STABILITY CRITERION FOR CONVOLUTION-DOMINATED INFINITE MATRICES

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Abstract. Let \( \ell^p \) be the space of all \( p \)-summable sequences on \( \mathbb{Z} \). An infinite matrix is said to have \( \ell^p \)-stability if it is bounded and has bounded inverse on \( \ell^p \). In this paper, a practical criterion is established for the \( \ell^p \)-stability of convolution-dominated infinite matrices.

1. Introduction

Let \( C \) be the Gohberg-Baskakov-Sjöstrand class of infinite matrices \( A := (a(j, j'))_{j, j' \in \mathbb{Z}} \) with
\[
\|A\|_C = \sum_{k \in \mathbb{Z}} \sup_{j-j'=k} |a(j, j')| < \infty.
\]
Let \( \ell^p := \ell^p(\mathbb{Z}) \) be the set of all \( p \)-summable sequences on \( \mathbb{Z} \) with the standard norm \( \| \cdot \|_p \). An infinite matrix \( A := (a(j, j'))_{j, j' \in \mathbb{Z}} \in C \) defines a bounded linear operator on \( \ell^p, 1 \leq p \leq \infty \), in the sense that
\[
(1.1) \quad Ac = \left( \sum_{j' \in \mathbb{Z}} a(j, j')c(j') \right)_{j \in \mathbb{Z}},
\]
where \( c = (c(j))_{j \in \mathbb{Z}} \in \ell^p \). Given a summable sequence \( h = (h(j))_{j \in \mathbb{Z}} \in \ell^1 \), define the convolution operator \( C_h \) on \( \ell^p, 1 \leq p \leq \infty \), by
\[
(1.2) \quad C_h : \ell^p \ni (b(j))_{j \in \mathbb{Z}} \mapsto \left( \sum_{k \in \mathbb{Z}} h(j-k)b(k) \right)_{j \in \mathbb{Z}} \in \ell^p.
\]
Observe that the linear operator associated with an infinite matrix \( A \in C \) is dominated by a convolution operator in the sense that
\[
(1.3) \quad \|(Ac)(j)\| \leq (C_h c)(j) := \sum_{j' \in \mathbb{Z}} h(j-j')|c(j')|, \quad j \in \mathbb{Z},
\]
for any sequence \( c = (c(j))_{j \in \mathbb{Z}} \in \ell^p, 1 \leq p \leq \infty \), where \( |c| = (|c(j)|)_{j \in \mathbb{Z}} \) and the sequence \( \sup_{j-j'=k} |a(j, j')|_{k \in \mathbb{Z}} \) can be chosen to be the sequence \( h = (h(j))_{j \in \mathbb{Z}} \) in (1.3). So infinite matrices in the set \( C \) are said to be convolution-dominated.

Convolution-dominated infinite matrices were introduced by Gohberg, Kaashoek, and Woerdeman [12] as a generalization of Toeplitz matrices. They showed that the class \( C \) equipped with the standard matrix multiplication and the above norm \( \| \cdot \|_C \)
is an inverse-closed Banach subalgebra of $\mathcal{B}(lp)$ for $p = 2$. Here $\mathcal{B}(lp)$, $1 \leq p \leq \infty$, is the space of all bounded linear operators on $lp$ with the standard operator norm, and a subalgebra $A$ of a Banach algebra $B$ is said to be inverse-closed if when an operator $T \in A$ has an inverse $T^{-1}$ in $B$, then $T^{-1} \in A$ ([7] [11] [21]). The inverse-closed property for convolution-dominated infinite matrices was rediscovered by Sjöstrand [25] with a completely different proof and an application to a deep theorem about pseudodifferential operators. Recently Shin and Sun [23] generalized Gohberg, Kaashoek and Woerdeman’s result and proved that the class $\mathcal{C}$ is an inverse-closed Banach subalgebra of $\mathcal{B}(lp)$ for any $1 \leq p \leq \infty$. The readers may refer to [5] [10] [20] [23] [25] [27] and the references therein for related results and various generalizations on the inverse-closed property for convolution-dominated infinite matrices.

Convolution-dominated infinite matrices arise and have been used in the study of spline approximation ([8] [9]), wavelets and affine frames ([6] [18]), Gabor frames ([1], [3], [14], [15], [16], [18], [19], [23], [24], [25]) and the references therein. Examples of convolution-dominated infinite matrices include the infinite matrix $(a(j-j'))_{j,j'\in\mathbb{Z}}$ associated with convolution operators and the infinite matrix $(a(j-j')e^{-2\pi\sqrt{-1}j\theta(j-j')})_{j,j'\in\mathbb{Z}}$ associated with twisted convolution operators, where $\theta \in \mathbb{R}$ and the sequence $a = (a(j))_{j\in\mathbb{Z}}$ satisfies $\sum_{j\in\mathbb{Z}}|a(j)| < \infty$ ([1] [14] [19] [27] [29]).

A convolution-dominated infinite matrix $A$ is said to have $lp$-stability if there are two positive constants $C_1$ and $C_2$ such that

$$C_1\|c\|_p \leq \|Ac\|_p \leq C_2\|c\|_p \quad \text{for all } c \in lp.$$  

The $lp$-stability is one of basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators; see [1] [3] [6] [8] [9] [10] [14] [15] [16] [18] [19] [23] [24] [25] [26] [27] [29] and the references therein. Practical criteria for the $lp$-stability of a convolution-dominated infinite matrix will play important roles in the further study of those topics.

However, up to the knowledge of the author, little is known about practical criteria for the $lp$-stability of an infinite matrix. For an infinite matrix $A = (a(j-j'))_{j,j'\in\mathbb{Z}}$ associated with convolution operators, there is a very useful criterion for its $lp$-stability. It states that $A$ has $lp$-stability if and only if the Fourier series $\hat{a}(\xi) := \sum_{j\in\mathbb{Z}}a(j)e^{-ij\xi}$ of the generating sequence $a = (a(j))_{j\in\mathbb{Z}} \in lp$ does not vanish on the real line, i.e.,

$$\hat{a}(\xi) \neq 0 \quad \text{for all } \xi \in \mathbb{R}.$$  

Applying this criterion for the $lp$-stability, one concludes that the spectrum $\sigma_p(C_a)$ of the convolution operator $C_a$ as an operator on $lp$ is independent of $1 \leq p \leq \infty$, i.e.,

$$\sigma_p(C_a) = \sigma_q(C_a) \quad \text{for all } 1 \leq p, q \leq \infty;$$  

see [4] [17] [22] [24] and the references therein for the discussion on spectrum of various convolution operators. Applying the above criterion again, together with the classical Wiener’s lemma ([29]), it follows that the inverse of an $lp$-stable convolution operator $C_a$ is a convolution operator $C_b$ associated with another summable sequence $b$. 


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For a convolution-dominated infinite matrix \( A = (a(j, j'))_{j,j' \in \mathbb{Z}} \), a popular sufficient condition for its \( \ell^1 \)-stability and \( \ell^\infty \)-stability is that \( A \) is diagonal-dominated, i.e.,

\[
(1.7) \quad \inf_{j \in \mathbb{Z}} \left( |a(j, j)| - \max \left( \sum_{j' \neq j} |a(j, j')|, \sum_{j' \neq j} |a(j', j)| \right) \right) > 0.
\]

In this paper, we provide a practical criterion for the \( \ell^p \)-stability of convolution-dominated infinite matrices. We show that a convolution-dominated infinite matrix \( A \) has \( \ell^p \)-stability if and only if it has certain “diagonal-blocks-dominated” property (see Theorem 2.1 for the precise statement).

2. Main theorem

To state our criterion for the \( \ell^p \)-stability of convolution-dominated infinite matrices, we introduce two concepts. Given an infinite matrix \( A \), define the truncation matrices \( A_s, s \geq 0 \), by

\[
A_s = (a(i, j)\chi_{(-s, s)}(i - j))_{i,j \in \mathbb{Z}},
\]

where \( \chi_E \) is the characteristic function on a set \( E \). Given \( y \in \mathbb{R} \) and \( 1 \leq N \in \mathbb{Z} \), define the operator \( \chi^N_y \) on \( \ell^p \) by

\[
\chi^N_y : \ell^p \ni (c(j))_{j \in \mathbb{Z}} \longmapsto (c(j)\chi_{(-N,N)}(j - y))_{j \in \mathbb{Z}} \in \ell^p.
\]

The operator \( \chi^N_y \) is a diagonal matrix \( \text{diag}(\chi_{(-N,N)}(j - y))_{j \in \mathbb{Z}} \).

**Theorem 2.1.** Let \( 1 \leq p \leq \infty \), and let \( A \) be a convolution-dominated infinite matrix in the class \( \mathcal{C} \). Then the following statements are equivalent:

(i) The infinite matrix \( A \) has \( \ell^p \)-stability.

(ii) There exist a positive constant \( C_0 \) and a positive integer \( N_0 \) such that

\[
(2.1) \quad \|\chi^N_{-n} A \chi^n_{-n} c\|_p \geq C_0 \|\chi^N_{-n} c\|_p, \quad c \in \ell^p,
\]

hold for all integers \( N \geq N_0 \) and \( n \in \mathbb{Z} \).

(iii) There exist a positive integer \( N_0 \) and a positive constant \( \alpha \) satisfying

\[
(2.2) \quad \alpha > 2(5 + 2^{1-p})^{1/p} \inf_{0 \leq s \leq N_0} (\|A - A_s\| + \frac{8}{N_0} \|A\|_c)
\]

such that

\[
(2.3) \quad \|\chi^{N_0}_{-n} A \chi^{N_0}_{-n} c\|_p \geq \alpha \|\chi^{N_0}_{-n} c\|_p, \quad c \in \ell^p,
\]

hold for all \( n \in N_0 \mathbb{Z} \).

Taking \( N_0 = 1 \) in (2.2) and (2.3), we obtain a sufficient condition \( (2.4) \), which is a strong version of the diagonal-domination condition \( (1.7) \), for the \( \ell^\infty \)-stability of a convolution-dominated infinite matrix.

**Corollary 2.2.** Let \( A = (a(j, j'))_{j,j' \in \mathbb{Z}} \) be a convolution-dominated infinite matrix in the class \( \mathcal{C} \). If

\[
(2.4) \quad \inf_{j \in \mathbb{Z}} \left( |a(j, j)| - 2 \sum_{0 \neq k \in \mathbb{Z}} \sup_{j' = k} |a(j, j')| \right) > 0,
\]

then \( A \) has \( \ell^\infty \)-stability.
We say that an infinite matrix $A = (a(i,j))_{i,j \in \mathbb{Z}}$ is a band matrix if $a(i,j) = 0$ for all $i,j \in \mathbb{Z}$ satisfying $j > i + k$ or $j < i - k$. The quantity $2k+1$ is the bandwidth of the matrix $A$. For a band matrix $A$ with bandwidth $2k+1$, $A - A_s$ is the zero matrix if $s > k$. Therefore for $N > k$,
\[
\inf_{0 \leq s \leq N} \left( \|A - A_s\|_C + \frac{s}{N} \|A\|_C \right) \leq \frac{k}{N} \|A\|_C.
\]
This, together with Theorem 2.1, gives the following sufficient condition for a band matrix to have $\ell^p$-stability.

**Corollary 2.3.** Let $1 \leq p \leq \infty$, and let $A$ be a convolution-dominated band matrix in the class $\mathcal{C}$ with bandwidth $2k+1$. If there exists an integer $N_0 > k$ such that
\[
\|A\chi_{N_0} c\|_p \geq \gamma \|\chi_{N_0} c\|_p, \quad c \in \ell^p,
\]
holds for some constant $\gamma$ strictly larger than $2(5 + 2^{1-p})^{1/p}k\|A\|_C/N_0$, then $A$ has $\ell^p$-stability.

If we further assume that the infinite matrix $A$ in Corollary 2.3 has the form $A = (a(j-j'))_{j,j' \in \mathbb{Z}}$ for some finite sequence $a = (a(j))_{j \in \mathbb{Z}}$ satisfying $a(j) = 0$ for $|j| > k$, then $\|A\|_C = \sum_{|j| \leq k} |a(j)|$ and the condition (2.5) can be reformulated as follows:
\[
\|A\|_C \geq \gamma \kappa_{N_0} \left( \sum_{|j| \leq k} |a(j)| \right), \quad c \in \mathbb{R}^{2N_0+1},
\]
holds for some $\gamma > 2(5 + 2^{1-p})^{1/p}$, where
\[
\hat{A}_{N_0} = (a(j-j'))_{-N_0 - k \leq j \leq N_0 + k, -N_0 \leq j' \leq N_0}
\]
and
\[
\|c\|_p = \left\{ \begin{array}{ll}
(\sum_{j=-k_1}^{k_2} |c(j)|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\sup_{-k_1 \leq j \leq k_2} |c(j)| & \text{if } p = \infty,
\end{array} \right.
\]
for $c = (c(-k_1), \ldots, c(0), \ldots, c(k_2))^T \in \mathbb{R}^{k_1 + k_2 + 1}$. As a conclusion from (2.6) and (2.7), we see that if $A = (a(j-j'))_{j,j' \in \mathbb{Z}}$ does not have $\ell^p$-stability, then for any large integer $N$,
\[
\inf_{0 \neq c \in \mathbb{R}^{2N+1}} \frac{\|\hat{A}_N c\|_p}{\|c\|_p} \leq \frac{2(5 + 2^{1-p})^{1/p} k}{N} \left( \sum_{|j| \leq k} |a(j)| \right).
\]
For the special case $p = 2$, the above inequality (2.8) can be interpreted as the minimal eigenvalue of $(\hat{A}_N)^T \hat{A}_N$ is less than or equal to $\frac{\sqrt{2}k}{N^2} (\sum_{|j| \leq k} |a(j)|)^2$, and it can also be rewritten as
\[
\inf_{0 \neq P_N \in \Pi_N} \left( \frac{\int_{-\pi}^{\pi} |\hat{a}(\xi)|^2 |P_N(\xi)|^2 d\xi}{\int_{-\pi}^{\pi} |P_N(\xi)|^2 d\xi} \right)^{1/2} \leq \frac{\sqrt{2}k}{N} \left( \sum_{|j| \leq k} |a(j)| \right),
\]
where $\hat{a}(\xi) = \sum_{j \in \mathbb{Z}} a(j)e^{-ij\xi}$ and $\Pi_N$ is the set of all trigonometrical polynomials of degree at most $N$.

If the sequence $a = (a(j))_{j \in \mathbb{Z}}$ satisfies $a(0) = 1, a(-1) = -1$, and $a(j) = 0$ otherwise, then the bandwidth of the infinite matrix $A = (a(j-j'))_{j,j' \in \mathbb{Z}}$ is equal to 3, the norm $\|A\|_C$ of the associated infinite matrix $A$ is equal to 2,
\( \tilde{A}_N = \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}, \)

and

\[
\inf_{0 \neq c \in \mathbb{R}^{2N+1}} \frac{\| \tilde{A}_N c \|_p}{\|c\|_p} \geq \frac{1}{N+1},
\]

where the last inequality holds since the matrix

\[
\tilde{B}_N := \begin{pmatrix} -1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ -1 & -1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ -1 & -1 & -1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \ldots & -1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}
\]

is a left inverse of the matrix \( \tilde{A}_N \). Therefore the order \( N^{-1} \) in (2.8) cannot be improved in general, but the author believes that the bound constant \( 2(5+2^{1-p})^{1/p} \) in (2.2) and (2.8) is not optimal and could be improved.

3. Proof

We say that a discrete subset \( \Lambda \) of \( \mathbb{R}^d \) is relatively-separated if

\[
R(\Lambda) := \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} \chi_{[-1/2,1/2]^d}(x) < \infty
\]

(\cite{1, 23, 27}). Clearly, the set \( \mathbb{Z} \) of all integers is a relatively-separated subset of \( \mathbb{R} \) with

\[
R(\mathbb{Z}) = 1.
\]

Given a discrete set \( \Lambda \), let \( \ell^p(\Lambda) \) be the set of all \( p \)-summable sequences on the set \( \Lambda \) with standard norm \( \| \cdot \|_{\ell^p(\Lambda)} \) or \( \| \cdot \|_p \) for brevity.

Given two relatively-separated subsets \( \Lambda \) and \( \Lambda' \) of \( \mathbb{R}^d \), define

\[
C(\Lambda, \Lambda') = \left\{ A := (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'} \mid \|A\|_{C(\Lambda, \Lambda')} < \infty \right\},
\]

where

\[
\|A\|_{C(\Lambda, \Lambda')} = \sum_{k \in \mathbb{Z}^d} \sup_{\lambda \in \Lambda, \lambda' \in \Lambda'} |a(\lambda, \lambda')| \chi_{k+[-1/2,1/2]^d}(\lambda - \lambda').
\]

It is obvious that

\[
C(\mathbb{Z}, \mathbb{Z}) = C.
\]
Given an infinite matrix \( A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda'} \), define its truncation matrices \( A_s, s \geq 0 \), by

\[
A_s = \left( a(\lambda, \lambda') \chi_{(-s,s)^d}(\lambda - \lambda') \right)_{\lambda, \lambda' \in \Lambda'}.
\]

For any \( y \in \mathbb{R}^d \) and a positive integer \( N \), define the operator \( \chi_N^y \) on \( \ell^p(\Lambda) \) by

\[
(\chi_N^y)_N = \mathcal{O} \subset \ell^p(\Lambda) \quad \text{for all } c \in \ell^p(\Lambda'),
\]

where \( \mathcal{O} \) are relatively-separated subsets of \( \mathbb{R}^d \).

In this section, we establish the following criterion for the \( \ell^p \)-stability of infinite matrices in the class \( C(\Lambda, \Lambda') \), which is a slight generalization of Theorem 2.1 by (3.2) and (3.3).

**Theorem 3.1.** Let \( 1 \leq p \leq \infty \), the subsets \( \Lambda, \Lambda' \) of \( \mathbb{R}^d \) be relatively-separated, and the infinite matrix \( A \) belong to \( C(\Lambda, \Lambda') \). Then the following statements are equivalent to each other:

(i) The infinite matrix \( A \) has \( \ell^p \)-stability, i.e., there exist positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 \| c \|_{\ell^p(\Lambda')} \leq \| Ac \|_{\ell^p(\Lambda)} \leq C_2 \| c \|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda').
\]

(ii) There exist a positive constant \( C_0 \) and a positive integer \( N_0 \) such that

\[
\| \chi_{N_0}^c A_n c \|_{\ell^p(\Lambda)} \geq C_0 \| \chi_{N_0}^c c \|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'),
\]

where \( N_0 \leq N \in \mathbb{Z} \) and \( n \in N\mathbb{Z}^d \).

(iii) There exist a positive integer \( N_0 \) and a positive constant \( \alpha \) satisfying

\[
\alpha > 2(5 + 2^{1-p})^{d/p} R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \inf_{0 \leq d \lesssim N_0} \left( \| A - A_s \|_{C(\Lambda, \Lambda')} + \frac{ds}{N_0} \| A \|_{C(\Lambda, \Lambda')} \right)
\]

such that

\[
\| \chi_{N_0}^{2N_0} A_{n_0} c \|_{\ell^p(\Lambda)} \geq \alpha \| \chi_{N_0}^{N_0} c \|_{\ell^p(\Lambda')}
\]

hold for all \( c \in \ell^p(\Lambda') \) and \( n \in N_0\mathbb{Z} \).

Using the above theorem, we obtain the following equivalence of \( \ell^p \)-stability for infinite matrices having certain off-diagonal decay, which is established in [2, 28, 23] for \( \gamma > d(1+p) \), \( \gamma > 0 \), and \( \gamma \geq 0 \) respectively.

**Corollary 3.2.** Let \( \Lambda, \Lambda' \) be relatively-separated subsets of \( \mathbb{R}^d \), and let \( A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda'} \) satisfy

\[
\| A \|_{c(\Lambda, \Lambda')} = \sum_{k \in \mathbb{Z}^d} (1 + |k|)\gamma \sup_{\lambda, \lambda' \in \Lambda'} |a(\lambda, \lambda')| \chi_{[k+[1/2, 1/2]^d]}(\lambda - \lambda') < \infty,
\]

where \( \gamma > 0 \). Then the \( \ell^p \)-stability of the infinite matrix \( A \) are equivalent to each other for different \( 1 \leq p \leq \infty \).

**Proof.** Let \( 1 \leq p \leq \infty \) and let \( A \) have \( \ell^p \)-stability. Then by Theorem 3.1, there exists a positive constant \( C_0 \) and a positive integer \( N_0 \) such that

\[
\| \chi_{N_0}^{2N_0} A_{n_0} c \|_{\ell^p(\Lambda)} \geq C_0 \| \chi_{N_0}^{N_0} c \|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'),
\]

where \( N_0 \leq N \in \mathbb{Z} \) and \( n \in N\mathbb{Z}^d \). From the equivalence of different norms on a finite-dimensional space, we have that

\[
((2N)^d R(\Lambda))^{\min(1/\gamma - 1/p, 0)} \| \chi_{N_0}^{N_0} c \|_{\ell^p(\Lambda)} \leq \| \chi_{N_0}^{N_0} c \|_{\ell^p(\Lambda)} \leq ((2N)^d R(\Lambda))^{\max(1/\gamma - 1/p, 0)} \| \chi_{N_0}^{N_0} c \|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'),
\]
where $1 \leq p, q \leq \infty$, $1 \leq N \in \mathbb{Z}$ and $n \in N\mathbb{Z}^d$ \((\ref{23})\). Therefore for $1 \leq q \leq \infty$, 
\[
\|\chi_n^{2N}A\chi_n^{N}c\|_{\ell^q(\Lambda)} \leq C_0(2N)^{-d[1/p-1/q]}R(\Lambda')^{\min(1/p-1/q, 0)}
\times R(\Lambda)^{-\max(1/p-1/q, 0)}\|\chi_n^{N}c\|_{\ell^q(\Lambda')} \quad \text{for all} \quad c \in \ell^q(\Lambda'),
\]
where $N_0 \leq N \in \mathbb{Z}$ and $n \in N\mathbb{Z}^d$. We notice that 
\[
\inf_{0 \leq s \leq N} (\|A - A_s\|_{\ell^q(\Lambda, \Lambda')} + \frac{ds}{N}\|A\|_{\ell^q(\Lambda, \Lambda')}) \leq \|A\|_{\ell^q(\Lambda, \Lambda')} \inf_{0 \leq s \leq N} (s^3 + \frac{ds}{N}) \leq (d + 1)\|A\|_{\ell^q(\Lambda)} N^{-\gamma/(1+\gamma)}.
\]
Thus for $1 \leq q \leq \infty$ with $d[1/p-1/q] < \gamma/(1+\gamma)$, it follows from \((\ref{3.14})\) and \((\ref{3.11})\)
that there exists a sufficiently large integer $N_0$ such that 
\[
\|\chi_n^{2N}A\chi_n^{N}c\|_{\ell^q(\Lambda)} \geq \alpha\|\chi_n^Nc\|_{\ell^q(\Lambda')}
\]
hold for all $c \in \ell^p(\Lambda')$, $N \geq N_0$ and $n \in N\mathbb{Z}^d$, where $\alpha$ is a positive constant larger than $2(5+2^{1-q})d/q(R(\Lambda')^{1/p}R(\Lambda')^{1-1/q})\inf_{0 \leq s \leq N_0} (\|A - A_s\|_{\ell^q(\Lambda, \Lambda')} + \frac{ds}{N}\|A\|_{\ell^q(\Lambda, \Lambda')}).$
Then by Theorem \ref{3.3} the infinite matrix $A$ has $\ell^p$-stability for all $1 \leq q \leq \infty$ with $d[1/q-1/p] < \gamma/(1+\gamma)$. Applying the above trick repeatedly, we prove the $\ell^p$-stability of the infinite matrix $A$ for any $1 \leq q \leq \infty$. 

To prove Theorem \ref{3.3}, we first recall some basic properties for infinite matrices $A$ in the class $C(\Lambda, \Lambda')$ and its truncation matrices $A_s, s \geq 0$.

Lemma \ref{3.3} \((\ref{23})\). Let $1 \leq p \leq \infty$, the subsets $\Lambda, \Lambda'$ of $\mathbb{R}^d$ be relatively-separated, $A$ be an infinite matrix in the class $C(\Lambda, \Lambda')$, and $A_s, s \geq 0$, be the truncation matrices of $A$. Then 
\[
\|Ac\|_{\ell^p(\Lambda)} \leq R(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\|A\|_{\ell^p(\Lambda, \Lambda')}\|c\|_{\ell^p(\Lambda')} \quad \text{for all} \quad c \in \ell^p(\Lambda'),
\]
\[
\lim_{s \to +\infty} \|A - A_s\|_{\ell^p(\Lambda, \Lambda')} = 0,
\]
\[
\lim_{N \to +\infty} \inf_{0 \leq s \leq N} (\|A - A_s\|_{\ell^p(\Lambda, \Lambda')} + \frac{ds}{N}\|A\|_{\ell^p(\Lambda, \Lambda')}) = 0,
\]
and 
\[
\|A_s\|c \leq \|A\|c \quad \text{for all} \quad s \geq 0.
\]

Let $\psi_0(x_1, \ldots, x_d) = \prod_{i=1}^d \max(\min(2 - 2|x_i|, 1), 0)$ be a cut-off function on $\mathbb{R}^d$. Then 
\[
0 \leq \chi_{[-1/2, 1/2]^d}(x) \leq \psi_0(x) \leq \chi_{[-1, 1]^d}(x) \leq 1 \quad \text{for all} \quad x \in \mathbb{R}^d,
\]
and 
\[
|\psi_0(x) - \psi_0(y)| \leq 2d\|x - y\|_{\infty} \quad \text{for all} \quad x, y \in \mathbb{R},
\]
where $\|x\|_{\infty} = \max_{1 \leq i \leq d}|x_i|$ for $x = (x_1, \ldots, x_d)$. Define the multiplication operator $\Psi_N^\Lambda$ on $\ell^p(\Lambda)$ by 
\[
\Psi_N^\Lambda : \ell^p(\Lambda) \ni (c(\lambda))_{\lambda \in \Lambda} \mapsto \left(\psi_0\left(\frac{\lambda-n}{N}\right)c(\lambda)\right)_{\lambda \in \Lambda} \in \ell^p(\Lambda).
\]
Applying \((\ref{3.17})\) and \((\ref{5.13})\) for the cut-off function $\psi_0$, we obtain the following properties for the multiplication operators $\Psi_N^\Lambda, n \in N\mathbb{Z}$. 

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Lemma 3.4. Let $1 \leq N \in \mathbb{Z}$, $A$ be a relatively-separated subset of $\mathbb{R}^d$, and the multiplication operators $\Psi^N_n, n \in N\mathbb{Z}^d$, be as in (3.19). Then
\begin{equation}
\|\Psi^N_n c\|_{\ell^p(A)} \leq \|\chi^N_n c\|_{\ell^p(A)} \quad \text{for all } c \in \ell^p(A),
\tag{3.20}
\end{equation}
where $1 \leq p \leq \infty$,
\begin{equation}
\|c\|_{\ell^p(A)} \leq \left( \sum_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^p(A)}^p \right)^{1/p} \leq 2^{d/p} \|c\|_{\ell^p(A)} \quad \text{for all } c \in \ell^p(A),
\tag{3.21}
\end{equation}
\begin{equation}
\left(\sum_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^p(A)}^p \right)^{1/p} \leq (5+2^{1-p})^{d/p} \|c\|_{\ell^p(A)} \quad \text{for all } c \in \ell^p(A),
\tag{3.22}
\end{equation}
where $1 \leq p < \infty$, and
\begin{equation}
\|c\|_{\ell^\infty(A)} = \sup_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^\infty(A)} = \inf_{0 \leq s \leq N} \left( \|A_N - A_s\|_{\ell^\infty(A')} + \frac{2ds}{N} \|A_s\|_{\ell^\infty(A')} \right).
\tag{3.23}
\end{equation}

To prove Theorem 2.1, we also need the following result.

Lemma 3.5 (23). Let $N \geq 1$, the subsets $\Lambda, \Lambda'$ of $\mathbb{R}^d$ be relatively-separated, $A$ be an infinite matrix in the class $C(\Lambda, \Lambda')$, $A_N$ be the truncation matrix of $A$, and $\Psi^N_n, n \in N\mathbb{Z}^d$, be the multiplication operators in (3.19). Then
\begin{equation}
\|\Psi^N_N A_N - A_N \Psi^N_N\|_{C(\Lambda, \Lambda')} \leq \inf_{0 \leq s \leq N} \left( \|A_N - A_s\|_{C(\Lambda, \Lambda')} + \frac{2ds}{N} \|A_s\|_{C(\Lambda, \Lambda')} \right).
\tag{3.24}
\end{equation}

Now we start to prove Theorem 3.1.

Proof of Theorem 3.1. (i)⇒(ii): By the $\ell^p$-stability of the infinite matrix $A$, there exists a positive constant $C_0$ (independent of $n \in N\mathbb{Z}^d$ and $1 \leq N \in \mathbb{Z}$) such that
\begin{equation}
\|A\chi^N_n c\|_{\ell^p(A)} \geq C_0 \|\chi^N_n c\|_{\ell^p(A)} \quad \text{for all } c \in \ell^p(A'),
\tag{3.25}
\end{equation}
where $n \in N\mathbb{Z}^d$ and $N \geq 1$. Noting that
\begin{equation}
\chi^N_n A_N \Psi^N_n = A_N \Psi^N_n
\tag{3.26}
\end{equation}
and applying (3.13) yield
\begin{equation}
\|A\chi^N_n c - \chi^N_n A_N \chi^N_n c\|_{\ell^p(A)} = \|(I - \chi^N_n)(A - A_N)\chi^N_n c\|_{\ell^p(A)} \leq R(A)^{1/p} R(A')^{1-1/p} \|A - A_N\|_{C(\Lambda, \Lambda')} \|\chi^N_n c\|_{\ell^p(A')},
\tag{3.27}
\end{equation}
where $I$ is the identity operator. Combining the estimates in (3.25) and (3.27) proves that
\begin{equation}
\|\chi^N_n A_N \Psi^N_n c\|_{\ell^p(A)} \geq \left( C_0 - R(A)^{1/p} R(A')^{1-1/p} \|A - A_N\|_{C(\Lambda, \Lambda')} \right) \|\chi^N_n c\|_{\ell^p(A')},
\tag{3.28}
\end{equation}
hold for all $c \in \ell^p(A')$, where $n \in N\mathbb{Z}^d$ and $N \geq 1$. The conclusion (ii) then follows from (3.14) and (3.28).

(ii)⇒(iii): The implication follows from (3.15).

(iii)⇒(i): Let $1 \leq p < \infty$. Take any $n \in N_0\mathbb{Z}^d$ and $c \in \ell^p(A')$. By the assumption (iii) for the infinite matrix $A$,
\begin{equation}
\|\chi^N_n A_N \Psi^N_n c\|_{\ell^p(A)} = \|\chi^N_n A_N \Psi^N_n c\|_{\ell^p(A')} \geq \alpha \|\Psi^N_n c\|_{\ell^p(A')}.
\tag{3.29}
\end{equation}
This together with (3.13) and (3.26) implies that

$$
\| A N_0 \Psi_n^N c \|_{\ell^p(\Lambda)}
= \| \chi_n^{2N_0} (A N_0 - A + A) \Psi_n^N c \|_{\ell^p(\Lambda)}
\geq \| \chi_n^{2N_0} A N_0 \Psi_n^N c \|_{\ell^p(\Lambda)} - \| \chi_n^{2N_0} (A N_0 - A) \Psi_n^N c \|_{\ell^p(\Lambda)}
\geq (\alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}) \| A - A_{N_0} \|_{C(\Lambda, \Lambda')} \| \Psi_n^N c \|_{\ell^p(\Lambda')}.
$$

From (3.13) and (3.24) it follows that

$$
\| (\Psi_n^N A N_0 - A N_0 \Psi_n^N) c \|_{\ell^p(\Lambda)}
= \| (\Psi_n^N A N_0 - A N_0 \Psi_n^N) \Psi_n^N c \|_{\ell^p(\Lambda)}
\leq R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \| \Psi_n^N A N_0 - A N_0 \Psi_n^N \|_{C(\Lambda, \Lambda')} \| (\Psi_n^N c) \|_{\ell^p(\Lambda')}
\leq R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}
\times \inf_{0 \leq s \leq N_0} \left( \| A N_0 - A_s \|_{C(\Lambda, \Lambda')} + \frac{2d_s}{N_0} \| A N_0 \|_{C(\Lambda, \Lambda')} \right) \| (\Psi_n^N c) \|_{\ell^p(\Lambda')}.
$$

Combining (3.21), (3.22), (3.30) and (3.31), we get

$$
2^{d/p} \| A N_0 c \|_{\ell^p(\Lambda)} \geq \left( \sum_{n \in N_0 Z} \| \Psi_n^N A N_0 c \|_{\ell^p(\Lambda)}^p \right)^{1/p}
\geq \left( \alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \| A - A_{N_0} \|_{C(\Lambda, \Lambda')} \left( \sum_{n \in N_0 Z} \| \Psi_n^N c \|_{\ell^p(\Lambda')}^p \right)^{1/p}
- R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N_0} \left( \| A N_0 - A_s \|_{C(\Lambda, \Lambda')} + \frac{2d_s}{N_0} \| A N_0 \|_{C(\Lambda, \Lambda')} \right)
\times \left( \sum_{n \in N_0 Z} \| \Psi_n^N c \|_{\ell^p(\Lambda')}^p \right)^{1/p}
\geq \left( \alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \| A - A_{N_0} \|_{C(\Lambda, \Lambda')} - (5 + 2^{1-p})^{1/p} R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}
\times \inf_{0 \leq s \leq N_0} \left( \| A N_0 - A_s \|_{C(\Lambda, \Lambda')} + \frac{2d_s}{N_0} \| A N_0 \|_{C(\Lambda, \Lambda')} \right) \right) \| c \|_{\ell^p(\Lambda')}.
$$

Therefore

$$
\| A c \|_{\ell^p(\Lambda)} \geq \| A N_0 c \|_{\ell^p(\Lambda)} - \| (A - A_{N_0}) c \|_{\ell^p(\Lambda)} \geq 2^{-1/p} \left( \alpha - (1 + 2^{d/p}) R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \| A - A_{N_0} \|_{C(\Lambda, \Lambda')} - (5 + 2^{1-p})^{1/p} R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}
\times \inf_{0 \leq s \leq N_0} \left( \| A N_0 - A_s \|_{C(\Lambda, \Lambda')} + \frac{2d_s}{N_0} \| A N_0 \|_{C(\Lambda, \Lambda')} \right) \right) \| c \|_{\ell^p(\Lambda')} \geq 2^{-d/p} \left( \alpha - 2(5 + 2^{1-p})^{1/p} R(\Lambda)^{1/p}
\times R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N_0} \left( \| A - A_s \|_{C(\Lambda, \Lambda')} + \frac{d_s}{N_0} \| A \|_{C(\Lambda, \Lambda')} \right) \right) \| c \|_{\ell^p(\Lambda')},
$$

and the conclusion (i) for $1 \leq p < \infty$ follows.

The conclusion (i) for $p = \infty$ can be proved by a similar argument. We omit the details here. \qed
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