donaldson [D05], is that the entire configuration can be imbedded equivariantly in $\mathbb{CP}^{N_k} \times \mathbb{C}$ in a way which respects a given L^2 metric on $H^0(X, L^k)$. The following formulation [PS07, Lemmas 2.1-2.3] is most convenient for our purposes.

Let $\underline{s}(k) = \{s_\alpha^{(k)}(z)\}_{\alpha=0}^{N_k}$ be a basis for $H^0(X, L^k)$. For all k sufficiently large, it defines a Kodaira imbedding

$$\iota_{\underline{s}(k)} : X \ni z \to [s_0^{(k)}(z) : s_1^{(k)}(z) : \cdots : s_{N_k}^{(k)}(z)] \in \mathbb{CP}^{N_k} \tag{2.5}$$

of X into $\mathbb{CP}^{N_k}$, with $O(1)$ pulled back to L^k. If h_0 is a fixed metric on L with $\omega_0 \equiv -\frac{i}{2}\partial\bar\partial \log h_0 > 0$, then $H^0(X, L^k)$ can be equipped with the L^2 metric defined by the metric h_0^k on sections of L^k and the volume form $\omega_0^n/n!$. For simplicity, we shall refer to this L^2 metric on $H^0(X, L^k)$ as just the "L^2 metric defined by h_0". Of particular importance then are the bases $\underline{s}(k)$ which are orthonormal with respect to this L^2 metric.

Lemma 2.1. *Let $\rho : \mathbb{C}^\times \to \mathrm{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C})$ be a test configuration, and fix a diagonal matrix B_k with the weights of ρ as diagonal entries as defined in §2.1. Fix a metric h_0 on L with positive curvature ω_0 and a corresponding L^2 metric on $H^0(X, L^k)$. Then there is an orthonormal basis $\underline{s}(k)$ of $H^0(X, L^k) = H^0(X_1, L_1^k)$ with respect to the L^2 metric defined by h_0 and an imbedding*

$$I_{\underline{s}} : (\mathcal{L} \to \mathcal{X} \to \mathbb{C}) \to (O(1) \times \mathbb{C} \to \mathbb{CP}^{N_k} \times \mathbb{C} \to \mathbb{C}) \tag{2.6}$$

satisfying

(1) $I_{\underline{s}(k)}|_X = \iota_{\underline{s}(k)}$;

(2) $I_{\underline{s}(k)}$ intertwines $\rho(\tau)$ and B_k,

$$I_{\underline{s}(k)}(\rho(\tau)\ell_w) = (\tau^{B_k} I_{\underline{s}(k)}(\ell_w), \tau w), \qquad \ell_w \in L_w,\ \tau \in \mathbb{C}^\times. \tag{2.7}$$

Let $E_k = \pi_*(\mathcal{L}^k)$ be the direct images of the bundles $\mathcal{L}^k$. Thus $E_k \to \mathbb{C}$ is a vector bundle over $\mathbb{C}$ of rank $N_k + 1$, and its sections $S(w)$ are holomorphic sections of L_w for each $w \in \mathbb{C}$. The action of $\mathbb{C}^\times$ on the sections S is given by

$$S^\tau(w) = \rho(\tau)^{-1} S(w\tau). \tag{2.8}$$

Then a third key statement in the equivariant imbedding lemma is:

(3) The functions $S_\alpha(w) \equiv w^{\eta_\alpha^{(k)}} \rho(w)\, s_\alpha$, $w \in \mathbb{C}^\times$, extend to a basis for the free $\mathbb{C}[w]$ module of all sections of $E_k \to \mathbb{C}$ and they have the property $S_\alpha(1) = s_\alpha$. This extension still satisfies the relation

$$\rho(\tau)^{-1} S_\alpha(w) = \tau^{\eta_\alpha^{(k)}} S_\alpha(w), \qquad w \in \mathbb{C}. \tag{2.9}$$

In the language of [PS07], the Hermitian generator Θ of the test configuration is the isomorphism $H^0(X_0, L_0^k) \to H^0(X_1, L_1^k)$ sending the basis $S_\alpha(0)$ to the basis $S_\alpha(1)$.

2.3. **The construction of geodesics.** We come now to the construction of geodesics by Bergman approximations. Let $\rho : \mathbb{C}^\times \to \mathrm{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C})$ be a test configuration for $L \to X$, and fix a metric h_0 on L with positive curvature ω_0. Let $\underline{s}(k) = \{s_\alpha^{(k)}(z)\}$ be an orthonormal basis for $H^0(X, L^k)$ with respect to the L^2 metric defined by h_0 as in Lemma 2.1. Let $D^\times = \{w \in \mathbb{C};\ 0 < |w| \le 1\}$ be the punctured disk. Define the functions $\Phi_k : X \times D^\times \to \mathbb{R}$ by

$$\Phi_k(z, w) = \frac{1}{k} \log \sum_{\alpha=0}^{N_k} |w|^{2\eta_\alpha^{(k)}} |s_\alpha^{(k)}(z)|^2_{h_0^k} - \frac{n}{k} \log k \tag{2.10}$$

and $\Phi(z, w)$ by

$$\Phi(z, w) = \lim_{k\to\infty} [\sup_{\ell \ge k} \Phi_\ell(z, w)]^* \tag{2.11}$$

where $\eta_\alpha^{(k)}$ are the weights of the test configuration ρ, * denotes the upper semi-continuous envelope, i.e.

$$f^*(z) = \lim_{\epsilon\to 0}\sup_{|w-z|<\epsilon} f(w),$$

and $|s_\alpha(z)|^2_{h_0^k} \equiv s_\alpha(z)\overline{s_\alpha(z)}h_0(z)^k$ denotes the norm-squared of $s_\alpha(z)$ with respect to the metric h_0^k. Then it is shown in [PS07][1] that $\Phi(z,w)$ is a generalized geodesic ray in the sense that

(a) $\pi_X^*(\omega_0) + \frac{i}{2}\partial\bar\partial\Phi \geq 0$ on $X \times D^\times$, where π_X is the projection $X \times D^\times \to X$;

(b) for each finite $T > 0$, we have

$$\sup_k|\Phi_k(z,w)|,\ |\Phi(z,w)| \leq C_T \text{ for } (z,w) \in X \times \{e^{-T} < |w| \leq 1\} \tag{2.13}$$

with C_T a constant independent of z, w and k, but possibly depending on T;

(c) $\Phi(z,w)$ is continuous when $|w| = 1$ and is a solution in the sense of pluripotential theory of the following Dirichlet problem:

$$(\pi_X^*(\omega_0) + \frac{i}{2}\partial\bar\partial\Phi)^{n+1} = 0 \text{ on } X \times D^\times;\ \Phi(z,w) = 0 \text{ when } |w| = 1. \tag{2.14}$$

The geodesic $\Phi(z,w)$ is non-constant if the test configuration is non-trivial, that is, not holomorphically equivalent to a product test configuration. We note that in the boundary value problem (2.14), the behavior of $\Phi(z,w)$ near $w = 0$ is not specifically assigned.

2.4. **Formulation in terms of equivariant imbeddings.** We come now to the main task in this chapter, which is to identify the solution (2.14) with the restriction to $X \times D^\times$ of the solution of a standard Dirichlet problem on a compact Kähler manifold $\tilde{\mathcal{X}}_D$ with boundary.

Let $\pi_{\text{red}} : \mathcal{X}_{\text{red}} \to \mathbb{C}$ be the projection map, and let $D = \{w \in \mathbb{C} : |w| \leq 1\}$. Here $\mathcal{X}_{\text{red}}$ is the variety underlying the scheme $\mathcal{X}$. Let $\mathcal{X}_D = \pi_{\text{red}}^{-1}(D)$, $\mathcal{X}_D^\times = \pi_{\text{red}}^{-1}(D^\times)$. The space $\mathcal{X}_D^\times$ is isomorphic to $X \times D^\times$ under the correspondence

$$X \times D^\times \ni (z,w) \to \rho(w)(z) \in X_w, \tag{2.15}$$

where $z \in X$ is viewed as a point in X_1. This correspondence lifts to a correspondence between $L \times D^\times$ and the restriction $\mathcal{L}_D^\times$ of $\mathcal{L}$ over $\mathcal{X}_D^\times$.

Let $p : \tilde{\mathcal{X}} \to \mathcal{X}_{\text{red}} \to \mathbb{C}$ be an S^1 equivariant smooth resolution and $\tilde{\mathcal{L}} = p^*\mathcal{L}$. The first step is to show that the functions $\Phi_k(z,w) - \Phi_1(z,w)$ of (2.10), which are defined on $X \times D^\times$, may be extended to plurisubharmonic functions on all of $\tilde{\mathcal{X}}_D = p^{-1}(\mathcal{X}_D)$.

Let us fix a metric h_0 on L with positive curvature ω_0. Let $\underline{s}(k)$ be the orthonormal basis for $H^0(X, L^k)$ with respect to h_0 provided by Lemma 2.1, and let $I_{\underline{s}(k)}$

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[1] Actually, in [PS07], the weights $\eta_\alpha^{(k)}$ in the definition of $\Phi_k(z,w)$ were replaced by the traceless weights $\lambda_\alpha^{(k)}$. If we denote by $\Phi_k^\#(z,w)$ the functions obtained in this manner with the traceless weights, then we have

$$\Phi_k(z,w) = \Phi_k^\#(z,w) + \frac{\operatorname{Tr} B_k}{k(N_k+1)} \log|w|^2. \tag{2.12}$$

It follows that the complex Hessians of $\Phi_k(z,w)$ and $\Phi_k^\#(z,w)$ are identical. However, the behaviors near $|w| = 0$ of $\Phi_k(z,w)$ and $\Phi_k^\#(z,w)$ are different, and for our purposes, it is important to work with $\Phi_k(z,w)$.

be a corresponding equivariant imbedding of the test configuration. Let $\Phi_k(z,w)$ be defined by (2.10). Define a closed $(1,1)$-form Ω_k on $\tilde{\mathcal{X}}_D$ by

$$\Omega_k = \frac{1}{k}(I_{\underline{s}(k)} \circ p)^* \omega_{FS} \tag{2.16}$$

where ω_{FS} is the Fubini-Study metric on $\mathbb{CP}^{N_k}$. Define as well a Hermitian metric H_k on $\tilde{\mathcal{L}}$ by $H_k = (I_{\underline{s}(k)} \circ p)^*(h_{\rm FS})^{1/k}$, where $h_{\rm FS}$ is the Fubini-Study metric on the hyperplane bundle $O(1)$ over $\mathbb{CP}^{N_k}$. Thus Ω_k is the curvature of H_k. The restriction of ω_k to $\mathcal{X}_D^\times$ can be readily worked out explicitly in terms of the coordinates (z,w). Using the intertwining property of the equivariant imbedding,

$$X \times D^\times \ni (z,w) \to \rho(w)z \to I_{\underline{s}(k)}(\rho(w)z) = (w^{B_k}\iota_{\underline{s}(k)}(z), w), \tag{2.17}$$

we find that $I_{\underline{s}(k)}$ is given by

$$X \times D^\times \ni (z,w) \to ([w^{\eta_0^{(k)}} s_0^{(k)}(z) : w^{\eta_1^{(k)}} s_1^{(k)}(z) : \cdots : w^{\eta_{N_k}^{(k)}} s_{N_k}^{(k)}(z)], w). \tag{2.18}$$

Since the Fubini-Study metric $h_{\rm FS}$ on $O(1)$ at $[s_0 : s_1 : \cdots : s_{N_k}] \in \mathbb{CP}^{N_k}$ is given by $h_{\rm FS} = (|s_0|^2 + \cdots + |s_{N_k}|^2)^{-1}$, we obtain the following expression for Ω_k:

$$\Omega_k|_{X\times D^\times} = \frac{1}{k}\frac{i}{2}\partial\bar\partial \log \sum_{\alpha=0}^{N_k} |w|^{2\eta_\alpha^{(k)}} |s_\alpha^{(k)}(z)|^2. \tag{2.19}$$

Recalling that the norm with respect to h_0^k of a section $s(z)$ of L^k is given by $|s(z)|^2_{h_0^k} = |s(z)|^2 h_0^k$, we find the following key relation between the $(1,1)$-forms Ω_k and the potentials $\Phi_k(z,w)$ defined earlier in (2.10):

$$\Omega_k|_{X\times D^\times} \ = \ \pi_X^*(\omega_0) + \frac{i}{2}\partial\bar\partial\Phi_k(z,w). \tag{2.20}$$

2.5. **The extension of Ψ_k to the total space $\tilde{\mathcal{X}}_D$.** The relation (2.20) that we have just obtained shows that the form $\pi_X^*(\omega_0) + \frac{i}{2}\partial\bar\partial\Phi_k(z,w)$, defined originally on $\mathcal{X}_D^\times$, admits the natural extension Ω_k to the whole of $\tilde{\mathcal{X}}_D$.

Since the form $\pi_X^*(\omega_0)$ does not extend by itself to $\tilde{\mathcal{X}}$, we rewrite Ω_k as

$$\Omega_k = \Omega_1 + \frac{i}{2}\partial\bar\partial\,(\Phi_k - \Phi_1) \equiv \Omega_1 + \frac{i}{2}\partial\bar\partial\,\Psi_k. \tag{2.21}$$

The function $\Psi_k = \Phi_k - \Phi_1$ has a simple interpretation that shows that it extends as a smooth function to the whole of $\tilde{\mathcal{X}}_D$: as we saw earlier in §2.2, under the maps $I_{\underline{s}(k)}$ and $I_{\underline{s}(1)}$ of the test configuration ρ, the Fubini-Study metric $h_{\rm FS}$ pulls back respectively to $H_k^k = (\sum_\alpha |w|^{2\eta_\alpha^{(k)}}|s_\alpha^{(k)}(z)|^2)^{-1}$ and $H_1 = (\sum_\alpha |w|^{2\eta_\alpha^{(1)}}|s_\alpha^{(1)}(z)|^2)^{-1}$ on $L \times D^\times$. Thus

$$\Psi_k = \ \log\frac{H_1}{H_k} - \frac{n}{k}\log\, k. \tag{2.22}$$

The right hand side is a well-defined, smooth scalar function over the whole of $\tilde{\mathcal{X}}_D$, since it is the logarithm of the ratio of two smooth metrics on the same line bundle $\tilde{\mathcal{L}} \to \tilde{\mathcal{X}}_D$.

Since Ω_k is non-negative as the pull-back of a non-negative form, the function Ψ_k is Ω_1-plurisubharmonic.[2] We also define

$$\Psi = \lim_{k\to\infty}[\sup_{\ell\geq k}\Psi_\ell]^*, \tag{2.23}$$

which is an extension of $\Phi - \Phi_1$ to $\tilde{\mathcal{X}}_D$.

2.6. **Uniform estimates for Ψ_k.** Recall that in [PS07], as quoted in (2.13) above, we only have bounds for the functions $\Phi_k(z,w)$ when $|w| > e^{-T}$, for some fixed finite $T > 0$. Since the function Ψ_k extends to a smooth function on $\tilde{\mathcal{X}}_D$, it follows that it is bounded on $\tilde{\mathcal{X}}_D$. However, the bound may a priori depend on k. The most important step in the extension to $\tilde{\mathcal{X}}_D$ is to show that this bound can actually be made uniform in k.

We need several lemmas, starting with the following essential "lower-triangular lemma":

Lemma 2.2. *Fix a test configuration ρ and a metric h_0 on L with positive curvature ω_0. For each k, let $\underline{s}(k) = \{s_\alpha^{(k)}\}_{\alpha=0}^{N_k}$ be an orthonormal basis for $H^0(X, L^k)$ as in Lemma 2.1. Then for any $s_\beta^{(1)}$ in $\underline{s}(1)$, we can write*

$$(s_\beta^{(1)})^k = \sum_{\eta_\alpha^{(k)}\leq k\eta_\beta^{(1)}} a_{\beta\alpha}\, s_\alpha^{(k)}, \tag{2.24}$$

where $a_{\beta\alpha} \in \mathbb{C}$ and the subindex indicates the range of indices α which are allowed. Furthermore, the coefficients $a_{\beta\alpha}$ satisfy the bound

$$|a_{\beta\alpha}| \leq V^{\frac{1}{2}} M^k, \tag{2.25}$$

where we have set $M = \sup_{0\leq\beta\leq N_1}\sup_X|s_\beta^{(1)}|_{h_0}$ and $V = \int_X \omega_0^n$.

Proof of Lemma 2.2. For each k, let $E_k = \pi_*(\mathcal{L}^k) \to \mathbb{C}$, and let $S_0(w), \cdots, S_{N_k}(w)$ be a basis for the free $\mathbb{C}[w]$ module of sections of $E_k \to \mathbb{C}$, as provided in Lemma 2.1. Now let S_β be an element of this basis for $E_1 \to \mathbb{C}$ and some β with $0 \leq \beta \leq N_1$. Then $\rho(\tau)^{-1}S_\beta(w\tau) = \tau^{\eta_\beta^{(1)}}S_\beta(w)$ which implies

$$\rho(\tau)^{-1}S_\beta^k(w\tau) = \tau^{k\eta_\beta^{(1)}}S_\beta^k(w). \tag{2.26}$$

On the other hand, S_β^k is a section of E_k, so we may write

$$S_\beta^k(w) \;=\; \sum_{\alpha=0}^{N_k} a_\alpha(w) S_\alpha(w) \tag{2.27}$$

for certain uniquely defined polynomials $a_\alpha(w) \in \mathbb{C}[w]$. Applying the $\mathbb{C}^\times$ action to both sides of (2.27) we obtain

$$\sum_{\alpha=0}^{N_k}\tau^{k\eta_\beta^{(1)}}a_\alpha(w)S_\alpha(w) \;=\; \tau^{k\eta_\beta^{(1)}}S_\beta^k(w) \;=\; \rho(\tau)^{-1}S_\beta^k(w\tau) \;=\; \sum_{\alpha=0}^{N_k}a_\alpha(w\tau)\tau^{\eta_\alpha^{(k)}}S_\alpha(w). \tag{2.28}$$

Comparing coefficients we obtain

$$\tau^{k\eta_\beta^{(1)}}a_\alpha(w) \;=\; a_\alpha(w\tau)\tau^{\eta_\alpha^{(k)}}. \tag{2.29}$$

[2] In general, given a non-negative smooth, closed $(1,1)$-form Ω on a complex manifold X, we say that a scalar function Φ is Ω-plurisubharmonic if $f_\alpha + \Phi$ is plurisubharmonic on U_α for each α if $X = \bigcup_\alpha U_\alpha$ is a covering of X by coordinate charts U_α with $\Omega = \frac{i}{2}\partial\bar\partial f_\alpha$ on U_α.

Setting $w=1$ we see that $a_\alpha(\tau)=a_{\beta\alpha}\tau^{r_\alpha}$ for some integer r_α and some $a_{\beta\alpha}\in\mathbb{C}$. But $a_\alpha(w)$ is a polynomial. Thus $r_\alpha\geq 0$ and $a_\alpha(w)=a_{\beta\alpha}w^{r_\alpha}$ for all $w\in\mathbb{C}$. The equation (2.29) implies that if $a_{\beta\alpha}\neq 0$, we have $k\eta_\beta^{(1)}=r_\alpha+\eta_\alpha^{(k)}$ and thus $\eta_\alpha^{(k)}\leq k\eta_\beta^{(1)}$. Evaluating (2.27) at $w=1$ we obtain the first part of the lemma.

Finally, the orthonormality of the sections $s_\alpha^{(k)}$ implies

$$(2.30)\qquad |a_{\beta\alpha}|=|\int\langle (s_\beta^{(1)})^k, s_\alpha^{(k)}\rangle_{h_0^k}\omega_0^n|\leq\int|s_\beta^{(1)}|_{h_0}^k\cdot|s_\alpha^{(k)}|_{h_0^k}\omega_0^n\leq M^k\,V^{\frac{1}{2}}.$$

Remark. It may happen that $\eta_\alpha^{(k)}<k\eta_\beta^{(1)}$ for all α with $a_{\beta\alpha}\neq 0$; that is, it may happen that $a_\alpha(w)$ vanishes at $w=0$ for all α. This would mean that $S_\beta(0)$ is a non-zero section of $H^0(X_0,L_0)$ but that $S_\beta^k(0)=0\in H^0(X_0,L_0^k)$; in other words, the section $S_\beta(0)$ is nilpotent (which is possible if X_0 is a non-reduced scheme, that is, if X_0 has nilpotent elements in its structure sheaf). $\square$

Lemma 2.3. *The complex manifold $\tilde{\mathcal{X}}_D$ always admits a Kähler metric.*

Proof of Lemma 2.3. This lemma is proved in [PS07a]. In fact, it is proved there that there exists a line bundle $\mathcal{M}$ on $\mathcal{X}_D$ which is trivial on $\mathcal{X}^\times$ and such that $\mathcal{L}^m\otimes\mathcal{M}$ is positive for some fixed positive power m. The desired Kähler metric on $\tilde{\mathcal{X}}_D$ can then be taken to be the ratio of the curvature of $\mathcal{L}^m\otimes\mathcal{M}$ by m. $\square$

Lemma 2.4. *There exists a finite constant C so that*

$$(2.31)\qquad \sup_{k\geq 1}\sup_{\tilde{\mathcal{X}}_D}|\Psi_k|\leq C<\infty.$$

In particular,

$$(2.32)\qquad \sup_{\tilde{\mathcal{X}}_D}|\Psi|\leq C<\infty.$$

Proof of Lemma 2.4. Let H be a Kähler metric on $\tilde{\mathcal{X}}_D$, which exists by Lemma 2.3. Since Ψ_k is Ω_1-plurisubharmonic, it follows that $\Delta_H\Psi_k\geq -C_1$, where Δ_H is the Laplacian with respect to H, and C_1 is an upper bound for the trace of Ω_1 with respect to the metric H. On the other hand, $\Psi_k|_{\partial\tilde{\mathcal{X}}_D}\to-\Phi_1$ uniformly as $k\to\infty$, and thus $\Psi_k|_{\partial\tilde{\mathcal{X}}_D}\leq C_2$. Let u be the smooth function on $\tilde{\mathcal{X}}_D$ which is the solution of the Dirichlet problem

$$(2.33)\qquad \Delta_H u=-C_1 \text{ on } \tilde{\mathcal{X}}_D,\qquad u=C_2 \text{ on } \partial\tilde{\mathcal{X}}_D.$$

By the maximum principle, we have $\Psi_k\leq u$ for all k, and this gives the upper bound.

To establish the lower bound, it suffices to prove that $\Psi_k\geq -C$ on $\mathcal{X}_D^\times$, where C is a constant independent of k, since each function Ψ_k is smooth on $\tilde{\mathcal{X}}_D$. On $\mathcal{X}^\times$, we can use the explicit expressions for $X\times D^\times$ and write

$$(2.34)\qquad \Psi_k=\log\frac{(\sum_{\alpha=0}^{N_k}|w|^{2\eta_\alpha^{(k)}}|s_\alpha^{(k)}|_{h_0^k}^2)^{\frac{1}{k}}}{\sum_{\beta=0}^{N_1}|w|^{2\eta_\beta^{(1)}}|s_\beta^{(1)}|_{h_0}^2}-\frac{n}{k}\log k.$$

Now fix w with $0<|w|\leq 1$, fix $z\in X$, and choose β_0 so that

$$(2.35)\qquad |w|^{2\eta_{\beta_0}^{(1)}}|s_{\beta_0}^{(1)}(z)|_{h_0}^2=\sup_{0\leq\beta\leq N_1}|w|^{2\eta_\beta^{(1)}}|s_\beta^{(1)}(z)|_{h_0}^2.$$

In view of Lemma 2.2, we can write

$$|(s^{(1)}_{\beta_0})^k|_{h_0^k} \le M^k V^{\frac{1}{2}} \sum_{k\eta^{(1)}_{\beta_0} \ge \eta^{(k)}_\alpha} |s^{(k)}_\alpha|_{h_0^k}. \tag{2.36}$$

Since $|w| \le 1$, we have then

$$\begin{aligned} |w|^{2k\eta^{(1)}_{\beta_0}} |s^k_{\beta_0}(z)|^2_{h_0^k} &\le M^{2k} V \Big(\sum_{k\eta^{(1)}_{\beta_0} \ge \eta^{(k)}_\alpha} |w|^{\eta^{(k)}_\alpha} |s^{(k)}_\alpha(z)|_{h_0^k} \Big)^2 \\ &\le M^{2k} V (N_k+1) \sum_{k\eta^{(1)}_{\beta_0} \ge \eta^{(k)}_\alpha} |w|^{2\eta^{(k)}_\alpha} |s^{(k)}_\alpha(z)|^2_{h_0^k}. \end{aligned} \tag{2.37}$$

Returning to Ψ_k, we can now write

$$\begin{aligned} \Psi_k(z,w) &\ge \log \frac{\big(\sum_{\alpha=0}^{N_k} |w|^{2\eta^{(k)}_\alpha} |s^{(k)}_\alpha(z)|^2_{h_0^k}\big)^{\frac{1}{k}}}{(N_1+1)|w|^{2\eta^{(1)}_{\beta_0}} |s^{(1)}_{\beta_0}(z)|^2_{h_0}} - \frac{n}{k}\log k \\ &\ge -\frac{1}{k}\log\big(V(N_k+1)\big) - 2\log M - \frac{n}{k}\log k - \log(N_1+1) \end{aligned} \tag{2.38}$$

in view of the preceding inequality. This establishes Lemma 2.4 since $N_k \le C\,k^n$. □

2.7. **The Monge-Ampère equation on the whole of $\tilde{\mathcal{X}}_D$.** We can now prove the main theorem of this section:

Theorem 2.5. *Let $L \to X$ be a positive line bundle over a compact complex manifold, let ρ be a test configuration, and let h_0 be a metric on L with positive curvature ω_0. Let $\tilde{\mathcal{X}}$ be an S^1 invariant resolution $p : \tilde{\mathcal{X}} \to \mathcal{X}_{\rm red} \to \mathbb{C}$ of $\mathcal{X}$, and let $\tilde{\mathcal{X}}_D = (\pi_{\rm red} \circ p)^{-1}(D)$. Let Φ_k, Φ be defined as in* (2.10) *and* (2.11)*. Set*

$$\Psi \;=\; \Phi - \Phi_1 \text{ on } X \times D^\times. \tag{2.39}$$

Then the function Ψ extends to a bounded, Ω_1-plurisubharmonic function on $\tilde{\mathcal{X}}_D$, which is a generalized solution of the following Dirichlet problem on $\tilde{\mathcal{X}}_D$,

$$(\Omega_1 + \frac{i}{2}\partial\bar\partial\Psi)^{n+1} = 0 \text{ on } \tilde{\mathcal{X}}_D, \qquad \Psi = -\Phi_1 \text{ on } \partial\tilde{\mathcal{X}}_D. \tag{2.40}$$

Here Ω_1 is the pull-back to $\tilde{\mathcal{X}}_D$ of the Fubini-Study metric by $I_{\underline{s}(1)} \circ p$.

Proof of Theorem 2.5*.* The function Ψ satisfies the completely degenerate Monge-Ampère equation on $\mathcal{X}_D^\times$. Since the singular set X_0 is an analytic subvariety and since Lemma 2.4 implies that the function Ψ defined by (2.23) is a bounded, Ω_1-plurisubharmonic function on $\tilde{\mathcal{X}}_D$, it follows from general pluripotential theory that Ψ satisfies the same completely degenerate equation on $\tilde{\mathcal{X}}_D$. Alternatively, a direct proof of this fact can also be given, since we already have at hand all the necessary ingredients. It suffices to observe that Ψ_k satisfies the following properties and apply Theorem 3 of [PS06]: □

Lemma 2.6. *The functions Ψ_k satisfy*
(a) $\sup_k \sup_{\tilde{\mathcal{X}}_D} |\Psi_k| \le C < \infty$.
(b)

$$\int_{\tilde{\mathcal{X}}_D} (\Omega_1 + \frac{i}{2}\partial\bar\partial\Psi_k)^{n+1} \le C\frac{1}{k}.$$

(c) Let T be the vector field $T = \frac{\partial}{\partial t}$ defined in a neighborhood of the boundary $|w| = 1$ on $\tilde{\mathcal{X}}_D$, where $t = \log |w|$. Then $\sup_U |T\Psi_k| \leq C$, where C is a constant and U is a neighborhood of the boundary $|w| = 1$, independent of k.
(d)

$$\sup_{\partial\tilde{\mathcal{X}}_D} |\Psi_k + \Phi_1| \leq a_k,$$

with a_k decreasing to 0 and $\sum_{k=1}^{\infty} a_k < \infty$.

Proof of Lemma 2.6. Part (a) is just the statement of Lemma 2.4. Part (b) follows from the fact that the form $\Omega_1 + \frac{i}{2}\partial\bar{\partial}\Psi_k$ is smooth on $\mathcal{X}_D$ and that its Monge-Ampère mass on $\mathcal{X}_D^\times$ coincides with the Monge-Ampère mass of $\omega_k = (\pi_X^*(\omega_0) + \frac{i}{2}\partial\bar{\partial}\Phi_k)$ on $X \times D^\times$. As we already observed in footnote 1, Φ_k and $\Phi_k^\#$ have the same complex Hessian. So the desired estimate follows from the analogous estimate for the Monge-Ampère mass of $(\pi_X^*(\omega_0) + \frac{i}{2}\partial\bar{\partial}\Phi_k^\#)$ established in Lemma 4.3 of [PS07]. Part (c) follows from the bound $|\eta_\alpha^{(k)}| \leq C\,k$, established in Lemma 3.1 of [PS07]. Finally, part (d), with $a_k = C\,k^{-2}$, follows from the Tian-Yau-Zelditch theorem [T90, Y93, Z] (see also Catlin [Ca] and Lu [L]) as shown in the case of geodesic segments in [PS06]. □

2.8. **Positivity of the background form away from $p^{-1}(X_0)$.** The equation (2.40) provides an extension of the degenerate complex Monge-Ampère equation to the compact manifold with boundary $\tilde{\mathcal{X}}_D$. It is however written with respect to a background $(1,1)$-form Ω_1 which may be degenerate. In preparation for uniqueness theorems for the complex Monge-Ampère equation, we rewrite it now with a background $(1,1)$-form which is non-negative everywhere and strictly positive away from $p^{-1}(X_0)$.

For this, we make use of Lemma 1 of [PS09], which asserts the existence of a S^1 invariant metric H_0 on $\tilde{\mathcal{L}}$ with $H_0|_{\partial\tilde{\mathcal{X}}_D} = h_0$ and

$$\Omega_0 \equiv -\frac{i}{2}\partial\bar{\partial}\log H_0 \geq 0 \quad \text{on } \tilde{\mathcal{X}}_D, \quad \Omega_0 > 0 \ \text{ on } \mathcal{X}_D^\times. \tag{2.41}$$

Let Ψ_0 be defined by

$$\Psi_0 = \log \frac{H_0}{(I_{\underline{s}(1)} \circ p)^*(h_{\mathrm{FS}})} = \log \frac{H_0}{H_1}\ ,$$

which is a smooth function on $\tilde{\mathcal{X}}_D$, since H_0, H_1 are two smooth metrics on the same line bundle $\tilde{\mathcal{L}}$. Restricted to $\partial\tilde{\mathcal{X}}_D$,

$$\Psi_0|_{\partial\tilde{\mathcal{X}}_D} = \log \frac{h_0}{(\sum_{\alpha=0}^{N_1} |s_\alpha^{(1)}|^2)^{-1}} = \log \sum_{\alpha=0}^{N_1} |s_\alpha^{(1)}|_{h_0}^2 = \Phi_1|_{\partial\tilde{\mathcal{X}}_D}. \tag{2.42}$$

Let Ψ be the solution on $\tilde{\mathcal{X}}_D$ of the completely degenerate Monge-Ampère equation with background form Ω_1 as given in Theorem 2.5. Define the function $\hat{\Phi}$ on $\tilde{\mathcal{X}}_D$ by

$$\hat{\Phi} = \Psi + \Psi_0. \tag{2.43}$$

Clearly $(\Omega_0 + \frac{i}{2}\partial\bar{\partial}\hat{\Phi})^{n+1} = 0$ on $\tilde{\mathcal{X}}_D$. Furthermore, restricted to the boundary $\partial\tilde{\mathcal{X}}_D$,

$$\hat{\Phi}|_{\partial\tilde{\mathcal{X}}_D} = \Psi|_{\partial\tilde{\mathcal{X}}_D} + \Psi_0|_{\partial\tilde{\mathcal{X}}_D} = -\Phi_1|_{\partial\tilde{\mathcal{X}}_D} + \Phi_1|_{\partial\tilde{\mathcal{X}}_D} = 0. \tag{2.44}$$

In summary, we have obtained the following alternative formulation of Theorem 2.5:

Theorem 2.7. *Let the setting be the same as in Theorem* 2.5*, and let* H_0 *be a metric on* $\mathcal{L}$ *as in* (2.41)*,* $\Psi_0 = \log \frac{H_0}{H_1}$*, and* $\hat{\Psi} \equiv \Phi - \Phi_1 + \Psi_0$*. Then the function* $\hat{\Psi}$ *is a bounded,* Ω_0*-plurisubharmonic generalized solution of the following Dirichlet problem:*

$$(\Omega_0 + \frac{i}{2}\partial\bar{\partial}\hat{\Phi})^{n+1} = 0 \text{ on } \tilde{\mathcal{X}}_D, \qquad \hat{\Phi}|_{\partial\tilde{\mathcal{X}}_D} = 0. \tag{2.45}$$

3. A uniqueness theorem for completely degenerate complex Monge-Ampère equations

There has been considerable progress recently on uniqueness theorems for the complex Monge-Ampère equation and in particular for certain broad classes of possibly unbounded solutions (see e.g. Blocki [B03], Blocki and Kolodziej [BK], Dinew [D], and references therein). For our purposes we need a version of the comparison principle of Bedford and Taylor which can be formulated as follows:

Theorem 3.1. *Let* (M, Ω) *be a compact Kähler manifold with smooth boundary* ∂M *and dimension* m*, and let* Ω_0 *be a smooth, non-negative, closed* $(1,1)$*-form. Then we have*

$$\int_{\{u<v\}} (\Omega_0 + \frac{i}{2}\partial\bar{\partial}v)^m \leq \int_{\{u<v\}} (\Omega_0 + \frac{i}{2}\partial\bar{\partial}u)^m \tag{3.1}$$

for all u, v *in* L^∞*,* Ω_0*-plurisubharmonic, and satisfying* $\operatorname{liminf}_{z\to\partial M}(u(z) - v(z)) \geq 0$*.*

The proof is a straightforward adaption of the original proof of Bedford-Taylor [BT82] in $\mathbb{C}^n$ to our setting using the approximation theorem for plurisubharmonic functions of Blocki and Kolodziej [BK].

Theorem 3.1 implies the following uniqueness theorem for Ω_0-plurisubharmonic solutions of completely degenerate Monge-Ampère equations, where the form Ω_0 is allowed to be degenerate along an analytic subvariety:

Theorem 3.2. *Let* (M, Ω) *be a Kähler manifold with smooth boundary* ∂M *and dimension* m*, and let* $u, v \in L^\infty$ *be* Ω_0*-plurisubharmonic functions satisfying*

$$(\Omega_0 + \frac{i}{2}\partial\bar{\partial}u)^m = (\Omega_0 + \frac{i}{2}\partial\bar{\partial}v)^m = 0, \qquad \operatorname{limsup}_{z\to\partial M}(u(z) - v(z)) = 0. \tag{3.2}$$

If Ω_0 *is* ≥ 0 *everywhere and* > 0 *away from an analytic subvariety of strictly positive codimension which does not intersect* ∂M*, then we must have* $u = v$ *on* M*.*

Proof. By adding the same large constant to both u and v, we may assume that $u, v > 0$. Arguing by contradiction we begin by assuming that $S = \{u < v\} \neq \emptyset$. Since u, v are Ω_0-plurisubharmonic, the set S must have strictly positive measure (it suffices to work in local coordinates and apply the corresponding well-known property of plurisubharmonic functions on $\mathbb{C}^m$). Furthermore, since we can write

$$S = \bigcup_{\varepsilon>0} \{u < (1-\varepsilon)v\} \equiv \bigcup_{\varepsilon>0} S_\varepsilon, \tag{3.3}$$

it follows that S_ε must have strictly positive measure for some $\varepsilon > 0$. Fix one such value of ε. Since $u \geq v \geq (1-\varepsilon)v$ on ∂M, we may apply the comparison principle for Kähler manifolds and obtain

$$\begin{aligned} 0 \geq \int_{S_\varepsilon} (\Omega_0 + \frac{i}{2}\partial\bar\partial u)^m &\geq \int_{S_\varepsilon} (\Omega_0 + (1-\varepsilon)\frac{i}{2}\partial\bar\partial v)^m \\ &= \int_{S_e} \{(1-\varepsilon)(\Omega_0 + \frac{i}{2}\partial\bar\partial v) + \varepsilon\Omega_0\}^m \geq \varepsilon^m \int_{S_e} \Omega_0^m \end{aligned} \tag{3.4}$$

since the form $\Omega_0 + \frac{i}{2}\partial\bar\partial v$ is non-negative. Let V_δ be the complement of a neighborhood of the divisor D, with $\Omega_0^m \geq \delta\Omega^m$ for each $\delta > 0$ small enough. Clearly for each $\delta > 0$

$$\int_{S_\varepsilon} \Omega_0^m \geq \int_{S_\varepsilon \cap V_\delta} \Omega_0^m \geq \delta \int_{S_\varepsilon \cap V_\delta} \Omega^m. \tag{3.5}$$

Since $M \setminus D = \bigcup_{\delta>0} V_\delta$ and D has measure 0 with respect to the volume form Ω^m, we have

$$0 < \int_{S_\varepsilon} \Omega^m = \lim_{\delta\to 0} \int_{S_\varepsilon \cap V_\delta} \Omega^m, \tag{3.6}$$

which implies that $\int_{S_\varepsilon \cap V_\delta} \Omega^m > 0$ for some $\delta > 0$. Altogether, we obtain a contradiction. Thus $\{u < v\}$ must be empty. Interchanging the roles of u and v completes the proof. $\square$

4. Proof of Theorem 1.1

We now prove Theorem 1.1. In Theorem 2.7, we have shown that the function $\hat\Phi$ is a bounded, Ω_0-plurisubharmonic solution of the Dirichlet problem (2.45) on $\tilde{\mathcal{X}}_D$. On the other hand, in [PS09] (Theorem 3), it was shown that the same Dirichlet problem admits a bounded, Ω_0-plurisubharmomic solution which is $C^{1,\alpha}$ for any $0 < \alpha < 1$ on $\mathcal{X}_D^\times$. By Theorem 3.2, it follows that the two solutions must coincide. Thus $\hat\Phi$ is $C^{1,\alpha}$ on $\mathcal{X}_D^\times$. Since $\hat\Phi = \Phi - \Phi_1 + \Psi_0$ and both Φ_1 and Ψ_0 are smooth on $\mathcal{X}_D^\times$, it follows that Φ is $C^{1,\alpha}$ on $\mathcal{X}_D^\times = X \times D^\times$. $\square$

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