REGULARITY OF GEODESIC RAYS
AND MONGE-AMPÈRE EQUATIONS

D. H. PHONG AND JACOB STURM

(Communicated by Richard A. Wentworth)

ABSTRACT. It is shown that the geodesic rays constructed as limits of Bergman geodesics from a test configuration are always of class $C^{1,\alpha}$, $0 < \alpha < 1$. An essential step is to establish that the rays can be extended as solutions of a Dirichlet problem for a Monge-Ampère equation on a Kähler manifold which is compact.

1. Introduction

The purpose of this paper is to establish the $C^{1,\alpha}$ regularity, $0 < \alpha < 1$, of the geodesic rays constructed in [PS07] from a test configuration by Bergman geodesic approximations. With the notation given in section 2 below, our main result can be stated as follows:

Theorem 1.1. Let $L \to X$ be a positive holomorphic line bundle over a compact complex manifold $X$ of dimension $n$. Let $\rho$ be a test configuration for the polarization $(X, L)$. Let $D^\times = \{0 < |w| \leq 1\}$ be the punctured unit disk, and let $\pi_X$ be the natural projection $X \times D^\times \to X$. For each metric $h_0$ on $L$ with positive curvature $\omega_0 \equiv -\frac{i}{2} \partial \bar{\partial} \log h_0 > 0$, let $\Phi(z, w)$ be the $\pi_X^*(\omega_0)$-plurisubharmonic function on $X \times D^\times$ defined by

\begin{equation}
\Phi(z, w) = \lim_{k \to \infty} \left[ \sup_{\ell \geq k} \Phi_\ell(z, w) \right]^*, \quad (z, w) \in X \times D^\times,
\end{equation}

where $\Phi_\ell(z, w)$ are the functions defined by (2.10) below. Then for any $0 < \alpha < 1$, $\Phi(z, w)$ is a $C^{1,\alpha}$ generalized solution of the Dirichlet problem

\begin{equation}
(\pi_X^*(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi)^n + 1 = 0 \text{ on } X \times D^\times, \quad \Phi(z, w) = 0 \text{ when } |w| = 1.
\end{equation}

The fact that $\Phi(z, w)$ is locally bounded and a solution of the Dirichlet problem was established in [PS07], so the new part of the theorem is the $C^{1,\alpha}$ regularity. In the case of toric varieties, the $C^{1,\alpha}$ regularity of geodesic rays was previously established by Song and Zelditch [SZ08], using an explicit analysis of orthonormal bases for $H^0(X, L^k)$ and the theory of large deviations. They also pointed out that, already for toric varieties, geodesic rays from test configurations can be at best $C^{1,1}$.

The interpretation of the completely degenerate Monge-Ampère equation in (1.2)
as the equation for geodesics in the space of Kähler potentials of class $c_1(L)$ on $X$ is well-known and due to Donaldson [D99], Semmes [S], and Mabuchi [M].

In [PS09], $C^{1,\alpha}$ geodesic rays were constructed in all generality from test configurations by a different approach, namely viscosity methods for the degenerate complex Monge-Ampère equation on a compactification $\tilde{X}_D$ of $X \times D^\times$. Thus our theorem can be established by showing that the above solution, more precisely $\Phi(z, w) - \Phi_1(z, w)$, can also be extended to $\tilde{X}_D$ and that such solutions must be unique. For this, it is essential to show that $\Phi(z, w) - \Phi_1(z, w)$ is uniformly bounded on $X \times D^\times$. We accomplish that with the help of a “lower-triangular” property of Donaldson’s equivariant imbeddings, relating $k$-th powers of sections of $H^0(X, L)$ to sections of $H^0(X, L^k)$, which may be of independent interest (see Lemma 2.2 below).

The uniqueness follows from a comparison theorem for Monge-Ampère equations on Kähler manifolds with boundary, using the approximation theorems for plurisubharmonic functions obtained recently by Blocki and Kolodziej [BK] (see also Demailly and Paun [DP] for other approximation theorems). It is well-known that such approximation theorems would imply comparison theorems by a straightforward adaptation to Kähler manifolds of the classic comparison theorem of Bedford and Taylor [RT82] for domains in $\mathbb{C}^m$.

The proof of Theorem 1.1 also shows that the function $\Phi_\ell - \Phi_1$ extends for all $\ell$ to a smooth function on $X_{\text{red}}$, the variety underlying the test configuration $X$. Thus its limsup envelope $\Phi - \Phi_1$ is an upper semi-continuous function on $X_{\text{red}}$ whose restriction to the central fiber of $X_{\text{red}}$ is a well-defined bounded upper semi-continuous function.

As has been stressed in [PS06], each test configuration defines a generalized vector field on the space of Kähler potentials, with the vector at each potential $h_0$ given by the tangent vector $\dot{\phi}$ to the geodesic at the initial time. This observation can now be given a precise formulation using the measures recently introduced by Berndtsson [B09a]: for each generalized $C^{1,\alpha}$ geodesic $(-\infty, 0] \to \phi(z, t) \equiv \Phi(z, e^t)$, the functional $\mu_\Phi : C^0_0(\mathbb{R}) \ni f \to \int_X f(\phi) \omega^n_{\phi(t)}$ defines a Borel measure on $\mathbb{R}$ which is independent of $t$. Taking $t = 0$, we can think of this measure as a way of characterizing $\dot{\phi}(0)$ by its moments. If $\Phi$ is the geodesic constructed in Theorem 1.1, the corresponding assignment $h_0 \to \mu_\Phi$ can be viewed as a precise realization of the generalized vector field defined by the test configuration $\rho$.

We note that Theorem 1.1 gives the regularity of the limiting function $\Phi(z, w)$, but it does not provide information on the precise rate of convergence of $\Phi_k$. For toric varieties, very precise rates of convergence have been provided by Song and Zelditch [SZ06, SZ08]. For general manifolds, in the case of geodesic segments, the precise rate of $C^0$ convergence was obtained a few years ago by Berndtsson [B09a] with an additional twisting by $1/2 K_X$ and very recently in [B09b] for the $\Phi_k$ themselves.

Finally, we would like to mention that geodesics have been constructed by Arezzo and Tian [AT], Chen [C00, C08], Chen and Tang [CT], Chen and Sun [CS], Blocki [B09] and others in various geometric situations. For geodesic segments, the $C^{1,\alpha}$ regularity has been established by Chen [C00]. Their construction by Bergman approximations is in [PS06]. This construction has also been extended by Rubinstein and Zelditch [RZ] to the construction of harmonic maps in the space of Kähler potentials in the case of toric varieties.
2. The extension to a compact Kähler manifold

In this section, we show how the generalized geodesic rays constructed in [PS07], originally defined on $X \times \{0 < |w| \leq 1\}$, actually extend as bounded solutions of a complex Monge-Ampère equation over a compact Kähler manifold $\tilde{X}_D \supset X \times \{0 < |w| \leq 1\}$. We begin by introducing the notation and recalling the results of [PS07].

2.1. Test configurations. Let $L \to X$ be a positive line bundle over a compact complex manifold $X$ of dimension $n$. A test configuration $\rho$ for $L \to X$ [D02] is a homomorphism $\rho : \mathbb{C}^\times \to \text{Aut}(\mathcal{L} \to \mathcal{X} \to \mathbb{C})$, where $\mathcal{L}$ is a $\mathbb{C}^\times$ equivariant line bundle with ample fibers over a scheme $\mathcal{X}$, and $\pi : \mathcal{X} \to \mathbb{C}$ is a flat $\mathbb{C}^\times$ equivariant map of schemes, with $(\pi^{-1}(1), \mathcal{L}_{|_{\pi^{-1}(1)}})$ isomorphic to $(X, L^r)$ for some fixed $r > 0$.

Replacing $L$ by $L^r$, we may assume that $r = 1$.

It is convenient to denote $(\pi^{-1}(w), \mathcal{L}_{|_{\pi^{-1}(w)}})$ by $(X_w, L_w)$. In particular, for each $\tau \neq 0$, $\rho(\tau)$ is an isomorphism between $(X_w, L_w)$ and $(X_{\tau w}, L_{\tau w})$.

The central fiber $(X_0, L_0)$ is fixed under the action of $\rho$. Thus, for each $k$, $\rho$ induces a one-parameter subgroup of automorphisms $\rho_k(\tau) : H^0(X_0, L_0^k) \to H^0(X_0, L_0^k), \quad \tau \in \mathbb{C}^\times$.

Since $\rho_k(\tau)$ is an algebraic one-parameter subgroup, there is a basis of $H^0(X_0, L_0^k)$ in which $\rho(\tau)$ is represented by a diagonal matrix with entries $\tau^{\eta_{\alpha}^{(k)}}$, where $\eta_{\alpha}^{(k)}$ are integers, $0 \leq \alpha \leq N_k \equiv \dim H^0(X_0, L_0^k) - 1$. Set

$$\lambda_{\alpha}^{(k)} = \eta_{\alpha}^{(k)} - \frac{1}{N_k + 1} \sum_{\beta=0}^{N_k} \eta_{\beta}^{(k)},$$

so that $(\lambda_{\alpha}^{(k)})$ is the traceless component of $(\eta_{\alpha}^{(k)})$.

For a fixed $k$, we shall refer to $\eta_{\alpha}^{(k)}$ and $\lambda_{\alpha}^{(k)}$ respectively as the weights and the traceless weights of the test configuration $\rho$.

It is convenient to introduce an $(N_k + 1) \times (N_k + 1)$ diagonal matrix $B_k$ whose diagonal entries are given by the weights $\eta_{\alpha}^{(k)}$. Such a matrix is determined up to a permutation of the diagonal entries $\eta_{\alpha}^{(k)}$, and we fix one choice once and for all.

Then the traceless weights $\lambda_{\alpha}^{(k)}$ are the diagonal entries of the matrix $A_k$ defined by $A_k = B_k - (N_k + 1)^{-1}(\text{Tr} B_k)I$, and

$$\text{Tr} B_k = \sum_{\alpha=0}^{N_k} \eta_{\alpha}^{(k)}, \quad \text{Tr} A_k = 0.$$

For sufficiently large $k$, the functions $k(N_k + 1)$ and $\text{Tr} B_k$ are polynomials in $k$ of degree $n + 1$, so we have an asymptotic expansion

$$\frac{\text{Tr} B_k}{k(N_k + 1)} \equiv F_0 + F_1 k^{-1} + F_2 k^{-2} + \cdots.$$ 

The Donaldson-Futaki invariant of $\rho$ is defined to be the coefficient $F_1$.

2.2. Equivariant imbeddings of test configurations. An essential property of test configurations, due to Donaldson [D05], is that the entire configuration can be imbedded equivariantly in $\mathbb{C}P^{N_k} \times \mathbb{C}$ in a way which respects a given $L^2$ metric on $H^0(X, L^k)$. The following formulation [PS07, Lemmas 2.1-2.3] is most convenient for our purposes.
Let $g(k) = \{ s^{(k)}_\alpha(z) \}_{\alpha=0}^{N_k}$ be a basis for $H^0(X, L^k)$. For all $k$ sufficiently large, it
defines a Kodaira imbedding
\begin{equation}
\iota_{g(k)} : X \ni z \rightarrow [s^{(k)}_0(z) : s^{(k)}_1(z) : \cdots : s^{(k)}_{N_k}(z)] \in \mathbb{CP}^{N_k}
\end{equation}
of $X$ into $\mathbb{CP}^{N_k}$, with $O(1)$ pulled back to $L^k$. If $h_0$ is a fixed metric on $L$ with
$\omega_0 \equiv -\frac{i}{2} \partial \bar{\partial} \log h_0 > 0$, then $H^0(X, L^k)$ can be equipped with the $L^2$ metric defined
by the metric $h^0_0$ on sections of $L^k$ and the volume form $\omega^k_0/n!$. For simplicity, we
shall refer to this $L^2$ metric on $H^0(X, L^k)$ as just the “$L^2$ metric defined by $h_0$”. Of particular
importance then are the bases $g(k)$ which are orthonormal with respect to this $L^2$ metric.

**Lemma 2.1.** Let $\rho : \mathbb{C}^\times \rightarrow \text{Aut}(\mathcal{L} \rightarrow \mathcal{X} \rightarrow \mathbb{C})$ be a test configuration, and fix
a diagonal matrix $B_k$ with the weights of $\rho$ as diagonal entries as defined in §2.1. Fix a metric $h_0$ on $L$ with positive curvature $\omega_0$ and a corresponding $L^2$ metric on
$H^0(X, L^k)$. Then there is an orthonormal basis $\tilde{g}(k)$ of $H^0(X, L^k) = H^0(X_1, L^k_1)$
with respect to the $L^2$ metric defined by $h_0$ and an imbedding
\begin{equation}
I_{\tilde{g}(k)} : (\mathcal{L} \rightarrow \mathcal{X} \rightarrow \mathbb{C}) \rightarrow (O(1) \times \mathbb{C} \rightarrow \mathbb{CP}^{N_k} \times \mathbb{C} \rightarrow \mathbb{C})
\end{equation}
satisfying
1. $I_{\tilde{g}(k)}|_X = \iota_{g(k)}$;
2. $I_{\tilde{g}(k)}$ intertwines $\rho(\tau)$ and $B_k$,
\begin{equation}
I_{\tilde{g}(k)}(\rho(\tau) \ell_w) = (\tau^{B_k} I_{\tilde{g}(k)}(\ell_w), \tau \ell_w), \quad \ell_w \in L_w, \quad \tau \in \mathbb{C}^\times.
\end{equation}

Let $E_k = \pi_0(\mathcal{L}^k)$ be the direct images of the bundles $\mathcal{L}^k$. Thus $E_k \rightarrow \mathbb{C}$ is a
vector bundle over $\mathbb{C}$ of rank $N_k + 1$, and its sections $S(w)$ are holomorphic sections
of $L_w$ for each $w \in \mathbb{C}$. The action of $\mathbb{C}^\times$ on the sections $S$ is given by
\begin{equation}
S^\tau(w) = \rho(\tau)^{-1} S(w \tau).
\end{equation}

Then a third key statement in the equivariant imbedding lemma is:
\begin{enumerate}
\item[(3)] The functions $S_\alpha(w) \equiv w^{\rho_0(\alpha)} \rho(w) s_\alpha, \ w \in \mathbb{C}^\times$, extend to a basis for the free
$\mathbb{C}[w]$ module of all sections of $E_k \rightarrow \mathbb{C}$ and they have the property $S_\alpha(1) = s_\alpha$. This
extension still satisfies the relation
\begin{equation}
\rho(\tau)^{-1} S_\alpha(w) = \tau^{\rho_0(\alpha)} S_\alpha(w), \quad w \in \mathbb{C}.
\end{equation}

In the language of [PS07], the Hermitian generator $\Theta$ of the test configuration
is the isomorphism $H^0(X_0, L^k_0) \rightarrow H^0(X_1, L^k_1)$ sending the basis $S_\alpha(0)$ to the basis $S_\alpha(1)$.

### 2.3. The construction of geodesics.

We come now to the construction of geodesics by Bergman approximations. Let $\rho : \mathbb{C}^\times \rightarrow \text{Aut}(\mathcal{L} \rightarrow \mathcal{X} \rightarrow \mathbb{C})$ be a
test configuration for $L \rightarrow X$, and fix a metric $h_0$ on $L$ with positive curvature $\omega_0$.
Let $g(k) = \{ s^{(k)}_\alpha(z) \}$ be an orthonormal basis for $H^0(X, L^k)$ with respect to the $L^2$
metric defined by $h_0$ as in Lemma 2.1. Let $D^\times = \{ w \in \mathbb{C}; 0 < |w| \leq 1 \}$ be the
punctured disk. Define the functions $\Phi_k : X \times D^\times \rightarrow \mathbb{R}$ by
\begin{equation}
\Phi_k(z, w) = \frac{1}{k} \log \sum_{\alpha=0}^{N_k} |w|^{2n(\alpha)} |s^{(k)}_\alpha(z)|^2 h_0 - \frac{n}{k} \log k
\end{equation}
and $\Phi(z, w)$ by
\begin{equation}
\Phi(z, w) = \lim_{k \to \infty} [\text{sup}_{|z| \leq k} \Phi_k(z, w)]^*
\end{equation}
where \( \eta^{(k)}_\alpha \) are the weights of the test configuration \( \rho \), \( \ast \) denotes the upper semi-continuous envelope, i.e.

\[
f^\ast(z) = \lim_{\epsilon \to 0} \sup_{|w-z|<\epsilon} f(w),
\]

and \( |s^\alpha(z)|^2_{h^0} \equiv s^\alpha(z)s^\alpha(z)h_0(z)^k \) denotes the norm-squared of \( s^\alpha(z) \) with respect to the metric \( h^0 \). Then it is shown in [PS07] that \( \Phi(z, w) \) is a generalized geodesic ray in the sense that

(a) \( \pi^X_N(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi \geq 0 \) on \( X \times D^\infty \), where \( \pi^X \) is the projection \( X \times D^\infty \to X \);

(b) for each finite \( T > 0 \), we have

\[
(2.13) \quad \sup_k |\Phi_k(z, w)|, |\Phi(z, w)| \leq C_T \text{ for } (z, w) \in X \times \{e^{-T} < |w| \leq 1\}
\]

with \( C_T \) a constant independent of \( z, w \) and \( k \), but possibly depending on \( T \);

(c) \( \Phi(z, w) \) is continuous when \( |w| = 1 \) and is a solution in the sense of pluripotential theory of the following Dirichlet problem:

\[
(2.14) \quad (\pi^X_N(\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi)^{n+1} = 0 \text{ on } X \times D^\infty; \quad \Phi(z, w) = 0 \text{ when } |w| = 1.
\]

The geodesic \( \Phi(z, w) \) is non-constant if the test configuration is non-trivial, that is, not holomorphically equivalent to a product test configuration. We note that in the boundary value problem \( (2.14) \), the behavior of \( \Phi(z, w) \) near \( w = 0 \) is not specifically assigned.

### 2.4. Formulation in terms of equivariant imbeddings

We come now to the main task in this chapter, which is to identify the solution \( (2.14) \) with the restriction to \( X \times D^\infty \) of the solution of a standard Dirichlet problem on a compact Kähler manifold \( X_D \) with boundary.

Let \( \pi_{\text{red}} : X_{\text{red}} \to \mathbb{C} \) be the projection map, and let \( D = \{ w \in \mathbb{C} : |w| \leq 1 \} \). Here \( X_{\text{red}} \) is the variety underlying the scheme \( X \). Let \( X_D = \pi^{-1}_{\text{red}}(D), \quad X^\infty_D = \pi^{-1}_{\text{red}}(D^\infty) \).

The space \( X^\infty_D \) is isomorphic to \( X \times D^\infty \) under the correspondence

\[
(2.15) \quad X \times D^\infty \ni (z, w) \to \rho(w)(z) \in X_w,
\]

where \( z \in X \) is viewed as a point in \( X_1 \). This correspondence lifts to a correspondence between \( L \times D^\infty \) and the restriction \( L^\infty_D \) of \( L \) over \( X^\infty_D \).

Let \( p : \tilde{X} \to X_{\text{red}} \to \mathbb{C} \) be an \( SL^1 \) equivariant smooth resolution and \( \tilde{\mathcal{L}} = p^* \mathcal{L} \). The first step is to show that the functions \( \Phi_k(z, w) - \Phi_1(z, w) \) of \( (2.10) \), which are defined on \( X \times D^\infty \), may be extended to plurisubharmonic functions on all of \( X_D = p^{-1}(X_D) \).

Let us fix a metric \( h_0 \) on \( L \) with positive curvature \( \omega_0 \). Let \( s^{(k)} \) be the orthonormal basis for \( H^0(X, L^k) \) with respect to \( h_0 \) provided by Lemma \( 2.1 \) and let \( I_{s^{(k)}} \)

---

\(^1\) Actually, in [PS07], the weights \( \eta^{(k)}_\alpha \) in the definition of \( \Phi_k(z, w) \) were replaced by the traceless weights \( \lambda^{(k)}_\alpha \). If we denote by \( \Phi^{(k)}_k(z, w) \) the functions obtained in this manner with the traceless weights, then we have

\[
(2.12) \quad \Phi_k(z, w) = \Phi^{(k)}_k(z, w) + \frac{\text{Tr} B_k}{k(N_k + 1)} \log |w|^2.
\]

It follows that the complex Hessians of \( \Phi_k(z, w) \) and \( \Phi^{(k)}_k(z, w) \) are identical. However, the behaviors near \( |w| = 0 \) of \( \Phi_k(z, w) \) and \( \Phi^{(k)}_k(z, w) \) are different, and for our purposes, it is important to work with \( \Phi_k(z, w) \).
be a corresponding equivariant imbedding of the test configuration. Let $\Phi_k(z, w)$ be defined by (2.10). Define a closed $(1, 1)$-form $\Omega_k$ on $\tilde{X}_D$ by

$$
(2.16) \quad \Omega_k = \frac{1}{k} (I_{z(k)} \circ p)^* \omega_{FS}
$$

where $\omega_{FS}$ is the Fubini-Study metric on $\mathbb{CP}^{N_k}$. Define as well a Hermitian metric $H_k$ on $\tilde{L}$ by $H_k = (I_{z(k)} \circ p)^* (h_{FS})^{1/k}$, where $h_{FS}$ is the Fubini-Study metric on the hyperplane bundle $O(1)$ over $\mathbb{CP}^{N_k}$. Thus $\Omega_k$ is the curvature of $H_k$. The restriction of $\omega_k$ to $X_D^*$ can be readily worked out explicitly in terms of the coordinates $(z, w)$.

Using the intertwining property of the equivariant imbedding,

$$
(2.22) \quad \Omega_k|_{X \times D^*} = \pi_X^*(\omega_0) + \frac{i}{2} \partial \overline{\partial} \Phi_k(z, w).
$$

Recalling that the norm with respect to $h_k^0$ of a section $s(z)$ of $L^k$ is given by $|s(z)|^2_{h_k^0} = |s(z)|^2 h_0^k$, we find the following key relation between the $(1, 1)$-forms $\Omega_k$ and the potentials $\Phi_k(z, w)$ defined earlier in (2.10):

$$
(2.20) \quad \Omega_k|_{X \times D^*} = \pi_X^*(\omega_0) + \frac{i}{2} \partial \overline{\partial} \Phi_k(z, w).
$$

2.5. The extension of $\Psi_k$ to the total space $\tilde{X}_D$. The relation (2.20) that we have just obtained shows that the form $\pi_X^*(\omega_0) + \frac{i}{2} \partial \overline{\partial} \Phi_k(z, w)$, defined originally on $X_D^*$, admits the natural extension $\Omega_k$ to the whole of $\tilde{X}_D$.

Since the form $\pi_X^*(\omega_0)$ does not extend by itself to $\tilde{X}$, we rewrite $\Omega_k$ as

$$
(2.21) \quad \Omega_k = \Omega_1 + \frac{i}{2} \partial \overline{\partial} (\Phi_k - \Phi_1) \equiv \Omega_1 + \frac{i}{2} \partial \overline{\partial} \Psi_k.
$$

The function $\Psi_k = \Phi_k - \Phi_1$ has a simple interpretation that shows that it extends as a smooth function to the whole of $\tilde{X}_D$: as we saw earlier in §2.2, under the maps $I_{z(k)}$ and $I_{z(1)}$ of the test configuration $\rho$, the Fubini-Study metric $h_{FS}$ pulls back respectively to $H_k^0 = (\sum \alpha |w|^{2\alpha} |s_{\alpha}^{(k)}(z)|^2)^{-1}$ and $H_1 = (\sum \alpha |w|^{2\alpha} |s_{\alpha}^{(1)}(z)|^2)^{-1}$ on $L \times D^*$. Thus

$$
(2.22) \quad \Psi_k = \log \frac{H_1}{H_k} - \frac{n}{k} \log k.
$$

The right hand side is a well-defined, smooth scalar function over the whole of $\tilde{X}_D$, since it is the logarithm of the ratio of two smooth metrics on the same line bundle $\tilde{L} \rightarrow \tilde{X}_D$. 

Since $\Omega_k$ is non-negative as the pull-back of a non-negative form, the function $\Psi_k$ is $\Omega_1$-plurisubharmonic. We also define
\begin{equation}
\Psi = \lim_{k \to \infty} \sup_{\ell \geq k} \Psi_k^*,
\end{equation}
which is an extension of $\Phi - \Phi_1$ to $X_D$.

2.6. **Uniform estimates for $\Psi_k$.** Recall that in [PS07], as quoted in (2.13) above, we only have bounds for the functions $\Phi_k(z, w)$ when $|w| > e^{-T}$, for some fixed finite $T > 0$. Since the function $\Psi_k$ extends to a smooth function on $X_D$, it follows that it is bounded on $X_D$. However, the bound may a priori depend on $k$. The most important step in the extension to $X_D$ is to show that this bound can actually be made uniform in $k$.

We need several lemmas, starting with the following essential “lower-triangular lemma”:

**Lemma 2.2.** Fix a test configuration $\rho$ and a metric $h_0$ on $L$ with positive curvature $\omega_0$. For each $k$, let $s(k) = \{ s_{\alpha}^{(k)} \}_{\alpha=0}^{N_k}$ be an orthonormal basis for $H^0(X, L^k)$ as in Lemma 2.1. Then for any $s_{\beta}^{(1)}$ in $s(1)$, we can write
\begin{equation}
(s_{\beta}^{(1)})^k = \sum_{\eta_\alpha^{(k)} \leq k \eta_\beta^{(1)}} a_{\beta \alpha} s_{\alpha}^{(k)},
\end{equation}
where $a_{\beta \alpha} \in \mathbb{C}$ and the subindex indicates the range of indices $\alpha$ which are allowed. Furthermore, the coefficients $a_{\beta \alpha}$ satisfy the bound
\begin{equation}
|a_{\beta \alpha}| \leq V \frac{1}{2} M^k,
\end{equation}
where we have set $M = \sup_{0 \leq \alpha \leq N_1} \sup_X |s_{\alpha}^{(1)}| h_0$ and $V = \int_X \omega_0^n$.

**Proof of Lemma 2.2.** For each $k$, let $E_k = \pi_* (L^k) \to \mathbb{C}$, and let $S_0(w), \cdots, S_{N_k}(w)$ be a basis for the free $\mathbb{C}[w]$ module of sections of $E_k \to \mathbb{C}$, as provided in Lemma 2.1. Now let $S_\beta$ be an element of this basis for $E_1 \to \mathbb{C}$ and some $\beta$ with $0 \leq \beta \leq N_1$. Then $\rho(\tau)^{-1} S_\beta(w \tau) = \tau^{\eta_\beta}_{\beta}(w \tau)$ which implies
\begin{equation}
\rho(\tau)^{-1} S_k^k(w \tau) = \tau^{\eta_\beta}_{\beta}(w \tau).
\end{equation}
On the other hand, $S_\beta^k$ is a section of $E_k$, so we may write
\begin{equation}
S_\beta^k(w) = \sum_{\alpha=0}^{N_k} a_\alpha(w) S_\alpha(w)
\end{equation}
for certain uniquely defined polynomials $a_\alpha(w) \in \mathbb{C}[w]$. Applying the $\mathbb{C}^\times$ action to both sides of (2.27) we obtain
\begin{equation}
\sum_{\alpha=0}^{N_k} \tau^{\eta_\beta}_{\beta}(w \tau) a_\alpha(w) S_\alpha(w) = \tau^{\eta_\beta}_{\beta}(w \tau) = \rho(\tau)^{-1} S_\beta^k(w \tau) = \sum_{\alpha=0}^{N_k} a_\alpha(w \tau) \tau^{\eta_\beta}_{\beta}(w \tau).
\end{equation}
Comparing coefficients we obtain
\begin{equation}
\tau^{\eta_\beta}_{\beta}(w \tau) = a_\alpha(w \tau) \tau^{\eta_\beta}_{\alpha}.
\end{equation}

\footnote{In general, given a non-negative smooth, closed (1,1)-form $\Omega$ on a complex manifold $X$, we say that a scalar function $\Phi$ is $\Omega$-plurisubharmonic if $f_\alpha + \Phi$ is plurisubharmonic on $U_\alpha$ for each $\alpha$ if $X = \bigcup U_\alpha$ is a covering of $X$ by coordinate charts $U_\alpha$ with $\Omega = \frac{1}{2} \partial \partial f_\alpha$ on $U_\alpha$.}
Setting $w = 1$ we see that $a_\alpha(\tau) = a_\beta(\tau)^{r_\alpha}$ for some integer $r_\alpha$ and some $a_\beta \in \mathbb{C}$. But $a_\alpha(w)$ is a polynomial. Thus $r_\alpha \geq 0$ and $a_\alpha(w) = a_\beta(w)^{r_\alpha}$ for all $w \in \mathbb{C}$. The equation (2.29) implies that if $a_\beta \neq 0$, we have $k\eta_\beta^{(1)} = r_\alpha + \eta_\alpha^{(k)}$ and thus $\eta_\alpha^{(k)} \leq k\eta_\beta^{(1)}$. Evaluating (2.27) at $w = 1$ we obtain the first part of the lemma.

Finally, the orthonormality of the sections $s_\alpha^{(k)}$ implies

$$|a_{\alpha|} = \int \left| (s_\beta^{(1)})^{k}, s_\alpha^{(k)} \right|_{h_{\frac{n}{2}}^n} \leq \int \left| s_\beta^{(1)} \right|_{h_0} \cdot \left| s_\alpha^{(k)} \right|_{h_{\frac{n}{2}}} \leq M^k \nu^\frac{1}{2}.$$  

Remark. It may happen that $\eta_\alpha^{(k)} < k\eta_\beta^{(1)}$ for all $\alpha$ with $a_\beta \neq 0$: that is, it may happen that $a_\alpha(w)$ vanishes at $w = 0$ for all $\alpha$. This would mean that $S_\beta(0)$ is a non-zero section of $H^0(X_0, L_0)$ but that $S_\beta^{(1)}(0) = 0 \in H^0(X_0, L_0)$; in other words, the section $S_\beta(0)$ is nilpotent (which is possible if $X_0$ is a non-reduced scheme, that is, if $X_0$ has nilpotent elements in its structure sheaf).

Lemma 2.3. **The complex manifold $\tilde{X}_D$ always admits a Kähler metric.**

Proof of Lemma 2.3. This lemma is proved in [PS07a]. In fact, it is proved there that there exists a line bundle $\mathcal{M}$ on $\tilde{X}_D$ which is trivial on $\tilde{X}^\times$ and such that $\mathcal{L}^m \otimes \mathcal{M}$ is positive for some fixed positive power $m$. The desired Kähler metric on $\tilde{X}_D$ can then be taken to be the ratio of the curvature of $\mathcal{L}^m \otimes \mathcal{M}$ by $m$. \hfill $\Box$

Lemma 2.4. **There exists a finite constant $C$ so that**

$$\sup_{k \geq 1} \sup_{\tilde{X}_D} |\Psi_k| \leq C < \infty.$$  

In particular,

$$\sup_{\tilde{X}_D} |\Psi| \leq C < \infty.$$  

Proof of Lemma 2.4. Let $H$ be a Kähler metric on $\tilde{X}_D$, which exists by Lemma 2.3. Since $\Psi_1$ is $\Omega_1$-plurisubharmonic, it follows that $\Delta_H \Psi_1 \geq -C_1$, where $\Delta_H$ is the Laplacian with respect to $H$, and $C_1$ is an upper bound for the trace of $\Omega_1$ with respect to the metric $H$. On the other hand, $\Psi_k |_{\partial \tilde{X}_D} \to -\Phi_1$ uniformly as $k \to \infty$, and thus $\Psi_k |_{\partial \tilde{X}_D} \leq C_2$. Let $u$ be the smooth function on $\tilde{X}_D$ which is the solution of the Dirichlet problem

$$\Delta_H u = -C_1 \text{ on } \tilde{X}_D, \quad u = C_2 \text{ on } \partial \tilde{X}_D.$$  

By the maximum principle, we have $\Psi_k \leq u$ for all $k$, and this gives the upper bound.

To establish the lower bound, it suffices to prove that $\Psi_k \geq -C$ on $\tilde{X}_D^\times$, where $C$ is a constant independent of $k$, since each function $\Psi_k$ is smooth on $\tilde{X}_D$. On $X^\times$, we can use the explicit expressions for $X \times D^\times$ and write

$$\Psi_k = \log \frac{\sum_{\alpha=0}^{N_k} \left| w^{2\eta_\alpha^{(k)}} \right|^2_{s_\alpha^{(k)}}} {\sum_{\beta=0}^{N_1} \left| w^{2\eta_\beta^{(1)}} \right|^2_{s_\beta^{(1)}}} - \frac{n}{k} \log k.$$  

Now fix $w$ with $0 < |w| \leq 1$, fix $z \in X$, and choose $\beta_0$ so that

$$\left| w^{2\eta_\beta^{(1)}} \right|^2_{s_\beta^{(1)}(z)} = \sup_{0 \leq \beta \leq N_1} \left| w^{2\eta_\beta^{(1)}} \right|^2_{s_\beta^{(1)}(z)}.$$  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
In view of Lemma 2.2, we can write
\[(2.36) \quad \left| (s^{(1)}_{\beta_0})_k \right|_{h_0^k} \leq M^k V^{1/2} \sum_{k \eta_{\beta_0}^{(1)} \geq \eta_{\alpha}^{(k)}} \left| s^{(k)}_{\alpha} \right|_{h_0^k}.
\]
Since \(|w| \leq 1\), we have then
\[|w|^{2k \eta_{\beta_0}^{(1)}} \left| s^{(k)}_{\beta_0}(z) \right|_{h_0^k}^2 \leq M^{2k} V \left( \sum_{k \eta_{\beta_0}^{(1)} \geq \eta_{\alpha}^{(k)}} |w|^{m^{(k)}} \left| s^{(k)}_{\alpha}(z) \right|_{h_0}^2 \right)^2
\]
\[(2.37) \quad \leq M^{2k} V (N_k + 1) \sum_{k \eta_{\beta_0}^{(1)} \geq \eta_{\alpha}^{(k)}} |w|^{2m^{(k)}} \left| s^{(k)}_{\alpha}(z) \right|_{h_0^k}^2.
\]
Returning to \(\Psi_k\), we can now write
\[\Psi_k(z, w) \geq \log \left( \sum_{\alpha=0}^{N_k} |w|^{2m^{(k)}} \left| s^{(k)}_{\alpha}(z) \right|_{h_0^k} \right)^{1/2} - \frac{n}{k} \log k
\]
\[(2.38) \quad \geq - \frac{1}{k} \log (V(N_k + 1)) - 2 \log M - \frac{n}{k} \log k - \log (N_k + 1)
\]
in view of the preceding inequality. This establishes Lemma 2.4 since \(N_k \leq C k^n\).

2.7. The Monge-Ampère equation on the whole of \(\hat{X}_D\). We can now prove the main theorem of this section:

**Theorem 2.5.** Let \(L \to X\) be a positive line bundle over a compact complex manifold, let \(\rho\) be a test configuration, and let \(h_0\) be a metric on \(L\) with positive curvature \(\omega_0\). Let \(\hat{X}\) be an \(S^1\) invariant resolution \(p : \hat{X} \to X_{\text{red}} \to \mathbb{C}\) of \(X\), and let \(\tilde{X}_D = (\pi_{\text{red}} \circ p)^{-1}(D)\). Let \(\Psi_k, \Phi\) be defined as in (2.10) and (2.11). Set
\[(2.39) \quad \Psi = \Phi - \Phi_1 \text{ on } X \times D^k.
\]
Then the function \(\Psi\) extends to a bounded, \(\Omega_1\)-plurisubharmonic function on \(\tilde{X}_D\), which is a generalized solution of the following Dirichlet problem on \(\tilde{X}_D\),
\[(2.40) \quad (\Omega_1 + \frac{i}{2} \bar{\partial} \partial \Psi)^{n+1} = 0 \text{ on } \tilde{X}_D, \quad \Psi = -\Phi_1 \text{ on } \partial \tilde{X}_D.
\]
Here \(\Omega_1\) is the pull-back to \(\tilde{X}_D\) of the Fubini-Study metric by \(I_{\hat{\omega}}^{(1)} \circ p\).

**Proof of Theorem 2.5.** The function \(\Psi\) satisfies the completely degenerate Monge-Ampère equation on \(X_D^\times\). Since the singular set \(X_0\) is an analytic subvariety and since Lemma 2.3 implies that the function \(\Psi\) defined by (2.22) is a bounded, \(\Omega_1\)-plurisubharmonic function on \(\tilde{X}_D\), it follows from general pluripotential theory that \(\Psi\) satisfies the same completely degenerate equation on \(\tilde{X}_D\). Alternatively, a direct proof of this fact can also be given, since we already have at hand all the necessary ingredients. It suffices to observe that \(\Psi_k\) satisfies the following properties and apply Theorem 3 of [PS06]:

**Lemma 2.6.** The functions \(\Psi_k\) satisfy
\[(a) \sup_k \sup_{\tilde{X}_D} |\Psi_k| \leq C < \infty.
\]
\[(b) \quad \int_{\tilde{X}_D} (\Omega_1 + \frac{i}{2} \bar{\partial} \partial \Psi_k)^{n+1} \leq C \frac{1}{k}.
\]
(c) Let \( T \) be the vector field \( T = \frac{\partial}{\partial m} \) defined in a neighborhood of the boundary \( |w| = 1 \) on \( \bar{X}_D \), where \( t = \log |w| \). Then \( \sup_U |T \Psi_k| \leq C \), where \( C \) is a constant and \( U \) is a neighborhood of the boundary \( |w| = 1 \), independent of \( k \).

(d) 

\[
\sup_{\partial \bar{X}_D} |\Psi_k + \Phi_1| \leq a_k,
\]

with \( a_k \) decreasing to 0 and \( \sum_{k=1}^{\infty} a_k < \infty \).

Proof of Lemma 2.6 Part (a) is just the statement of Lemma 2.4. Part (b) follows from the fact that the form \( \Omega_1 + \frac{i}{2} \partial \bar{\partial} \Psi_k \) is smooth on \( \mathcal{X}_D \) and that its Monge-Ampère mass on \( \mathcal{X}_D^\infty \) coincides with the Monge-Ampère mass of \( \omega_k = (\pi_X^* (\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi_k) \) on \( X \times D^\infty \). As we already observed in footnote 1, \( \Phi_k \) and \( \Phi_k^{\#} \) have the same complex Hessian. So the desired estimate follows from the analogous estimate for the Monge-Ampère mass of \( (\pi_X^* (\omega_0) + \frac{i}{2} \partial \bar{\partial} \Phi_k^{\#}) \) established in Lemma 4.3 of [PS07].

Part (c) follows from the bound \( |\eta^{(k)}_s| \leq C \) established in Lemma 3.1 of [PS07]. Finally, part (d), with \( a_k = Ck^{-2} \), follows from the Tian-Yau-Zelditch theorem [T90] [Y93] [Z] (see also Catlin [Ca] and Lu [L]) as shown in the case of geodesic segments in [PS06].

2.8. Positivity of the background form away from \( p^{-1}(X_0) \). The equation (2.40) provides an extension of the degenerate complex Monge-Ampère equation to the compact manifold with boundary \( \bar{X}_D \). It is however written with respect to a background \((1,1)\)-form \( \Omega_1 \) which may be degenerate. In preparation for uniqueness theorems for the complex Monge-Ampère equation, we rewrite it now with a background \((1,1)\)-form which is non-negative everywhere and strictly positive away from \( p^{-1}(X_0) \).

For this, we make use of Lemma 1 of [PS09], which asserts the existence of a \( S^1 \) invariant metric \( H_0 \) on \( \bar{L} \) with \( H_0|_{\partial \bar{X}_D} = h_0 \) and

\[
\Omega_0 = -\frac{i}{2} \partial \bar{\partial} \log H_0 \geq 0 \quad \text{on} \quad \bar{X}_D, \quad \Omega_0 > 0 \quad \text{on} \quad \mathcal{X}_D^\infty.
\]

Let \( \Psi_0 \) be defined by

\[
\Psi_0 = \log \frac{H_0}{(\mathcal{L}^{(1)} \circ p)^*(h_{FS})} = \log \frac{H_0}{H_1},
\]

which is a smooth function on \( \bar{X}_D \), since \( H_0, H_1 \) are two smooth metrics on the same line bundle \( \bar{L} \). Restricted to \( \partial \bar{X}_D \),

\[
(2.42) \quad \Psi_0|_{\partial \bar{X}_D} = \log \frac{h_0}{(\sum_{\alpha=0}^{N_1} |s_{\alpha}^{(1)}|_0^2)^{-1}} = \log \sum_{\alpha=0}^{N_1} \frac{|s_{\alpha}^{(1)}|_0^2}{H_0} = \Phi_1|_{\partial \bar{X}_D}.
\]

Let \( \Psi \) be the solution on \( \bar{X}_D \) of the completely degenerate Monge-Ampère equation with background form \( \Omega_1 \) as given in Theorem 2.5. Define the function \( \Phi \) on \( \mathcal{X}_D \) by

\[
(2.43) \quad \Phi = \Psi + \Psi_0.
\]

Clearly \( (\Omega_0 + \frac{i}{2} \partial \bar{\partial} \Phi)^{n+1} = 0 \) on \( \bar{X}_D \). Furthermore, restricted to the boundary \( \partial \bar{X}_D \),

\[
(2.44) \quad \Phi|_{\partial \bar{X}_D} = \Psi|_{\partial \bar{X}_D} + \Psi_0|_{\partial \bar{X}_D} = -\Phi_1|_{\partial \bar{X}_D} + \Phi_1|_{\partial \bar{X}_D} = 0.
\]

In summary, we have obtained the following alternative formulation of Theorem 2.6.
Theorem 2.7. Let the setting be the same as in Theorem 2.5 and let \( H_0 \) be a metric on \( L \) as in (2.41), \( \Phi_0 = \log H_0 \), and \( \Phi = \Phi_1 + \Psi_0 \). Then the function \( \hat{\Psi} \) is a bounded, \( \Omega_0 \)-plurisubharmonic generalized solution of the following Dirichlet problem:

\[
(\Omega_0 + \frac{i}{2} \partial \bar{\partial} \hat{\Psi})^{n+1} = 0 \text{ on } \tilde{\mathcal{X}}_D, \quad \hat{\Psi}|_{\partial \tilde{\mathcal{X}}_D} = 0.
\]  

3. A uniqueness theorem for completely degenerate complex Monge-Ampère equations

There has been considerable progress recently on uniqueness theorems for the complex Monge-Ampère equation and in particular for certain broad classes of possibly unbounded solutions (see e.g. Blocki [B03], Blocki and Kolodziej [BK], Dinew [D], and references therein). For our purposes we need a version of the comparison principle of Bedford and Taylor which can be formulated as follows:

Theorem 3.1. Let \((M, \Omega)\) be a compact Kähler manifold with smooth boundary \( \partial M \) and dimension \( m \), and let \( \Omega_0 \) be a smooth, non-negative, closed \((1,1)\)-form. Then we have

\[
\int_{\{u < v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} v)^m \leq \int_{\{u < v\}} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m
\]

for all \( u, v \) in \( L^\infty \), \( \Omega_0 \)-plurisubharmonic, and satisfying \( \liminf_{z \to \partial M} (u(z) - v(z)) \geq 0 \).

The proof is a straightforward adaption of the original proof of Bedford-Taylor [BT82] in \( \mathbb{C}^n \) to our setting using the approximation theorem for plurisubharmonic functions of Blocki and Kolodziej [BK].

Theorem 3.1 implies the following uniqueness theorem for \( \Omega_0 \)-plurisubharmonic solutions of completely degenerate Monge-Ampère equations, where the form \( \Omega_0 \) is allowed to be degenerate along an analytic subvariety:

Theorem 3.2. Let \((M, \Omega)\) be a Kähler manifold with smooth boundary \( \partial M \) and dimension \( m \), and let \( u, v \in L^\infty \) be \( \Omega_0 \)-plurisubharmonic functions satisfying

\[
(\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m = (\Omega_0 + \frac{i}{2} \partial \bar{\partial} v)^m = 0, \quad \limsup_{z \to \partial M} (u(z) - v(z)) = 0.
\]

If \( \Omega_0 \) is \( \geq 0 \) everywhere and \( > 0 \) away from an analytic subvariety of strictly positive codimension which does not intersect \( \partial M \), then we must have \( u = v \) on \( M \).

Proof. By adding the same large constant to both \( u \) and \( v \), we may assume that \( u, v > 0 \). Arguing by contradiction we begin by assuming that \( S = \{u < v\} \neq \emptyset \).

Since \( u, v \) are \( \Omega_0 \)-plurisubharmonic, the set \( S \) must have strictly positive measure (it suffices to work in local coordinates and apply the corresponding well-known property of plurisubharmonic functions on \( \mathbb{C}^m \)). Furthermore, since we can write

\[
S = \bigcup_{\varepsilon > 0} \{u < (1 - \varepsilon)v\} \equiv \bigcup_{\varepsilon > 0} S_\varepsilon,
\]
it follows that \( S_\varepsilon \) must have strictly positive measure for some \( \varepsilon > 0 \). Fix one such value of \( \varepsilon \). Since \( u \geq v \geq (1 - \varepsilon)v \) on \( \partial M \), we may apply the comparison principle for Kähler manifolds and obtain

\[
0 \geq \int_{S_\varepsilon} (\Omega_0 + \frac{i}{2} \partial \bar{\partial} u)^m = \int_{S_\varepsilon} (\Omega_0 + (1 - \varepsilon) \frac{i}{2} \partial \bar{\partial} v)^m \geq \int_{S_\varepsilon} \Omega_0^m \geq \int_{S_\varepsilon} \Omega_0^m
\]

since the form \( \Omega_0 + \frac{i}{2} \partial \bar{\partial} v \) is non-negative. Let \( V_\delta \) be the complement of a neighborhood of the divisor \( D \), with \( \Omega_0^m \geq \delta \Omega^m \) for each \( \delta > 0 \) small enough. Clearly for each \( \delta > 0 \)

\[
\int_{S_\varepsilon} \Omega_0^m \geq \int_{S_\varepsilon \cap V_\delta} \Omega_0^m \geq \delta \int_{S_\varepsilon \cap V_\delta} \Omega^m.
\]

Since \( M \setminus D = \bigcup_{\delta > 0} V_\delta \) and \( D \) has measure 0 with respect to the volume form \( \Omega^m \), we have

\[
0 < \int_{S_\varepsilon} \Omega^m = \lim_{\delta \rightarrow 0} \int_{S_\varepsilon \cap V_\delta} \Omega^m,
\]

which implies that \( \int_{S_\varepsilon \cap V_\delta} \Omega^m > 0 \) for some \( \delta > 0 \). Altogether, we obtain a contradiction. Thus \( \{ u < v \} \) must be empty. Interchanging the roles of \( u \) and \( v \) completes the proof. \( \square \)

4. Proof of Theorem 1.1

We now prove Theorem 1.1. In Theorem 2.7, we have shown that the function \( \hat{\Phi} \) is a bounded, \( \Omega_0 \)-plurisubharmonic solution of the Dirichlet problem (2.45) on \( \tilde{X}_D \). On the other hand, in [PS09] (Theorem 3), it was shown that the same Dirichlet problem admits a bounded, \( \Omega_0 \)-plurisubharmonic solution which is \( C^{1,\alpha} \) for any \( 0 < \alpha < 1 \) on \( X_D^\times \). By Theorem 3.2, it follows that the two solutions must coincide. Thus \( \hat{\Phi} \) is \( C^{1,\alpha} \) on \( X_D^\times \). Since \( \hat{\Phi} = \Phi - \Phi_1 + \Psi_0 \) and both \( \Phi_1 \) and \( \Psi_0 \) are smooth on \( X_D^\times \), it follows that \( \Phi \) is \( C^{1,\alpha} \) on \( X_D^\times = X \times D^\times \). \( \square \)

Acknowledgement

We would like to acknowledge some useful comments by the referee.

References


DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027
E-mail address: phong@math.columbia.edu

DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEWARK, NEW JERSEY 07102
E-mail address: sturm@rutgers.edu