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INSTANTON HOMOLOGY AND THE ALEXANDER POLYNOMIAL

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Abstract. We prove that the instanton knot homology $KHI(K)$ as defined by Kronheimer-Mrowka recovers the Alexander polynomial for knots K in the 3-sphere.

1. Introduction

In a recent paper [8], Kronheimer and Mrowka revisit an instanton knot homology first defined and studied by A. Floer [4]. For simplicity we assume the case of knots K in the 3-sphere. Kronheimer and Mrowka refined the instanton knot homology theory by introducing a bigrading on the groups. The bigraded groups are denoted by $KHI(K)$. They conjecture that $KHI(K)$ is isomorphic to other knot homologies, such as the one defined by Ozsvath and Szabo. A small step in this direction is to show that $KHI(K)$ recovers the Alexander polynomial.[1] In this note we prove this. In addition we state two simple corollaries of the result.

1.1. **Instanton homology for knots.** We review the version of instanton Floer homology for an oriented knot K in the 3-sphere S^3, as considered in [8]. It is defined to be the instanton homology of a certain closure K^T of the knot complement $S^3 - \nu K^\circ$ by $(T - D^\circ) \times S^1$. Here T is a 2-torus and D is a small disk in T. Let Σ_K be a Seifert surface for K, oriented so that the boundary, with the induced orientation, matches the orientation of K. In the closure, the boundaries of $\Sigma_K \cap (S^3 - \nu K^\circ)$ and $(T - D^\circ) \times \{1\}$ are identified with each other, as well as the meridians for K on $\partial\nu(K)$ and the fibers $\{\mathrm{pt}\} \times S^1$ on the boundary of $(T - D^\circ) \times S^1$. An orientation of the 3-manifold is required in the definition of instanton homology. K^T is given the orientation induced from $S^3 - \nu K^\circ$, and S^3 is given the standard orientation on $\mathbb{R}^3$, regarding it as $\mathbb{R}^3 \cup \{\infty\}$.

To define instanton homology in the non-homology 3-sphere case we also need to specify a complex line bundle $\omega \to K^T$ whose Chern class has a non-zero mod 2 evaluation on at least one integral homology class in K^T. Let α be a homologically non-trivial oriented simple closed curve in T. Choose ω_0 to be the complex line with Chern class Poincaré dual to α, thinking of α as living on $(T - D^\circ) \times \{1\}$ in K^T. $I_*(K^T)_{\omega_0}$ denotes the instanton homology for (K^T, ω_0) with complex coefficients.

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[1]Concurrently, the main result of this paper was also obtained by Kronheimer and Mrowka in [9].

1.2. **Generalized eigenspace decomposition.** There is only a relative mod 8 grading on $I_*(K^T)_{\omega_0}$, but there is an absolute mod 2 grading due to Froyshov [5] (details below). We shall always assume the canonical mod 2 grading, and any mod 8 grading used will be assumed to be consistent with the mod 2 grading. This makes the groups

$$\widetilde{I}_0(K^T)_{\omega_0} = \bigoplus I_{2i}(K^T)_{\omega_0}, \quad \widetilde{I}_1(K^T)_{\omega_0} = \bigoplus I_{2i-1}(K^T)_{\omega_0}$$

well-defined.

Let x_0 be a point in K^T. The action of the μ-map evaluated at x_0, $\mu(x_0)$, sends $I_*(K^T)_{\omega_0}$ to $I_{*-4}(K^T)_{\omega_0}$. By [8] this determines a splitting of $\widetilde{I}_*(K^T)_{\omega_0}$ into ± 2-eigenspaces, in their normalization. Then by definition

$$KHI_*(K) = \widetilde{I}_*^+(K^T)_{\omega_0} = +2\text{-eigenspace}.$$

Let $\widehat{\Sigma}$ denote the surface $(\Sigma_K \cap (S^3 - \nu K^\circ)) \cup (T - D^\circ) \times \{1\}$, with the induced orientation from Σ_K. The action of the μ-map evaluated on $\widehat{\Sigma}$, $\mu(\widehat{\Sigma})$, sends $I_*(K^T)_{\omega_0}$ to $I_{*-2}(K^T)_{\omega_0}$. The actions of $\mu(\widehat{\Sigma})$ and $\mu(x_0)$ commute; $\widetilde{I}_*^+(K^T)_{\omega_0}$ is preserved under the action of $\mu(\widehat{\Sigma})$. In [8] it is shown (with the normalization used there) that the eigenvalues of $\mu(\widehat{\Sigma})$ are the even integers n satisfying the bound $|n| \leq 2g-2$, g being the genus of $\widehat{\Sigma}$. Then there is a decomposition of $KHI_*(K)$ by the generalized eigenspaces of $\mu(\widehat{\Sigma})$, which in the notation of [8] is written

$$KHI_*(K) = \bigoplus_{i=-g}^{i=g} KHI_*(K, i). \tag{1.1}$$

Define the finite Laurent polynomial

$$P_K(t) = P_{\omega_0}(K^T, \widehat{\Sigma})(t) = \chi_{-g}t^{-g} + \cdots + \chi_0 + \chi_1 t + \cdots + \chi_g t^g,$$

where χ_i is the Euler characteristic of $KHI_*(K, i)$.

1.3. **Statement of results.**

Theorem 1.1. *For any knot K in the 3-sphere, $P_K(t)$ is exactly the symmetrized and normalized Alexander polynomial $\Delta_K(t)$ of K.*

This proves Conjecture 7.26 of [8]. The symmetrized and normalized Alexander polynomial $\Delta_K(t)$ of K is the unique representative that satisfies $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$. The proof of the theorem involves a straightforward application of Floer's surgery exact triangle.

Instead of the closure of the knot complement by $(T - D^\circ) \times S^1$ we can consider the standard closure by $D^2 \times S^1$ given by 0-surgery on K, which we denote by K^D. We consider the instanton homology of K^D but this time choose the complex line ω' with Chern class Poincaré dual to an oriented meridian of K, regarding the meridian as a curve in K^D.

Corollary 1.2. *For an oriented K, let $Q_K(t)$ be the finite Laurent polynomial defined analogously to $P_K(t)$ but with K^D and the complex line ω' instead. Then*

$$Q_K(t) = \frac{\Delta_K(t) - 1}{t - 2 + t^{-1}}$$

where $\Delta_K(t)$ is as above. In particular $Q_K(1) = \frac{1}{2}\Delta_K''(1)$.

For example the unknot has $Q_K(t) = 0$, the trefoil knot has $Q_K(t) = 1$ and the figure-8 knot has $Q_K(t) = -1$. We mention that for $I_*(K^D)_{\omega'}$ we do not actually know that $\mu(x_0)$ splits $I_*(K^D)_{\omega'}$ into ± 2-eigenspaces. However this is not needed, and we can simply use the generalized $+2$-eigenspace in place of the $+2$-eigenspace.

Corollary 1.3. *If the symmetrized and normalized Alexander polynomial $\Delta_K(t)$ of a knot K in S^3 is non-trivial, i.e. $\Delta_K(t) \neq 1$, then ± 1-surgery on K never yields a simply connected 3-manifold.*

Remark 1.4. This corollary is a special case of the property P conjecture. (Property P is proven in [7] and [8], independently of Perelman's proof of the Poincaré conjecture. It uses a result in [2] that states that consideration of ± 1-surgery is sufficient.)

Proof. If $\Delta_K(t) \neq 1$, then $Q_K(t) \neq 0$ and the instanton homology groups $I_*(K^D)_{\omega'}$ are non-trivial. By Floer's surgery exact triangle the instanton homology for the integral homology sphere $K_{\pm 1}$ obtained by ± 1-surgery on K is also non-trivial. Thus there must be at least one non-trivial representation of $\pi_1(K_{\pm 1})$ into $SU(2)$. □

2. Preliminaries

2.1. **Mod 2 grading.** We briefly review the canonical mod 2 grading since we will be using it throughout this note (see [5] and also [3, Sect. 6.5]). Let $[\varrho] \in I_*(Y)_\omega$. Suppose that $Y = \partial X$ as oriented manifolds and that $E \to X$ is a $U(2)$-bundle with connection A that extends ϱ on Y.

Let $\widehat{X}$ be the cylindrical-end manifold obtained by joining the semi-infinite tube $Y \times [0, \infty)$ to the boundary. Likewise, extend E to $\widehat{E}$ and also extend A to $\widehat{A}$ by the pullback of ϱ over the cylindrical end. We let $\mathrm{Ind}E$ be the index of the anti-self-dual operator on $\widehat{X}$ coupled to $\widehat{A}$. We also have indices $\mathrm{Ind}^{\pm}X$ of the anti-self-dual operator on forms on $\widehat{X}$, on the positive/negative δ-weighted spaces where the weight is non-zero and sufficiently small in absolute value [3, Sect. 3.3.1].

Define the mod 2 grading of $[\varrho]$ to be

$$\nu[\varrho] = \mathrm{Ind}E - 3\mathrm{Ind}^{-}X \bmod 2.$$

Index calculations (see for instance [3, Sect. 3.3.1]) show that $\mathrm{Ind}^{-}X = b_1(X) - b_2^{+}(X)$. (Here we assume that Y is connected.) b_2^{+} is the dimension of a maximal positive definite subspace for the (possibly degenerate) intersection form on the image of $H_2(X)$ in $H_2(X, \partial X)$.

Let us now suppose that W is a cobordism between Y and Y' that induces a map $I_W \colon I_*(Y)_\omega \to I_{*+k}(Y')_{\omega'}$. We wish to determine the value of $k \bmod 2$, the mod 2 degree of I_W.

Lemma 2.1. *The degree k of the map I_W above satisfies*

$$k = 3(b_1(W) - b_1(Y) + b_0(Y') - b_0(W) - b_2^{+}(W)) \bmod 2.$$

Remark 2.2. If Y or Y' is disconnected the lemma is still valid. We need however to interpret the instanton homology of a disjoint union $I_*(Y_0 \cup Y_1)_{\omega\omega'}$ as the tensor product $I_*(Y_0)_\omega \otimes I_*(Y_1)_{\omega'}$. In particular the grading satisfies $\nu([\varrho] \otimes [\varrho']) = \nu[\varrho] + \nu[\varrho'] \bmod 2$.

Proof. Let $[\varrho] \in I_*(Y)_\omega$ and $I_W([\varrho]) = [\varrho'] \in I_{*+k}(Y')_{\omega'}$. Let E_W be the $U(2)$-bundle with connection that limits to ϱ and ϱ' at the ends. Then by assumption $\mathrm{Ind}E_W = 0$. Let $Y = \partial X$ and E be as above. The additivity property of the index [3, Sect. 3.3] tells us that

$$\begin{aligned} \mathrm{Ind}(E \cup E_W) &= \mathrm{Ind}E + \mathrm{Ind}E_W, \\ \mathrm{Ind}^-(X \cup W) &= \mathrm{Ind}^+ X + \mathrm{Ind}^- W. \end{aligned}$$

On the other hand, according to [3, Prop. 3.10],

$$\mathrm{Ind}^+ X - \mathrm{Ind}^- X = -(b_0(\partial X) + b_1(\partial X))$$

and, by [3, Prop. 3.15] (with additional terms added for a non-connected boundary),

$$\mathrm{Ind}^- W = b_1(W) - (b_0(W) - b_0(\partial W)) - b_2^+(W).$$

Since the difference $\nu[\varrho] - \nu[\varrho']$ is given by

$$\mathrm{Ind}(E \cup E_W) - \mathrm{Ind}E - 3(\mathrm{Ind}^-(X \cup W) - \mathrm{Ind}^- X) \bmod 2,$$

the result follows. □

2.2. **Instanton invariants for 2-component links.** We will need to introduce versions of instanton homology associated to an oriented 2-component link L in the 3-sphere. The definitions are parallel to those for knots. They are defined as the instanton homology of certain closures of the link complement. Let $C = [0, 1] \times S^1$. Let D_1 and D_2 be two small disjoint disks in T and set $T_2 = T - D_1^\circ - D_2^\circ$. Let Σ_L be an oriented (connected) Seifert surface with induced orientation on the boundary equal to the orientation on L. Set

$$\begin{aligned} L^C &= (S^3 - \nu L^\circ) \cup (C \times S^1), \\ L^{T_2} &= (S^3 - \nu L^\circ) \cup (T_2 \times S^1). \end{aligned}$$

Again the union is taken along the boundaries such that the oriented boundary of $\Sigma_L \cap (S^3 - \nu L^\circ)$ matches up with the oriented boundary of $C \times \{1\}$ or $T_2 \times \{1\}$. Additionally the meridians of the link should match up with the fibers $\{\mathrm{pt}\} \times S^1$ on the boundaries. In the case of L^C the genus of $\widehat{\Sigma}$, the extension of Σ_L, is one greater than that of Σ_L, but in the case of L^{T_2} it is two greater.

As in the case of knots we can take the instanton homology groups $I_*^+(L^C)_\omega$ and $I_*^+(L^{T_2})_\omega$ as the basis for the definition of instanton link homology. We shall define our choices $\omega = \omega_1, \omega_2$ below. Using the action of $\mu(\widehat{\Sigma})$ to give a decomposition in terms of $\widetilde{I}_*^+(L^C)_{\omega_1}$ and $\widetilde{I}_*^+(L^{T_2})_{\omega_2}$, we can again define the finite Laurent polynomials

$$P_{\omega_1}(L^C, \widehat{\Sigma})(t) \quad \text{and} \quad P_{\omega_2}(L^{T_2}, \widehat{\Sigma})(t).$$

Let α_0 be an oriented simple arc in $S^3 - \nu L^\circ$ connecting the two boundary components. Let α_1 be an oriented simple arc in $C \times S^1$ connecting the two boundary components such that the boundary of α_1 matches up with the boundary of α_0 and the union is an oriented simple closed curve α'. Then in the case of L^C choose $\omega = \omega_1$ where ω_1 has Chern class Poincaré dual to α'.

Think of T_2 as the connected sum of surfaces $C \sharp T$. Let α'' be an oriented simple closed curve of the form $\alpha_0 \cup \alpha_1$ where α_1 is the arc as before but living in the C factor of $C \sharp T$. Let α be a homologically non-trivial simple closed curve in the T factor of the connected sum. We think of α as living on $T_2 \times \{1\}$. Then for L^{T_2} choose $\omega = \omega_2$ where ω_2 has Chern class Poincaré dual to $\alpha'' + \alpha$.

2.3. **The excision principle and instanton homology for L^{T_2}.** With our choices for ω, $P_{\omega_2}(L^{T_2}, \widehat{\Sigma})(t)$ is actually determined by $P_{\omega_1}(L^C, \widehat{\Sigma})(t)$. This is a consequence of Floer's excision principle.

Lemma 2.3. *Let Σ_2 be the surface of genus 2. The finite Laurent polynomial is such that*

$$P_{\omega_2}(L^{T_2}, \widehat{\Sigma})(t) = P_{\omega_1}(L^C, \widehat{\Sigma})(t) \cdot P_{\omega_3}(\Sigma_2 \times S^1, \Sigma_2)(t),$$

where $P_{\omega_3}(\Sigma_2 \times S^1, \Sigma_2)(t)$ is the finite Laurent polynomial for $\Sigma_2 \times S^1$ derived from the instanton homology of $\Sigma_2 \times S^1$ and defined in an analogous manner. The complex line ω_3 has Chern class the Poincaré dual of the curve α above.

Proof. Let F_1 and F_2 be the two tori that form the boundary of $(S^3 - \nu L^\circ) \subset L^{T_2}$. Since ω_2 evaluates non-trivally mod 2 on F_i we may apply Floer's excision principle (see [8, Theorem 7.7]) to conclude that there is an isomorphism

$$\widetilde{I}^+_*(L^{T_2})_{\omega_2} \cong \widetilde{I}^+_*(\Sigma_2 \times S^1)_{\omega_3} \otimes \widetilde{I}^+_*(L^C)_{\omega_1}.$$

However it is not immediately evident that this isomorphism preserves the mod 2 grading; we shall prove this below but only in this specific situation.

In the above isomorphism the action of $\mu(\widehat{\Sigma})$ on $\widetilde{I}^+_*(L^{T_2})_{\omega_2}$ corresponds to the action of $\mu(\widehat{\Sigma}) \otimes 1 + 1 \otimes \mu(\Sigma_2)$ on the tensor product space. It follows that the generalized λ-eigenspaces W_λ in $\widetilde{I}^+_*(L^{T_2})_{\omega_2}$ obey a relation of the form

$$W_\lambda \cong \bigoplus_{\lambda = \lambda_0 + \lambda_1} U_{\lambda_0} \otimes V_{\lambda_1},$$

where U_{λ_0} and V_{λ_1} are corresponding generalized eigenspaces in $\widetilde{I}^+_*(\Sigma_2 \times S^1)_{\omega_3}$ and $\widetilde{I}^+_*(L^C)_{\omega_1}$ respectively. The lemma follows easily, assuming the mod 2 grading claim.

To establish the mod 2 grading, consider the surgery cobordism W between $L^C \cup (\Sigma_2 \times S^1)$ and L^{T_2} in the proof of the excision principle. Let $H = T \times [-1,1] \times [-1,1]$. The boundary is $T \times \{-1,1\} \times [-1,1] \cup T \times [-1,1] \times \{-1,1\}$. We regard $T \times \{0\} \times [-1,1]$ as the 'core' of H. Then W is obtained from $(L^C \cup (\Sigma_2 \times S^1)) \times [0,1]$ by identifying $\nu F_1 \cup \nu F_2$ in $(L^C \cup (\Sigma_2 \times S^1)) \times \{1\}$ with $T \times [-1,1] \times \{-1,1\}$ in H. Clearly W deformation retracts onto W_0, the union of $L^C \cup (\Sigma_2 \times S^1)$ and the core of H, where F_1 and F_2 are identified with $\partial(T \times \{0\} \times [-1,1])$.

In L^C let a be an oriented simple closed curve corresponding to $\{0\} \times S^1 \times \{1\} \subset C \times S^1$. Let b be an oriented simple closed curve corresponding to $\{\mathrm{pt}\} \times S^1 \subset C \times S^1$. Then thinking of a and b as homology classes, we have $a = 0$ in $H_1(L^C)$ and that b is a generator of $H_1(L^C)$. On the other hand, by the identification via the core of H, a is a generator of $H_1(\Sigma_2 \times \{\mathrm{pt}\})$ and b corresponds to a fiber $\{\mathrm{pt}\} \times S^1$.

Since $H_1(L^C) \cong \mathbb{Z} \oplus \mathbb{Z}$ we have that the rank of $H_1(\Sigma_2 \times S^1) \oplus H_1(L^C)$ is 7. Then $H_1(W) = H_1(W_0)$ is obtained from $H_1(\Sigma_2 \times S^1) \oplus H_1(L^C)$ by introducing the two relations above. Thus $H_1(W_0)$ has rank 5. It is easily seen that $b_2^+(W) = 0$; thus by Lemma 2.1 the dimension shift is

$$\begin{aligned} k &= b_1(W) - b_1(Y) + b_0(Y') - b_0(W) - b_2^+(W) \\ &= 5 - (2+5) + 1 - 1 - 0 \equiv 0 \bmod 2. \end{aligned}$$

This completes the proof. $\square$

2.4. **Some calculations.** Lemma 2.3 above necessitates evaluation of $P_{\omega_3}(\Sigma_2 \times S^1, \Sigma_2)(t)$. For later use we will also need the evaluation of $P_{\omega_4}(T \times S^1)(t)$ where ω_4 has Chern class dual to any homologically non-trivial oriented simple closed curve in T.

Lemma 2.4. *$P_{\omega_4}(T \times S^1)(t) = 1$ for either orientation on $T \times S^1$.*

Proof. After applying a diffeomorphism, the line bundle ω_4 can be assumed to have Chern class Poincaré dual to the fiber $\{\mathrm{pt}\} \times S^1$. It is well-known (see for instance [1, Prop. 1.14]) that the $SO(3)$-bundle with 2nd Steifel-Whitney class $w_2 \equiv c_1(\omega_4) \bmod 2$ carries a unique flat connection, up to gauge equivalence. In terms of $U(2)$-bundles this gives two flat connections ϱ_0 and ϱ_1 on the adjoint bundle. The instanton homology is such that $I_*(T \times S^1)_{\omega_4} \cong \mathbb{C} \oplus \mathbb{C}$, where the dimensions differ by 4. The rest of the lemma follows easily once we can show that the generators lie in even dimensions.

Let $X = T \times D^2$ so that $Y = \partial X$. Let $E_{\varrho_i} \to X$ be a $U(2)$-bundle carrying connection A that extends ϱ_i. Then $\mathrm{Ind} E_{\varrho_1} = \mathrm{Ind} E_{\varrho_0} + 4 \bmod 8$ since the Floer dimensions differ by 4. We wish to evaluate $\mathrm{Ind} E_{\varrho_0}$. Let $\varphi\colon T \times S^1 \to T \times S^1$ be the orientation reversing diffeomorphism that reverses the S^1 factor; denote by $\tilde{\varphi}$ a lift to the bundle level. Then $\tilde{\varphi}^*(\varrho_0)$ is equivalent in instanton homology to either ϱ_0 or ϱ_1. Now form the double of X, identifying the boundaries via φ, and let $E = E_{\varrho_0} \cup E_{\tilde{\varphi}^*(\varrho_0)}$ be the corresponding bundle over the double. There are two ways of forming E. We choose the one that has c_1 dual to the surface $\{y_0\} \times D^2 \cup \{y_0\} \times D^2$ in the double. Then

$$\mathrm{Ind} E = 2(4c_2 - c_1^2) - 3(1 - b_1 + b_2^+) \equiv 0 \bmod 8.$$

On the other hand, by the additivity of the index we have

$$\mathrm{Ind} E_{\varrho_0} + \mathrm{Ind} E_{\tilde{\varphi}^*(\varrho_0)} = \mathrm{Ind} E_{\varrho_0} + \mathrm{Ind} E_{\varrho_0} + 4n \equiv 0 \bmod 8.$$

It follows that $\mathrm{Ind} E_{\varrho_0} \equiv 0 \bmod 2$. Clearly $b_1(X) - b_2^+(X) = 2$, and so $\nu[\varrho_0] \equiv 0 \bmod 2$. □

Lemma 2.5. *$P_{\omega_3}(\Sigma_2 \times S^1, \Sigma)(t) = t^{-1} - 2 + t$ for either orientation on $\Sigma_2 \times S^1$.*

Proof. According to [1, Prop. 1.15] we have, in increasing order of dimension,

$$I_*(\Sigma_2 \times S^1)_{\omega_3} = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \oplus \{0\} \oplus \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C} \oplus \{0\}.$$

We shall fix this grading, beginning with zero for the first group and reading from left to right. We assume for the moment that this is consistent with the mod 2 grading. We shall prove this below by showing that the group $\mathbb{C}^2$ in dimension 1 above must be an odd graded group.

The action of $\mu(\Sigma_2)$ has eigenvalues -2, 0 or $+2$ on $\widetilde{I}_*^+(\Sigma_2 \times S^1)_{\omega_3}$. On the odd dimensions $\mu(\Sigma_2)$ necessarily shifts each summand of the odd groups $\mathbb{C}^2 \oplus \{0\} \oplus \mathbb{C}^2 \oplus \{0\}$ to the next. Thus $\mu(\Sigma_2)$ is zero on $\widetilde{I}_1^+(\Sigma_2 \times S^1)_{\omega_3}$ and this makes up its entire generalized eigenspace decomposition. On the other hand, the ± 2-eigenvalues for $\mu(\Sigma_2)$ are simple on $\widetilde{I}_*^+(\Sigma_2 \times S^1)$ [8, Prop. 7.4], so only the ± 2-eigenvalues on $\widetilde{I}_0^+(\Sigma_2 \times S^1)_{\omega_3}$ have non-trivial eigenspaces, each of dimension 1. This completely decomposes $\widetilde{I}_0^+(\Sigma_2 \times S^1)_{\omega_3}$ and the result follows, modulo the claim regarding the mod 2 grading.

Let ϱ be the (perturbed) flat connection that is a generator $I_1(\Sigma_2 \times S^1)_{\omega_3}$. Let $\varphi\colon \Sigma \times S^1 \to \Sigma_2 \times S^1$ be the orientation reversing diffeomorphism that reverses the S^1 factor. Let X be such that $\partial X = Y$ and let $E_\varrho \to X$ be the bundle with connection that extends ϱ. Let $\tilde{\varphi}$ be a lift to a bundle map. It must be the case then that $\tilde{\varphi}^*(\varrho)$ is a generator in dimension 1 or 5, so by the same reasoning as in the preceding lemma, $\mathrm{Ind}E_{\tilde{\varphi}^*(\varrho)} \equiv \mathrm{Ind}E_\varrho \bmod 4$. Repeating the doubling argument, we find that

$$2\mathrm{Ind}E_\varrho \equiv 2(4c_2 - c_1^2) - 3(1 - b_1 + b_2^+) \bmod 4, \tag{2.1}$$

where the right-hand terms refer to the doubled manifold (and bundle). We now need to construct an appropriate X. This is done as follows.

Let H be a handlebody whose boundary is Σ_2. Let a_1, b_1, a_2, b_2 be the standard oriented simple closed curves in Σ_2 that represent a basis for H_1 with the property that a_1 and a_2 bound disks D_1 and D_2 respectively in H. Without loss of generality we may assume that ω_3 has Chern class Poincaré dual to $a_1 \times \{\mathrm{pt}\}$ in $\Sigma_2 \times S^1$. We now choose $X = H \times S^1$. The bundles E_ϱ and $E_{\tilde{\varphi}^*(\varrho)}$ extend over X. For the bundle $E_\varrho \cup E_{\tilde{\varphi}^*(\varrho)}$ over the doubled manifold, we may choose c_1 Poincaré dual to $D_1 \cup D_1$. Thus $c_1^2 = 0$. The second homology of the doubled manifold (over $\mathbb{Z}$) is generated by $S_i = D_i \cup D_i$ and $b_i \times S^1$ $(i = 1, 2)$. The intersection form can then be written as two copies of

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This gives $b_2^+(X \cup X) = 2$. It is seen that $b_1(X \cup X) = 3$. Therefore the left-hand side of (2.1) evaluates to 0 mod 4 so that $\mathrm{Ind}E_\varrho \equiv 0 \bmod 2$.

From the preceding considerations it is also straightforward to see that $b_1(X) = 3$ and $b_2^+(X) = 0$. So

$$\nu[\varrho] = \mathrm{Ind}E_\varrho - 3(b_1(X) - b_2^+(X)) \equiv 1 \bmod 2,$$

and the lemma follows. □

2.5. **Skein relations.** Let J denote an oriented knot or link in the 3-sphere. Let J_+, J_- and J_0 denote in the usual way the knots or links that in a projection differ in a neighborhood of a single crossing. We use the conventions for J_+ and J_- in [6]. If $J_\pm$ is modeled on the x- and y-axes in the plane (the strands agreeing with the standard orientation of the axes), then in J_+ the x-axis strand goes over the y-axis strand.

If $J_\pm$ are knots, then J_0 is a 2-component link. If $J_\pm$ are 2-component links and the crossing is between different components of $J_\pm$, then J_0 is a knot. We have a version of the following well-known theorem that applies only to knots and 2-component links.

Theorem 2.6. *Let J denote either an oriented knot or an oriented 2-component link. Let $\Delta_J(t)$ be a finite Laurent polynomial in powers of $t^{1/2}$ that is an oriented isotopy invariant of J. Assume that (1) $\Delta_J(t) = 1$ for the unknot, (2) $\Delta_J(t) = 0$ for split 2-component links, and (3) the following skein rule holds:*

$$\Delta_{J_+}(t) - \Delta_{J_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{J_0}(t),$$

where if J is a 2-component link, then the crossing change is between different strands. Then for knots K, $\Delta_K(t)$ is exactly the symmetrized and normalized Alexander polynomial of K; i.e. it satisfies $\Delta_K(t^{-1}) = \Delta_K(t)$ and $\Delta_K(1) = 1$.

Proof. See [6, Proof of Theorem 1.5]. □

3. Proofs

3.1. **Theorem 1.1.** The strategy is to show that for oriented knots K, $P_K(t) = P_{\omega_0}(K^T, \widehat{\Sigma})(t)$ satisfies the conditions of Theorem 2.6 upon applying Floer's exact triangle ([1] will be our general reference). However this will require, for oriented 2-component links L, a slight alteration of $P_{\omega_1}(L^C, \widehat{\Sigma})(t)$ before the skein relation can be satisfied.

We let

$$\widehat{P}_J(t) = \begin{cases} P_{\omega_0}(J^T, \widehat{\Sigma})(t) & \text{if } J \text{ is a knot,} \\ -(t^{1/2} - t^{-1/2})P_{\omega_1}(J^C, \widehat{\Sigma})(t) & \text{if } J \text{ is a link.} \end{cases}$$

According to Floer, for oriented knots K there is an exact sequence of the form

$$\longrightarrow \widetilde{I}_i(K_+^T)_{\omega_0} \xrightarrow{a} \widetilde{I}_i(K_-^T)_{\omega_0} \xrightarrow{b} \widetilde{I}_i(K_0^{T_2})_{\omega_2} \xrightarrow{c} \widetilde{I}_{i-1}(K_+^T)_{\omega_0} \longrightarrow$$

where the maps are induced by the various surgery cobordisms, essentially adding 2-handles to the product cobordism along a knot. The maps a and b are of degree zero and c is of degree -1, by applying Lemma 2.1 to the surgery cobordisms W_a, W_b and W_c respectively. W_a and W_b are obtained by adding two handles along homologically trivial knots with -1-framing. Therefore b_1 is the same as the ends of the cobordism. Clearly, for W_a and W_b, $b_2^+ = 0$. By Lemma 2.1 this gives a dimension shift of 0 mod 2. On the other hand, W_c is obtained by adding a 2-handle along a knot which is a generator for 1st homology, so b_1 drops by 1 in W_c, but $b_2^+ = 0$. This results in a dimension shift of -1 mod 2.

Since all the surgery cobordisms are connected, the action of $\mu(x_0)$ in each group commutes with a, b and c. A similar statement is true for the action of $\mu(\widehat{\Sigma})$ on each group, because all the various oriented surfaces $\widehat{\Sigma}$ are homologous in the surgery cobordisms.

Therefore the above exact sequence respects the decomposition of each group into the ± 2-eigenspaces of $\mu(x_0)$ and the generalized eigenspaces for $\mu(\widehat{\Sigma})$ contained therein. Thus

$$\begin{aligned} P_{\omega_0}(K_+^T, \widehat{\Sigma}) - P_{\omega_0}(K_-^T, \widehat{\Sigma}) &= -P_{\omega_2}(K_0^{T_2}, \widehat{\Sigma}) \\ &= -(t^{-1} - 2 + t)P_{\omega_1}(K_0^C, \widehat{\Sigma}), \end{aligned}$$

where the last equality is obtained from Lemmas 2.3 and 2.5. It immediately follows that

$$\widehat{P}_{K_+}(t) - \widehat{P}_{K_-}(t) = (t^{1/2} - t^{-1/2})\widehat{P}_{K_0}(t). \tag{3.1}$$

In the case of a 2-component oriented link L where the crossing change is between different components of the link, we have the exact sequence

$$\longrightarrow \widetilde{I}_i(L_+^C)_{\omega_1} \xrightarrow{a} \widetilde{I}_i(L_-^C)_{\omega_1} \xrightarrow{b} \widetilde{I}_i(L_0^T)_{\omega_0} \xrightarrow{c} \widetilde{I}_{i-1}(L_+^C)_{\omega_1} \longrightarrow .$$

A similar argument gives

$$P_{\omega_1}(L_+^C, \widehat{\Sigma}) - P_{\omega_1}(L_-^C, \widehat{\Sigma}) = -P_{\omega_0}(L_0^T, \widehat{\Sigma}),$$

and again relation (3.1) holds. For split 2-component links L', the instanton homology must vanish because ω_1 is non-trivial on the 2-sphere S that separates the components of the link; there are no flat $SO(3)$-connections over S. Thus $0 = P_{\omega_1}(L')(t) = \widehat{P}_{L'}(t)$. For the unknot U, U^T is clearly $T \times S^1$ so that $\widehat{P}_U(t) = P_{\omega_0}(U^T)(t) = 1$ by Lemma 2.4. This completes the proof. ∎

3.2. **Corollary 1.2.** In our definition of ω_0 (defined for K^T) we could have also twisted it by the complex line that has Chern class Poincaré dual to an oriented fiber $\{\mathrm{pt}\} \times S^1$ in the subset $(T - D^\circ) \times S^1$. We could have made corresponding changes to ω_1 and ω_2 (recall that these are defined for L^C and L^{T_2} respectively), ensuring that these are compatible over the knot or link complement with each other. We will not need to change ω_3 (defined for $\Sigma_2 \times S^1$) but will need a similar change to ω_4 (defined for $T \times S^1$). Denote the changed complex lines by ω_0', ω_1', ω_2' and ω_4' respectively.

Lemma 2.3 remains unchanged with ω_1 and ω_2 replaced by ω_1' and ω_2' respectively. Excision does not require ω_3 to be changed.

Lemma 2.4 remains the same with ω_4 replaced by ω_4'. There is a diffeomorphism that moves ω_4' back to ω_4.

Then the proof of Theorem 1.1 goes through with ω_i replaced by ω_i'. In particular the conclusion about $\widehat{P}_J(t)$, defined using ω_i', remains the same. Therefore we now assume the ω_i's are changed to ω_i''s.

Then, according to Floer, for oriented knots K there is an exact sequence of the form

$$\longrightarrow \widetilde{I}_i(K_+^D)_{\omega'} \xrightarrow{a} \widetilde{I}_i(K_-^D)_{\omega'} \xrightarrow{b} \widetilde{I}_i(K_0^C)_{\omega_1'} \xrightarrow{c} \widetilde{I}_{i-1}(K_+^D)_{\omega'} \longrightarrow .$$

Repeating the argument of the proof of Theorem 1.1, we have

$$Q_{K_+}(t) - Q_{K_-}(t) = -P_{\omega_1'}(K_0^C, \widehat{\Sigma})(t),$$

and therefore

$$(t - 2 + t^{-1})(Q_{K_+}(t) - Q_{K_-}(t)) = (t^{1/2} - t^{-1/2})\Delta_{K_0}(t).$$

Here we make use of the proof of Theorem 1.1 (with the new ω_i'), which identifies $-(t^{1/2} - t^{-1/2})P(K_0^C, \widehat{\Sigma})_{\omega_1'}(t)$ with $\Delta_{K_0}(t)$. Thus there is a universal constant C such that

$$(t - 2 + t^{-1})Q_K(t) + C = \Delta_K(t),$$

where $\Delta_K(t)$ is the symmetrized and normalized Alexander polynomial. By considering the unknot, which has $Q_K(t) = 0$, we must have $C = 1$. The corollary follows immediately. □

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