

DIHEDRAL BLOCKS WITH TWO SIMPLE MODULES

FRAUKE M. BLEHER

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ABSTRACT. Let k be an algebraically closed field of characteristic 2, and let G be a finite group. Suppose B is a block of kG with dihedral defect groups such that there are precisely two isomorphism classes of simple B -modules. The description by Erdmann of the quiver and relations of the basic algebra of B is usually only given up to a certain parameter c whose value is either 0 or 1. In this article, we show that $c = 0$ if there exists a central extension \hat{G} of G by a group of order 2 together with a block \hat{B} of $k\hat{G}$ with generalized quaternion defect groups such that B is contained in the image of \hat{B} under the natural surjection from $k\hat{G}$ onto kG . As a special case, we obtain that $c = 0$ if $G = \mathrm{PGL}_2(\mathbb{F}_q)$ for some odd prime power q and B is the principal block of $k\mathrm{PGL}_2(\mathbb{F}_q)$.

1. INTRODUCTION

Let k be an algebraically closed field of arbitrary characteristic, and let Λ be a finite dimensional k -algebra. One of the fundamental problems in representation theory is to determine the indecomposable Λ -modules up to isomorphism. Depending on the complexity of the set of isomorphism classes of indecomposable Λ -modules, one distinguishes between three representation types: finite, tame and wild. If G is a finite group and B is a block of kG , then B has finite representation type if and only if the defect groups of B are cyclic, whereas B has tame representation type if and only if the characteristic of k is 2 and the defect groups of B are either dihedral, semidihedral or generalized quaternion. In a series of articles culminating in the monograph [8], Erdmann described the quivers and relations of the basic algebras of all blocks of group algebras of tame representation type, by introducing the larger classes of algebras of dihedral, semidihedral and quaternion type. For example, if k has characteristic 2, then the class of all k -algebras Λ of dihedral type includes, up to Morita equivalence, all blocks of kG with dihedral defect groups.

Suppose now that k has characteristic 2. Let $\mathcal{B}_{\ell,d}$ consist of all blocks B of group algebras of finite groups over k with dihedral defect groups of order 2^d and precisely ℓ isomorphism classes of simple B -modules. In [4, Thm. 2], Brauer showed that

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$\ell \leq 3$. Let Λ be a k -algebra of dihedral type such that Λ is Morita equivalent to a block $B \in \mathcal{B}_{\ell,d}$. In [8, Chap. VI], the possible quivers for Λ have been determined; it follows from the tables in [8, pp. 294–297] that they depend only on the decomposition matrix of B . If $\ell = 3$, then the relations of Λ given in [8, pp. 294–297] depend only on d . If $\ell = 2$, then the relations in [8, pp. 294–297] additionally depend on a parameter $c \in \{0, 1\}$, and it is shown in [8, Sect. VI.8] that algebras with the same quiver, the same d but different c are not Morita equivalent. If B is a block in $\mathcal{B}_{2,d}$, it is usually difficult to decide whether $c = 0$ or $c = 1$. To our knowledge there are only a few cases of blocks $B \in \mathcal{B}_{2,d}$ for which c has been determined, such as the principal 2-modular block of the symmetric group S_4 (see [8, Cor. V.2.5.1]) or the principal 2-modular blocks of certain quotients of the general unitary group $\mathrm{GU}_2(\mathbb{F}_q)$ where $q \equiv 3 \pmod{4}$ (see [9, Sect. 1.5]).

In this paper, we consider the following situation, where as before k has characteristic 2. Let G be a finite group, and suppose B is a block of kG belonging to $\mathcal{B}_{2,d}$. Suppose further that there exists a central extension \hat{G} of G by a group of order 2 together with a block \hat{B} of $k\hat{G}$ with generalized quaternion defect groups of order 2^{d+1} such that B is contained in the image of \hat{B} under the natural projection $\pi : k\hat{G} \rightarrow kG$. We will prove in Theorem 2 that in this case the parameter c in the description of the relations in [8] must be equal to 0. If \hat{G} is a finite group with generalized quaternion Sylow 2-subgroups such that \hat{G} has no non-trivial normal subgroups of odd order, it follows from a result of Brauer and Suzuki [5] that the center of \hat{G} has order 2. Therefore, if $G = \hat{G}/Z(\hat{G})$ and the principal block of kG belongs to $\mathcal{B}_{2,d}$, then the corresponding parameter c must also be zero (see Corollary 3). As a special case, we obtain that if q is an odd prime power and 2^d is the maximal 2-power dividing $(q^2 - 1)$, then the parameter of the principal block of $k\mathrm{PGL}_2(\mathbb{F}_q)$, which belongs to $\mathcal{B}_{2,d}$, must also be equal to 0 (see Corollary 4).

The main idea for the proof of Theorem 2 is as follows. Let $\hat{\Lambda}$ (resp. Λ) be the basic algebra of \hat{B} (resp. B). Since B belongs to $\mathcal{B}_{2,d}$, it follows that there are precisely two isomorphism classes of simple \hat{B} -modules and that their representatives are given by the inflations of simple B -modules. In particular, if B is the principal block of kG , then \hat{B} is the principal block of $k\hat{G}$. By [8, pp. 303–304], we know the quiver and relations of $\hat{\Lambda}$ up to a parameter $\hat{c} \in k$. Since there is a surjective k -algebra homomorphism $\pi_B : \hat{B} \rightarrow B$, there is a surjective k -algebra homomorphism $\pi_\Lambda : \hat{\Lambda} \rightarrow \Lambda$ which is compatible with π_B when we view $\hat{\Lambda}$ (resp. Λ) as a (non-unitary) subalgebra of \hat{B} (resp. B). We then determine possible generators of the kernel $\mathrm{Ker}(\pi_\Lambda)$. To find these generators, we use the fact that all isomorphism classes of indecomposable Λ -modules are known, since Λ modulo its socle is a special biserial algebra by [8, Thm. VI.10.1]. This knowledge gives us enough control on $\mathrm{Ker}(\pi_\Lambda)$ to be able to deduce from the relations for $\hat{\Lambda}$ and the generators of $\mathrm{Ker}(\pi_\Lambda)$ that the constant c occurring in the relations for $\Lambda \cong \hat{\Lambda}/\mathrm{Ker}(\pi_\Lambda)$ in [8] must be zero. The surprising result is that although much less is known about 2-modular blocks with generalized quaternion defect groups than about 2-modular blocks with dihedral defect groups, the relationship we establish between the basic algebras $\hat{\Lambda}$ and Λ is enough to determine the parameter c occurring in the description of Λ .

We finish this introduction with one application of our results. Let $B \in \mathcal{B}_{2,d}$ be a block of kG for some finite group G and let V be a finitely generated kG -module belonging to B with stable endomorphism ring k . If there exists a central

extension \hat{G} of G by a group of order 2 and a block \hat{B} of $k\hat{G}$ with generalized quaternion defect groups as above, then the universal deformation ring $R(G, V)$ is isomorphic to a subquotient ring of the group ring of a defect group D of B over the ring $W(k)$ of infinite Witt vectors over k (see [3]). In fact, the conclusion about the structure of $R(G, V)$ for all V belonging to an arbitrary block B in $\mathcal{B}_{2,d}$ with stable endomorphism ring k is equivalent to the statement that the parameter c associated with each block B in $\mathcal{B}_{2,d}$ is equal to 0. This was our original motivation for studying these parameters. In [2, Question 1.1] we asked whether the conclusion about the structure of $R(G, V)$ above holds for all V having stable endomorphism ring k , without any condition on the block to which V belongs. We believe that further study of this question will lead either to a counterexample or to new results about the structure of blocks of group rings of finite groups.

For background on group algebras and blocks, we refer the reader to [7, Chaps. 1, 2 and 7]. For background on finite dimensional algebras, and in particular algebras given by quivers and relations, we refer the reader to [1, Chaps. I–III]. We would like to thank the referee for helpful comments.

2. DIHEDRAL BLOCKS WITH TWO SIMPLE MODULES

Throughout this section we make the following assumptions.

Hypothesis 1. Let k be an algebraically closed field of characteristic 2, and let $d \geq 3$ be a fixed integer. Suppose G is a finite group and B is a block of kG with dihedral defect groups of order 2^d such that there are precisely two isomorphism classes of simple B -modules.

Under these assumptions, it follows from [8, Chaps. VI, IX and pp. 294–295] that there exist $c \in \{0, 1\}$ and $i \in \{1, 2\}$ such that the basic algebra of B is isomorphic to the symmetric algebra $\Lambda_{i,c}$ as defined in Figure 1 and the decomposition matrix of B is as in Figure 2. In particular, if q is an odd prime power, 2^d is the maximal 2-power dividing $(q^2 - 1)$ and B is the principal block of $k \text{PGL}_2(\mathbb{F}_q)$, then there exists $c \in \{0, 1\}$ such that B is Morita equivalent to $\Lambda_{1,c}$ (resp. $\Lambda_{2,c}$) if $q \equiv 1 \pmod 4$ (resp. $q \equiv 3 \pmod 4$).

$$\begin{aligned}
 Q_1 &= \alpha \begin{array}{ccc} & 0 & \xrightarrow{\beta} 1 \\ \curvearrowright & \bullet & \xleftarrow{\gamma} \bullet \\ & & \end{array} \text{ and} \\
 I_{1,c} &= \langle \beta\gamma, \alpha^2 - c(\gamma\beta\alpha)^{2^{d-2}}, (\gamma\beta\alpha)^{2^{d-2}} - (\alpha\gamma\beta)^{2^{d-2}} \rangle, \\
 Q_2 &= \alpha \begin{array}{ccc} & 0 & \xrightarrow{\beta} 1 \\ \curvearrowright & \bullet & \xleftarrow{\gamma} \bullet \\ & & \end{array} \curvearrowright \eta \text{ and} \\
 I_{2,c} &= \langle \eta\beta, \gamma\eta, \beta\gamma, \alpha^2 - c\gamma\beta\alpha, \gamma\beta\alpha - \alpha\gamma\beta, \eta^{2^{d-2}} - \beta\alpha\gamma \rangle.
 \end{aligned}$$

FIGURE 1. The basic algebras $\Lambda_{1,c} = kQ_1/I_{1,c}$ and $\Lambda_{2,c} = kQ_2/I_{2,c}$.

Let $i \in \{1, 2\}$. The projective indecomposable $\Lambda_{i,c}$ -module corresponding to the vertex 0 (resp. 1) of the quiver Q_i is generated by all paths in kQ_i modulo $I_{i,c}$ that start at the vertex 0 (resp. 1). Since $\Lambda_{i,c}$ is a symmetric algebra, it follows that every projective indecomposable $\Lambda_{i,c}$ -module has a simple socle (see for example

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} (*) \qquad \left(\text{resp.} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} (*) \right)$$

FIGURE 2. The decomposition matrix for a block B of kG that is Morita equivalent to $\Lambda_{1,c}$ (resp. $\Lambda_{2,c}$), where $(*)$ means that the last row is repeated $(2^{d-2} - 1)$ times.

[7, Prop. 9.12]). An elementary combinatorial argument using the generators of the ideal $I_{1,c}$ shows that the element $(\gamma\beta\alpha)^{2^{d-2}}$ modulo $I_{1,c}$ is not zero and lies in the socle of $\Lambda_{1,c}$. Similarly, $(\beta\alpha\gamma)^{2^{d-2}}$ modulo $I_{1,c}$ is not zero and lies in the socle of $\Lambda_{1,c}$. It follows that the socle of the projective indecomposable $\Lambda_{1,c}$ -module corresponding to the vertex 0 (resp. 1) of the quiver Q_1 is generated by $(\gamma\beta\alpha)^{2^{d-2}} = (\alpha\gamma\beta)^{2^{d-2}}$ modulo $I_{1,c}$ (resp. $(\beta\alpha\gamma)^{2^{d-2}}$ modulo $I_{1,c}$). Thus the radical series length of the projective indecomposable $\Lambda_{1,c}$ -modules is at least $3 \cdot 2^{d-2} + 1$. Using the generators of the ideal $I_{1,c}$, it follows that all paths of length at least $3 \cdot 2^{d-2} + 1$ are zero modulo $I_{1,c}$. Hence the radical series length of the projective indecomposable $\Lambda_{1,c}$ -modules is exactly $3 \cdot 2^{d-2} + 1$. Analyzing the projective indecomposable $\Lambda_{2,c}$ -module corresponding to the vertex 0 of the quiver Q_2 , one similarly sees that its socle is generated by $\gamma\beta\alpha = \alpha\gamma\beta$ modulo $I_{2,c}$ and its radical series length is 4. For the projective indecomposable $\Lambda_{2,c}$ -module corresponding to the vertex 1 of the quiver Q_2 , one has that its socle is generated by $\eta^{2^{d-2}} = \beta\alpha\gamma$ modulo $I_{2,c}$ and its radical series length is 4 (resp. $2^{d-2} + 1$) if $d = 3$ (resp. $d > 3$).

It follows from the definition of string algebras in [6, Sect. 3] and from the above description of the socle of $\Lambda_{i,c}$ for $i \in \{1, 2\}$ that $\Lambda_{i,c}/\text{soc}(\Lambda_{i,c})$ is a string algebra. Therefore, one can see as in [8, Sect. I.8.11] that the isomorphism classes of all non-projective indecomposable $\Lambda_{i,c}$ -modules are given by string and band modules as defined in [6, Sect. 3].

If $i \in \{1, 2\}$ and B is a block of kG that is Morita equivalent to $\Lambda_{i,c}$, we can use the decomposition matrix of B in Figure 2 to compute the composition series length of the projective indecomposable B -modules (see [7, Thm. 18.26]). It follows that if $i = 1$, then the projective indecomposable B -module corresponding to the vertex 0 (resp. 1) of the quiver Q_1 has $6 + 6 \cdot (2^{d-2} - 1) = 6 \cdot 2^{d-2}$ (resp. $4 + 3 \cdot (2^{d-2} - 1) = 1 + 3 \cdot 2^{d-2}$) composition factors. If $i = 2$, then the projective indecomposable B -module corresponding to the vertex 0 (resp. 1) of the quiver Q_2 has 6 (resp. $4 + (2^{d-2} - 1) = 3 + 2^{d-2}$) composition factors.

Theorem 2. *Assume Hypothesis 1. Suppose there exists a central extension \hat{G} of G by a group of order 2 together with a block \hat{B} of $k\hat{G}$ with generalized quaternion defect groups such that B is contained in the image of \hat{B} under the natural projection $\pi : k\hat{G} \rightarrow kG$. Then B is Morita equivalent to either $\Lambda_{1,0}$ or $\Lambda_{2,0}$; i.e. the parameter c is equal to 0.*

Proof. Let $z \in \hat{G}$ be a central element of order 2 such that G is isomorphic to $\hat{G}/\langle z \rangle$. In the following, we identify G with $\hat{G}/\langle z \rangle$. Let $\pi : k\hat{G} \rightarrow kG$ be the natural projection. Since by assumption B is contained in the image of \hat{B} under π , there

is a simple kG -module S belonging to B whose inflation via π belongs to \hat{B} . Let T be an arbitrary simple $k\hat{G}$ -module. Since z has order 2, it acts trivially on T . Thus T is inflated via π from a simple kG -module, which we also denote by T . By [7, Thm. 56.12], T belongs to B (resp. \hat{B}) if and only if there is a sequence of simple kG -modules (resp. simple $k\hat{G}$ -modules) E_1, E_2, \dots, E_n such that $E_1 = S, E_n = T$ and, for $1 \leq j < n$, E_j and E_{j+1} are equal or there is a non-split G -extension (resp. a non-split \hat{G} -extension) of one of them by the other. If E and E' are simple kG -modules whose inflations from G to \hat{G} are also denoted by E and E' , then the Lyndon-Hochschild-Serre spectral sequence gives an exact sequence

$$0 \rightarrow \text{Ext}_{kG}^1(E, E') \rightarrow \text{Ext}_{k\hat{G}}^1(E, E') \rightarrow \text{Hom}_{kG}(E, E').$$

This implies immediately that $\text{Ext}_{k\hat{G}}^1(E, E')$ and $\text{Ext}_{kG}^1(E, E')$ have the same k -dimension for each choice of non-isomorphic E and E' . Therefore, the above characterization of the simple modules which belong to B (resp. \hat{B}) implies that \hat{B} has the same number of isomorphism classes of simple modules as B . We can thus identify the simple \hat{B} -modules with the simple B -modules. Therefore, the restriction of the natural projection π to \hat{B} gives a surjective k -algebra homomorphism

$$(1) \quad \pi_B : \hat{B} \rightarrow B.$$

Moreover, since $\text{Ker}(\pi) = (1 + z)k\hat{G}$, it follows that $\text{Ker}(\pi_B) = (1 + z)\hat{B}$. Thus if \hat{P} is a projective indecomposable \hat{B} -module and $P = B \otimes_{\hat{B}} \hat{P}$ is the corresponding projective indecomposable B -module, then we have a short exact sequence of \hat{B} -modules

$$(2) \quad 0 \rightarrow (1 + z)\hat{P} \rightarrow \hat{P} \rightarrow P \rightarrow 0.$$

Since the map $f : (1 + z)\hat{P} \rightarrow B \otimes_{\hat{B}} \hat{P} = P$ defined by $f((1 + z)x) = 1 \otimes x$ for all $x \in \hat{P}$ is a \hat{B} -module isomorphism, \hat{P} is a non-trivial \hat{B} -extension of P by itself. In particular, \hat{P} has twice as many composition factors as P .

Considering the possible decomposition matrices for blocks with generalized quaternion defect groups and two isomorphism classes of simple modules as given in [8, pp. 303–304] and using that the projective indecomposable \hat{B} -modules must have twice as many composition factors as the corresponding projective indecomposable B -modules, we conclude the following: If B is Morita equivalent to $\Lambda_{1,c}$ (resp. $\Lambda_{2,c}$), then there exists a constant $\hat{c} \in k$ such that the basic algebra of \hat{B} is isomorphic to $\hat{\Lambda}_{1,\hat{c}}$ (resp. $\hat{\Lambda}_{2,\hat{c}}$) as defined in Figure 3 and the decomposition matrix of \hat{B} is as in Figure 4.

$$\begin{aligned} \hat{I}_{1,\hat{c}} &= \langle \gamma\beta\gamma - \alpha\gamma(\beta\alpha\gamma)^{2^{d-1}-1}, \beta\gamma\beta - \beta\alpha(\gamma\beta\alpha)^{2^{d-1}-1}, \\ &\quad \alpha^2 - \gamma\beta(\alpha\gamma\beta)^{2^{d-1}-1} - \hat{c}(\alpha\gamma\beta)^{2^{d-1}}, \beta\alpha^2 \rangle, \\ \hat{I}_{2,\hat{c}} &= \langle \eta\beta - \beta\alpha(\gamma\beta\alpha), \gamma\eta - \alpha\gamma(\beta\alpha\gamma), \beta\gamma - \eta^{2^{d-1}-1}, \\ &\quad \alpha^2 - \gamma\beta(\alpha\gamma\beta) - \hat{c}(\alpha\gamma\beta)^2, \beta\alpha^2 \rangle. \end{aligned}$$

FIGURE 3. The basic algebras $\hat{\Lambda}_{1,\hat{c}} = kQ_1/\hat{I}_{1,\hat{c}}$ and $\hat{\Lambda}_{2,\hat{c}} = kQ_2/\hat{I}_{2,\hat{c}}$, where Q_1 and Q_2 are as in Figure 1.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \quad (**)$$

$$\left(\text{resp.} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} (**)\right)$$

FIGURE 4. The decomposition matrix for a block \hat{B} of $k\hat{G}$ that is Morita equivalent to $\hat{\Lambda}_{1,\hat{c}}$ (resp. $\hat{\Lambda}_{2,\hat{c}}$), where $(**)$ means that the last row is repeated $(2^{d-1} - 1)$ times.

As we did for B , we can use the decomposition matrix of \hat{B} in Figure 4 to compute the composition series length of the projective indecomposable \hat{B} -modules. If \hat{B} is Morita equivalent to $\hat{\Lambda}_{1,\hat{c}}$, then the projective indecomposable \hat{B} -module corresponding to the vertex 0 (resp. 1) of the quiver Q_1 has $6+6\cdot(2^{d-1}-1) = 6\cdot 2^{d-1}$ (resp. $5+3\cdot(2^{d-1}-1) = 2+3\cdot 2^{d-1}$) composition factors. If \hat{B} is Morita equivalent to $\hat{\Lambda}_{2,\hat{c}}$, then the projective indecomposable \hat{B} -module corresponding to the vertex 0 (resp. 1) of the quiver Q_2 has 12 (resp. $7+(2^{d-1}-1) = 6+2^{d-1}$) composition factors.

Let E and F be non-isomorphic simple B -modules. Then the inflations of E and F to \hat{B} are representatives of the isomorphism classes of simple \hat{B} -modules, which we again denote by E and F . Let \hat{e} and \hat{f} be two orthogonal primitive idempotents in \hat{B} such that $\hat{B}\hat{e}$ (resp. $\hat{B}\hat{f}$) is a projective \hat{B} -module cover of E (resp. F). Then $e = \pi_B(\hat{e})$ and $f = \pi_B(\hat{f})$ are orthogonal primitive idempotents in B , and Be (resp. Bf) is a projective B -module cover of E (resp. F). Let $\hat{\varepsilon} = \hat{e} + \hat{f}$ and $\varepsilon = e + f$. Then π_B from (1) restricts to a surjective k -algebra homomorphism

$$(3) \quad \pi_\varepsilon : \hat{\varepsilon}\hat{B}\hat{\varepsilon} \rightarrow \varepsilon B \varepsilon.$$

Let $i \in \{1, 2\}$, $\hat{c} \in k$ and $c \in \{0, 1\}$ be such that there are k -algebra isomorphisms $\hat{f}_\Lambda : \hat{\varepsilon}\hat{B}\hat{\varepsilon} \rightarrow \hat{\Lambda}_{i,\hat{c}}$ and $f_\Lambda : \varepsilon B \varepsilon \rightarrow \Lambda_{i,c}$. Then there exists a surjective k -algebra homomorphism

$$(4) \quad \pi_\Lambda : \hat{\Lambda}_{i,\hat{c}} \rightarrow \Lambda_{i,c}$$

such that the diagram in Figure 5 commutes. In the diagram, nat.proj. stands for the natural projection of $\hat{\Lambda}_{i,\hat{c}}$ onto $\hat{\Lambda}_{i,\hat{c}}/\text{Ker}(\pi_\Lambda)$ and $\bar{\pi}_\Lambda$ is the k -algebra isomorphism induced by π_Λ . As in Figure 5, define $\Lambda'_{i,c} = \hat{\Lambda}_{i,\hat{c}}/\text{Ker}(\pi_\Lambda)$. We introduce this extra notation because in the following we will need to distinguish between the isomorphic k -algebras $\Lambda_{i,c}$ and $\Lambda'_{i,c}$.

We next show that the kernel of π_Λ is contained in the square of the radical of $\hat{\Lambda}_{i,\hat{c}}$. This basically follows from the fact that $\hat{\Lambda}_{i,\hat{c}}$ and $\Lambda_{i,c}$ have the same quiver Q_i . However, for the convenience of the reader, we now give a more detailed explanation. Since π_Λ is surjective, $\pi_\Lambda(\text{rad}^n(\hat{\Lambda}_{i,\hat{c}})) \subseteq \text{rad}^n(\Lambda_{i,c})$ for all $n \geq 1$ by [7, Prop. 5.6]. Thus π_Λ induces a surjective k -algebra homomorphism $\pi_{\Lambda,n} : \hat{\Lambda}_{i,\hat{c}}/\text{rad}^n(\hat{\Lambda}_{i,\hat{c}}) \rightarrow \Lambda_{i,c}/\text{rad}^n(\Lambda_{i,c})$ for all $n \geq 1$. Let kQ_i^+ be the ideal of kQ_i generated by all arrows. By [1, Prop. III.1.6], $\text{rad}(\hat{\Lambda}_{i,\hat{c}})$ (resp. $\text{rad}(\Lambda_{i,c})$) is the image of kQ_i^+ in $\hat{\Lambda}_{i,\hat{c}}$ (resp. $\Lambda_{i,c}$). Because $\hat{I}_{i,\hat{c}}$ (resp. $I_{i,c}$) is contained in $(kQ_i^+)^2$,

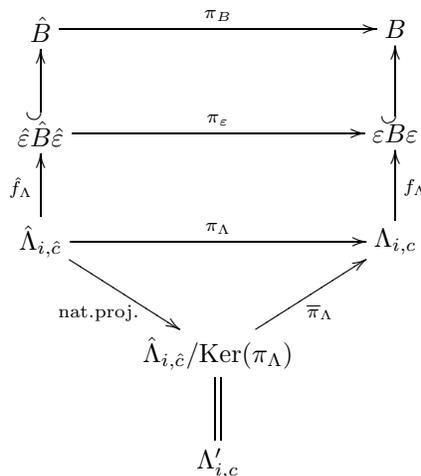


FIGURE 5. The commutative diagram relating \hat{B} , $\hat{\Lambda}_{i,\hat{c}}$, B and $\Lambda_{i,c}$.

it follows that for $n \in \{1, 2\}$ there is an isomorphism between $kQ_i/(kQ_i^+)^n$ and $\hat{\Lambda}_{i,\hat{c}}/\text{rad}^n(\hat{\Lambda}_{i,\hat{c}})$ (resp. $\Lambda_{i,c}/\text{rad}^n(\Lambda_{i,c})$). Hence $\pi_{\Lambda,n}$ is a k -algebra isomorphism for $n \in \{1, 2\}$, which implies $\text{Ker}(\pi_\Lambda) \subset \text{rad}^2(\hat{\Lambda}_{i,\hat{c}})$. In particular, [1, Prop. III.1.6] implies that the radical of $\Lambda'_{i,c}$ is the image of kQ_i^+ in $\Lambda'_{i,c}$.

Even though π_B in (1) is explicit and natural and the domain and range of π_Λ in (4) are explicitly given, we do not have an explicit description of π_Λ , owing to the fact that Morita equivalences, given by f_Λ and \hat{f}_Λ , were used to define π_Λ in Figure 5. Therefore, we need to show that π_Λ matches up the simple modules corresponding to the vertices of Q_i . Let S_0 (resp. S_1) be the simple $\Lambda'_{i,c}$ -module corresponding to the vertex 0 (resp. 1) in Q_i . Then the inflations of S_0 and S_1 to $\hat{\Lambda}_{i,\hat{c}}$ are the simple $\hat{\Lambda}_{i,\hat{c}}$ -modules, which also correspond to the vertices 0 and 1 of Q_i , respectively. Let T_0 (resp. T_1) be the simple $\Lambda_{i,c}$ -modules corresponding to the vertex 0 (resp. 1) in Q_i . We computed the composition series lengths of the projective indecomposable $\Lambda_{i,c}$ -modules $P(T_0)$ and $P(T_1)$ in the paragraph before the statement of Theorem 2, and we computed the composition series lengths of the projective indecomposable $\hat{\Lambda}_{i,\hat{c}}$ -modules $\hat{P}(S_0)$ and $\hat{P}(S_1)$ in the paragraph after defining $\hat{\Lambda}_{1,\hat{c}}$ and $\hat{\Lambda}_{2,\hat{c}}$. Comparing these composition series lengths, we see that $\hat{P}(S_u)$ has twice as many composition factors as $P(T_v)$ if and only if $u = v$. Thus the Morita equivalence between $\Lambda'_{i,c}$ and $\Lambda_{i,c}$ induced by the isomorphism $\bar{\pi}_\Lambda$ must send S_u to T_u for $u \in \{0, 1\}$.

To complete the proof of Theorem 2, we will make use of $\hat{\Lambda}_{i,\hat{c}}$ -modules corresponding to certain paths in kQ_i . Let $w = \zeta_n \cdots \zeta_2 \zeta_1$ be a path of length $n \geq 1$ in kQ_i whose image modulo $\hat{\Lambda}_{i,\hat{c}}$ does not lie in $\text{soc}_2(\hat{\Lambda}_{i,\hat{c}})$. For $1 \leq j \leq n$, let v_j be the end vertex of ζ_j , and let v_0 be the starting vertex of ζ_1 . Define a kQ_i -module M_w of k -dimension $n + 1$ with respect to a given k -basis $\{b_0, \dots, b_n\}$ as follows. Let $0 \leq j \leq n$. If v is a vertex in Q_i , define $vb_j = b_j$ if $v = v_j$ and $vb_j = 0$ otherwise. If ζ is an arrow in Q_i , define $\zeta b_j = b_{j+1}$ if $\zeta = \zeta_{j+1}$ and $j \leq n - 1$, and otherwise define $\zeta b_j = 0$. By our assumption on w , the ideal $\hat{\Lambda}_{i,\hat{c}}$ of kQ_i acts as zero on M_w .

Hence M_w defines a $\hat{\Lambda}_{i,\hat{e}}$ -module, which we also denote by M_w . Moreover, M_w is a uniserial $\hat{\Lambda}_{i,\hat{e}}$ -module with descending composition factors $S_{v_0}, S_{v_1}, \dots, S_{v_n}$.

Suppose first that $i = 1$. As seen in the second paragraph after Hypothesis 1, the radical series lengths of the projective $\Lambda_{1,c}$ -module covers of T_0 and T_1 are both $3 \cdot 2^{d-2} + 1$. Using the Morita equivalence between $\Lambda'_{1,c}$ and $\Lambda_{1,c}$ induced by $\bar{\pi}_\Lambda$, it follows that the radical series lengths of the projective $\Lambda'_{1,c}$ -module covers of S_0 and S_1 are also both $3 \cdot 2^{d-2} + 1$. Since both the radical of $\hat{\Lambda}_{1,\hat{e}}$ and the radical of $\Lambda'_{1,c}$ are generated by arrows, all paths in kQ_1 of length greater than or equal to $3 \cdot 2^{d-2} + 1$ modulo $\hat{I}_{1,\hat{e}}$ must lie in $\text{Ker}(\pi_\Lambda)$. In particular,

$$\gamma\beta(\alpha\gamma\beta)^{2^{d-1}-1} + \hat{e}(\alpha\gamma\beta)^{2^{d-1}} \quad \text{modulo } \hat{I}_{1,\hat{e}}$$

lies in $\text{Ker}(\pi_\Lambda)$. Thus it follows from the description of $\hat{I}_{1,\hat{e}}$ in Figure 3 that α^2 modulo $\hat{I}_{1,\hat{e}}$ lies in $\text{Ker}(\pi_\Lambda)$. Using similar arguments, we see that

$$(5) \quad \alpha^2, \beta\gamma\beta, \gamma\beta\gamma, (\alpha\gamma\beta)^{2^{d-2}}\alpha, (\gamma\beta\alpha)^{2^{d-2}}\gamma, (\beta\alpha\gamma)^{2^{d-2}}\beta \quad \text{modulo } \hat{I}_{1,\hat{e}}$$

all lie in $\text{Ker}(\pi_\Lambda)$. Taking the path $\beta\gamma$ in kQ_1 , we can define the uniserial $\hat{\Lambda}_{1,\hat{e}}$ -module $M_{\beta\gamma}$ as above with descending composition factors S_1, S_0, S_1 . Using that $\Lambda_{1,c}/\text{soc}(\Lambda_{1,c})$ is a string algebra, we see that there is no uniserial $\Lambda_{1,c}$ -module with descending composition factors T_1, T_0, T_1 . Because of the Morita equivalence between $\Lambda'_{1,c}$ and $\Lambda_{1,c}$ induced by $\bar{\pi}_\Lambda$, it follows that there is also no uniserial $\Lambda'_{1,c}$ -module with descending composition factors S_1, S_0, S_1 . Therefore, there must exist an element $x \in \text{Ker}(\pi_\Lambda)$ that acts non-trivially on $M_{\beta\gamma}$. The only elements of $\hat{\Lambda}_{1,\hat{e}}$ that act non-trivially on $M_{\beta\gamma}$ are k -linear combinations of the paths $1_0, 1_1, \beta, \gamma$ and $\beta\gamma$ modulo $\hat{I}_{1,\hat{e}}$. Since $\text{Ker}(\pi_\Lambda) \subset \text{rad}^2(\hat{\Lambda}_{1,\hat{e}})$, it follows that $x = \beta\gamma + x'$ modulo $\hat{I}_{1,\hat{e}}$ where x' is a (possibly zero) k -linear combination of paths z of length at least 2 that are different from $\beta\gamma$. Multiplying x on both sides by 1_1 modulo $\hat{I}_{1,\hat{e}}$, we may assume that all the paths z coming up in the description of x' both start and end at the vertex 1. Moreover, we may omit all paths z that modulo $\hat{I}_{1,\hat{e}}$ lie in $\text{Ker}(\pi_\Lambda)$ according to (5). Thus we may assume that x' is a k -linear combination of $(\beta\alpha\gamma)^j$ for $1 \leq j \leq 2^{d-2}$. It follows that there exist constants $a_1, \dots, a_{2^{d-2}}$ in k such that

$$(6) \quad x = \beta\gamma + \sum_{j=1}^{2^{d-2}} a_j (\beta\alpha\gamma)^j \quad \text{modulo } \hat{I}_{1,\hat{e}}$$

lies in $\text{Ker}(\pi_\Lambda)$. Suppose that not all the a_j are zero, and let $1 \leq j_0 \leq 2^{d-2}$ be minimal with $a_{j_0} \neq 0$. If $j_0 < 2^{d-2}$, we can multiply x on the left by $\gamma(\beta\alpha\gamma)^{2^{d-2}-j_0-1}$ modulo $\hat{I}_{1,\hat{e}}$ (resp. on the right by $(\beta\alpha\gamma)^{2^{d-2}-j_0-1}\beta$ modulo $\hat{I}_{1,\hat{e}}$) and use (5) to see that

$$\gamma(\beta\alpha\gamma)^{2^{d-2}-1} \quad \text{and} \quad (\beta\alpha\gamma)^{2^{d-2}-1}\beta \quad \text{modulo } \hat{I}_{1,\hat{e}}$$

lie in $\text{Ker}(\pi_\Lambda)$. But this implies that all paths of length $3 \cdot 2^{d-2}$ modulo $\hat{I}_{1,\hat{e}}$ lie in $\text{Ker}(\pi_\Lambda)$, and hence that $\text{rad}^{3 \cdot 2^{d-2}}(\Lambda'_{1,c}) = 0$, which is a contradiction. Thus $a_j = 0$ for $1 \leq j < 2^{d-2}$ and we obtain

$$(7) \quad \pi_\Lambda \left(\beta\gamma - c_1 (\beta\alpha\gamma)^{2^{d-2}} \text{ modulo } \hat{I}_{1,\hat{e}} \right) = 0$$

for some $c_1 \in k$. Using that $\Lambda_{1,c}/\text{soc}(\Lambda_{1,c})$ is a string algebra, we see that there is no uniserial $\Lambda_{1,c}$ -module of length $3 \cdot 2^{d-2} + 1$ with descending composition factors

$$(T_0, T_0, T_1, T_0, T_0, \dots, T_1, T_0, T_0, T_1, T_0)$$

(resp. $(T_0, T_1, T_0, T_0, T_1, \dots, T_0, T_0, T_1, T_0, T_0)$).

Because of the Morita equivalence between $\Lambda'_{1,c}$ and $\Lambda_{1,c}$ induced by $\bar{\pi}_\Lambda$, we see that there is also no uniserial $\Lambda'_{1,c}$ -module of length $3 \cdot 2^{d-2} + 1$ with descending composition factors

$$(S_0, S_0, S_1, S_0, S_0, \dots, S_1, S_0, S_0, S_1, S_0)$$

(resp. $(S_0, S_1, S_0, S_0, S_1, \dots, S_0, S_0, S_1, S_0, S_0)$).

On the other hand, we can define a uniserial $\hat{\Lambda}_{1,\hat{c}}$ -module M_w corresponding to the path $w = (\gamma\beta\alpha)^{2^{d-2}}$ (resp. $w = (\alpha\gamma\beta)^{2^{d-2}}$) in kQ_1 with such descending composition factors. Thus there must exist an element $y \in \text{Ker}(\pi_\Lambda)$ that acts non-trivially on M_w . The only elements of $\hat{\Lambda}_{1,\hat{c}}$ that act non-trivially on M_w are k -linear combinations of $1_0, 1_1, \alpha, \beta, \gamma$ and subpaths of w of length at least 2 modulo $\hat{I}_{1,\hat{c}}$. Since $\text{Ker}(\pi_\Lambda) \subset \text{rad}^2(\hat{\Lambda}_{1,\hat{c}})$, there exists a subpath w' of w of length at least 2 such that $y = w' + y'$ modulo $\hat{I}_{1,\hat{c}}$ and y' is a (possibly zero) k -linear combination of paths $z \neq w'$ of length at least 2. Since $w = w_1 w' w_2$ for certain subpaths w_1 and w_2 of w , we can multiply y on the left by w_1 modulo $\hat{I}_{1,\hat{c}}$ and on the right by w_2 modulo $\hat{I}_{1,\hat{c}}$ to be able to assume that $w' = w$. Multiplying y on both sides by 1_0 modulo $\hat{I}_{1,\hat{c}}$, we may assume that all the paths z coming up in the description of y' both start and end at the vertex 0. Moreover, we may omit all paths z that modulo $\hat{I}_{1,\hat{c}}$ lie in $\text{Ker}(\pi_\Lambda)$ according to (5). If \tilde{w} is the path such that $\{w, \tilde{w}\} = \{(\gamma\beta\alpha)^{2^{d-2}}, (\alpha\gamma\beta)^{2^{d-2}}\}$, we may thus assume that y' is a k -linear combination of $\tilde{w}, (\gamma\beta\alpha)^j$ and $(\alpha\gamma\beta)^j$ for $1 \leq j \leq 2^{d-2} - 1$. It follows that there exist constants $a, a_{j,1}, a_{j,2}$ in k for $1 \leq j \leq 2^{d-2} - 1$ such that

$$(8) \quad y = w + a\tilde{w} + \sum_{j=1}^{2^{d-2}-1} a_{j,1} (\gamma\beta\alpha)^j + \sum_{j=1}^{2^{d-2}-1} a_{j,2} (\alpha\gamma\beta)^j \quad \text{modulo } \hat{I}_{1,\hat{c}}$$

lies in $\text{Ker}(\pi_\Lambda)$. Arguing in a similar way to (7), we see that $a_{j,1} = 0 = a_{j,2}$ for all $1 \leq j \leq 2^{d-2} - 1$. We obtain

$$(9) \quad \pi_\Lambda \left((\gamma\beta\alpha)^{2^{d-2}} - c_2 (\alpha\gamma\beta)^{2^{d-2}} \text{ modulo } \hat{I}_{1,\hat{c}} \right) = 0,$$

$$(10) \quad \pi_\Lambda \left((\alpha\gamma\beta)^{2^{d-2}} - c_3 (\gamma\beta\alpha)^{2^{d-2}} \text{ modulo } \hat{I}_{1,\hat{c}} \right) = 0$$

for certain $c_2, c_3 \in k$. If one of c_2 and c_3 were zero, we could also choose the other to be zero. This would imply that the radical series length of the projective $\Lambda'_{1,c}$ -module cover of S_0 is at most $3 \cdot 2^{d-2}$, which is a contradiction. Thus c_2 and c_3 both have to be non-zero. Therefore, there exist certain $c_1 \in k$ and $c_2 \in k^*$ such that $\text{Ker}(\pi_\Lambda)$ contains the image \hat{J}_{c_1, c_2} in $\hat{\Lambda}_{1,\hat{c}}$ of the ideal

$$J_{c_1, c_2} = \langle \alpha^2, \beta\gamma - c_1(\beta\alpha\gamma)^{2^{d-2}}, (\gamma\beta\alpha)^{2^{d-2}} - c_2(\alpha\gamma\beta)^{2^{d-2}} \rangle \subset kQ_1.$$

We can compute the k -dimension of $kQ_1/I_{1,c} = \Lambda_{1,c}$ (resp. of $kQ_1/J_{c_1, c_2} \cong \hat{\Lambda}_{1,\hat{c}}/\hat{J}_{c_1, c_2}$) by counting k -linearly independent paths in kQ_1 modulo $I_{1,c}$ (resp. modulo J_{c_1, c_2}). Since these k -dimensions are equal, it follows that $\text{Ker}(\pi_\Lambda) = \hat{J}_{c_1, c_2}$

and thus $\Lambda'_{1,c} \cong kQ_1/J_{c_1,c_2}$. Because $\Lambda'_{1,c} \cong kQ_1/J_{c_1,c_2}$ is symmetric, there exists a k -linear map $\varphi : kQ_1/J_{c_1,c_2} \rightarrow k$ such that $\varphi(ab) = \varphi(ba)$ for all $a, b \in kQ_1/J_{c_1,c_2}$ and such that $\text{Ker}(\varphi)$ does not contain any non-zero left ideal of $kQ_1/J_{c_1,c_2}$. As in the second paragraph after Hypothesis 1, one sees that the socle of the projective indecomposable $kQ_1/J_{c_1,c_2}$ -module corresponding to the vertex 0 is generated by $(\gamma\beta\alpha)^{2^{d-2}}$ modulo J_{c_1,c_2} . It follows that

$$\varphi \left((\gamma\beta\alpha)^{2^{d-2}} \text{ modulo } J_{c_1,c_2} \right) = \varphi \left((\alpha\gamma\beta)^{2^{d-2}} \text{ modulo } J_{c_1,c_2} \right)$$

is non-zero, which implies $c_2 = 1$. Hence $\Lambda'_{1,c} \cong kQ_1/J_{c_1,1}$ for a certain $c_1 \in k$. Since the k -algebra homomorphism

$$\begin{aligned} \Lambda_{1,0} = kQ_1/I_{1,0} &\longrightarrow kQ_1/J_{c_1,1} \\ z \text{ modulo } I_{1,0} &\mapsto z \text{ modulo } J_{c_1,1} \quad \text{for } z \in \{1_0, 1_1, \alpha, \beta\}, \\ \gamma \text{ modulo } I_{1,0} &\mapsto \gamma - c_1\alpha\gamma(\beta\alpha\gamma)^{2^{d-2}-1} \text{ modulo } J_{c_1,1} \end{aligned}$$

is a k -algebra isomorphism for all $c_1 \in k$, it follows that $\Lambda'_{1,c} \cong \Lambda_{1,0}$. Therefore, the parameter c must be equal to zero.

Suppose next that $i = 2$. As seen in the second paragraph after Hypothesis 1, the radical series length of the projective $\Lambda_{2,c}$ -module cover of T_0 (resp. T_1) is 4 (resp. 4 if $d = 3$ and $2^{d-2} + 1$ if $d > 3$). Using the Morita equivalence between $\Lambda'_{2,c}$ and $\Lambda_{2,c}$ induced by $\bar{\pi}_\Lambda$, it follows that the radical series length of the projective $\Lambda'_{2,c}$ -module cover of S_0 (resp. S_1) is also 4 (resp. 4 if $d = 3$ and $2^{d-2} + 1$ if $d > 3$). Since both the radical of $\hat{\Lambda}_{2,\hat{c}}$ and the radical of $\Lambda'_{2,c}$ are generated by arrows, all paths in kQ_2 starting at vertex 0 of length greater than or equal to 4 modulo $\hat{I}_{2,\hat{c}}$ must lie in $\text{Ker}(\pi_\Lambda)$. Also, if $d = 3$ (resp. $d > 3$), then all paths in kQ_2 starting at vertex 1 of length greater than or equal to 4 (resp. $2^{d-2} + 1$) modulo $\hat{I}_{2,\hat{c}}$ must lie in $\text{Ker}(\pi_\Lambda)$. Thus we can argue similarly to (5), using the description of $\hat{I}_{2,\hat{c}}$ in Figure 3, and obtain that

$$(11) \quad \alpha(\gamma\beta\alpha), \beta(\alpha\gamma\beta), \alpha^2, \eta\beta, \gamma\eta, \gamma(\beta\alpha\gamma), \eta^4 \text{ modulo } \hat{I}_{2,\hat{c}} \quad \text{if } d = 3$$

$$(12) \text{ (resp. } \alpha(\gamma\beta\alpha), \beta(\alpha\gamma\beta), \alpha^2, \eta\beta, \gamma\eta, \beta\gamma, \eta^{2^{d-2}+1} \text{ modulo } \hat{I}_{2,\hat{c}} \quad \text{if } d > 3)$$

all lie in $\text{Ker}(\pi_\Lambda)$. We first show that for $d = 3$, we also have that $\beta\gamma$ and $\eta^{2^{d-2}+1}$ modulo $\hat{I}_{2,\hat{c}}$ lie in $\text{Ker}(\pi_\Lambda)$. Using that $\Lambda_{2,c}/\text{soc}(\Lambda_{2,c})$ is a string algebra, we see that if $d = 3$, there is no uniserial $\Lambda_{2,c}$ -module with descending composition factors T_1, T_1, T_1 . Because of the Morita equivalence between $\Lambda'_{2,c}$ and $\Lambda_{2,c}$ induced by $\bar{\pi}_\Lambda$, there is then also no uniserial $\Lambda'_{2,c}$ -module with descending composition factors S_1, S_1, S_1 . However, we can define the uniserial $\hat{\Lambda}_{2,\hat{c}}$ -module M_{η^2} corresponding to the path η^2 in kQ_2 with such descending composition factors. Since $\beta\gamma = \eta^3$ modulo $\hat{I}_{2,\hat{c}}$ if $d = 3$, we can use (11) to argue similarly to (6) that there are constants b_1 and b_2 in k such that

$$(13) \quad \eta^2 + b_1\beta\alpha\gamma + b_2\eta^3 \text{ modulo } \hat{I}_{2,\hat{c}}$$

lies in $\text{Ker}(\pi_\Lambda)$ if $d = 3$. Multiplying (13) on the left by η modulo $\hat{I}_{2,\hat{c}}$ and using that $\eta\beta$ and η^4 modulo $\hat{I}_{2,\hat{c}}$ lie in $\text{Ker}(\pi_\Lambda)$ by (11), we see that η^3 and thus $\beta\gamma$ modulo $\hat{I}_{2,\hat{c}}$ lie in $\text{Ker}(\pi_\Lambda)$ if $d = 3$. Thus it follows from (11) and (12) that for all $d \geq 3$,

$$(14) \quad \alpha(\gamma\beta\alpha), \beta(\alpha\gamma\beta), \alpha^2, \eta\beta, \gamma\eta, \beta\gamma, \eta^{2^{d-2}+1} \text{ modulo } \hat{I}_{2,\hat{c}}$$

all lie in $\text{Ker}(\pi_\Lambda)$. Using that $\Lambda_{2,c}/\text{soc}(\Lambda_{2,c})$ is a string algebra, we see that there is no uniserial $\Lambda_{2,c}$ -module with descending composition factors T_1, T_0, T_0, T_1 . Because of the Morita equivalence between $\Lambda'_{2,c}$ and $\Lambda_{2,c}$ induced by $\bar{\pi}_\Lambda$, we see that there is also no uniserial $\Lambda'_{2,c}$ -module with descending composition factors S_1, S_0, S_0, S_1 . On the other hand, we can define the uniserial $\hat{\Lambda}_{2,\hat{c}}$ -module $M_{\beta\alpha\gamma}$ corresponding to the path $\beta\alpha\gamma$ in kQ_2 with such descending composition factors. Thus we can argue similarly to (6) and (8), using (14), that there exist constants $a_2, \dots, a_{2^{d-2}}$ in k such that

$$(15) \quad \beta\alpha\gamma + \sum_{j=2}^{2^{d-2}} a_j \eta^j \quad \text{modulo } \hat{I}_{2,\hat{c}}$$

lies in $\text{Ker}(\pi_\Lambda)$. Suppose that not all the a_j are zero, and let $2 \leq j_0 \leq 2^{d-2}$ be minimal with $a_{j_0} \neq 0$. If $j_0 < 2^{d-2}$, then $d > 3$. Since $\eta\beta$ and $\eta^{2^{d-2}+1}$ modulo $\hat{I}_{2,\hat{c}}$ lie in $\text{Ker}(\pi_\Lambda)$ by (14), we can multiply (15) on the left by $\eta^{2^{d-2}-j_0}$ modulo $\hat{I}_{2,\hat{c}}$ to see that $\eta^{2^{d-2}}$ modulo $\hat{I}_{2,\hat{c}}$ lies in $\text{Ker}(\pi_\Lambda)$. But this implies that all paths of length 2^{d-2} modulo $\hat{I}_{2,\hat{c}}$ lie in $\text{Ker}(\pi_\Lambda)$, and hence that $\text{rad}^{2^{d-2}}(\Lambda'_{2,c}) = 0$, which is a contradiction. Thus $a_j = 0$ for $2 \leq j < 2^{d-2}$, and we obtain

$$(16) \quad \pi_\Lambda \left(\beta\alpha\gamma - c_1 \eta^{2^{d-2}} \text{ modulo } \hat{I}_{2,\hat{c}} \right) = 0$$

for some $c_1 \in k$. Similarly, one sees that

$$(17) \quad \pi_\Lambda \left(\eta^{2^{d-2}} - c_2 \beta\alpha\gamma \text{ modulo } \hat{I}_{2,\hat{c}} \right) = 0$$

for some $c_2 \in k$. If one of c_1 and c_2 were zero, we could also choose the other to be zero. This would imply that the radical series length of the projective $\Lambda'_{2,c}$ -module cover of S_1 is at most 3 (resp. 2^{d-2}) if $d = 3$ (resp. $d > 3$), which is a contradiction. Thus both c_1 and c_2 have to be non-zero. Using that $\Lambda_{2,c}/\text{soc}(\Lambda_{2,c})$ is a string algebra, we see that there is no uniserial $\Lambda_{2,c}$ -module with descending composition factors

$$(T_0, T_0, T_1, T_0) \quad (\text{resp. } (T_0, T_1, T_0, T_0)).$$

Because of the Morita equivalence between $\Lambda'_{2,c}$ and $\Lambda_{2,c}$ induced by $\bar{\pi}_\Lambda$, we see that there is also no uniserial $\Lambda'_{2,c}$ -module with descending composition factors

$$(S_0, S_0, S_1, S_0) \quad (\text{resp. } (S_0, S_1, S_0, S_0)).$$

On the other hand, we can define a uniserial $\hat{\Lambda}_{2,\hat{c}}$ -module M_w with $w = \gamma\beta\alpha$ (resp. $w = \alpha\gamma\beta$) with such descending composition factors. Hence we obtain in a similar way to (16) and (17) that

$$(18) \quad \pi_\Lambda \left(\gamma\beta\alpha - c_3 \alpha\gamma\beta \text{ modulo } \hat{I}_{2,\hat{c}} \right) = 0,$$

$$(19) \quad \pi_\Lambda \left(\alpha\gamma\beta - c_4 \gamma\beta\alpha \text{ modulo } \hat{I}_{2,\hat{c}} \right) = 0$$

for certain $c_3, c_4 \in k$. Moreover, since the radical series length of the projective $\Lambda'_{2,c}$ -module cover of S_0 is 4, we can argue as above that c_3 and c_4 have to be non-zero. Therefore, there exist certain $c_1, c_3 \in k^*$ such that $\text{Ker}(\pi_\Lambda)$ contains the image \hat{J}_{c_1, c_3} in $\hat{\Lambda}_{2,\hat{c}}$ of the ideal

$$J_{c_1, c_3} = \langle \alpha^2, \eta\beta, \gamma\eta, \beta\gamma, \beta\alpha\gamma - c_1 \eta^{2^{d-2}}, \gamma\beta\alpha - c_3 \alpha\gamma\beta \rangle \subset kQ_2.$$

We can compute the k -dimension of $kQ_2/I_{2,c} = \Lambda_{2,c}$ (resp. of $kQ_2/J_{c_1,c_3} \cong \hat{\Lambda}_{2,\hat{c}}/\hat{J}_{c_1,c_3}$) by counting k -linearly independent paths in kQ_2 modulo $I_{2,c}$ (resp. modulo J_{c_1,c_3}). Since these k -dimensions are equal, it follows that $\text{Ker}(\pi_\Lambda) = \hat{J}_{c_1,c_3}$ and thus $\Lambda'_{2,c} \cong kQ_2/J_{c_1,c_3}$. Because $\Lambda'_{2,c}$ is symmetric, we can argue in the same way as for $\Lambda'_{1,c}$ that $c_3 = 1$. Hence $\Lambda'_{2,c} \cong kQ_2/J_{c_1,1}$ for a certain $c_1 \in k^*$. Since the k -algebra homomorphism

$$\begin{aligned} \Lambda_{2,0} = kQ_2/I_{2,0} &\longrightarrow kQ_2/J_{c_1,1} \\ z \text{ modulo } I_{2,0} &\mapsto z \text{ modulo } J_{c_1,1} \quad \text{for } z \in \{1_0, 1_1, \alpha, \beta, \eta\}, \\ \gamma \text{ modulo } I_{2,0} &\mapsto c_1^{-1} \gamma \text{ modulo } J_{c_1,1} \end{aligned}$$

is a k -algebra isomorphism for all $c_1 \in k^*$, it follows that $\Lambda'_{2,c} \cong \Lambda_{2,0}$. Therefore, the parameter c must be equal to zero. \square

Corollary 3. *Let k be an algebraically closed field of characteristic 2, and let $d \geq 3$ be an integer. Suppose \hat{G} is a finite group with generalized quaternion Sylow 2-subgroups of order 2^{d+1} such that \hat{G} has no non-trivial normal subgroups of odd order. Let $Z(\hat{G})$ denote the center of \hat{G} and define $G = \hat{G}/Z(\hat{G})$. Then G has dihedral Sylow 2-subgroups of order 2^d . If the principal block B of kG has precisely two isomorphism classes of simple modules, then B is Morita equivalent to either $\Lambda_{1,0}$ or $\Lambda_{2,0}$.*

Proof. By a result of Brauer and Suzuki [5], the center of \hat{G} is generated by an element of order 2. Since the principal block B of kG is contained in the image of the principal block \hat{B} of $k\hat{G}$ under the natural projection $\pi : k\hat{G} \rightarrow kG$, Corollary 3 follows from Theorem 2. \square

Corollary 4. *Let k be an algebraically closed field of characteristic 2. Suppose q is an odd prime power and B is the principal block of $k\text{PGL}_2(\mathbb{F}_q)$. Further, let $2^d \geq 8$ be the maximal 2-power dividing $(q^2 - 1)$. Then B is Morita equivalent to $\Lambda_{1,0}$ (resp. $\Lambda_{2,0}$) if $q \equiv 1 \pmod{4}$ (resp. $q \equiv 3 \pmod{4}$).*

Proof. By a result of Dickson (see for example [10, Hauptsatz II.8.27]), the projective special linear group $\text{PSL}_2(\mathbb{F}_{q^2})$ contains a subgroup H_q that is isomorphic to $\text{PGL}_2(\mathbb{F}_q)$. Consider the short exact sequence of groups

$$(20) \quad 1 \rightarrow \{\pm I\} \rightarrow \text{SL}_2(\mathbb{F}_{q^2}) \xrightarrow{\sigma} \text{PSL}_2(\mathbb{F}_{q^2}) \rightarrow 1,$$

where I is the 2×2 identity matrix in $\text{SL}_2(\mathbb{F}_{q^2})$ and σ is the natural surjection. Define \hat{G}_q to be the full preimage in $\text{SL}_2(\mathbb{F}_{q^2})$ of H_q under the map σ . Then $\hat{G}_q/\{\pm I\}$ is isomorphic to $\text{PGL}_2(\mathbb{F}_q)$. Moreover, the center of \hat{G}_q is $\{\pm I\}$, since the center of $H_q \cong \text{PGL}_2(\mathbb{F}_q)$ is trivial. The order of \hat{G}_q is twice the order of $\text{PGL}_2(\mathbb{F}_q)$ and thus equal to $2 \cdot q(q^2 - 1)$. Since the order of $\text{SL}_2(\mathbb{F}_{q^2})$ is equal to $q^2(q^2 - 1)(q^2 + 1)$, it follows that the Sylow 2-subgroups of $\text{SL}_2(\mathbb{F}_{q^2})$ and the Sylow 2-subgroups of \hat{G}_q have order 2^{d+1} . Because $\text{SL}_2(\mathbb{F}_{q^2})$ has generalized quaternion Sylow 2-subgroups (see for example [10, Satz II.8.10]), it follows that the Sylow 2-subgroups of \hat{G}_q are also generalized quaternion. Hence Corollary 4 is a consequence of Corollary 3. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, IOWA 52242-1419
E-mail address: fbleher@math.uiowa.edu