STABILITY CRITERION FOR CONVOLUTION-DOMINATED INFINITE MATRICES

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Abstract. Let $\ell^p$ be the space of all $p$-summable sequences on $\mathbb{Z}$. An infinite matrix is said to have $\ell^p$-stability if it is bounded and has bounded inverse on $\ell^p$. In this paper, a practical criterion is established for the $\ell^p$-stability of convolution-dominated infinite matrices.

1. Introduction

Let $\mathcal{C}$ be the Gohberg-Baskakov-Sjöstrand class of infinite matrices $A := (a(j,j'))_{j,j' \in \mathbb{Z}}$ with

$$\|A\|_C = \sum_{k \in \mathbb{Z}} \sup_{j - j' = k} |a(j,j')| < \infty.$$ 

Let $\ell^p := \ell^p(\mathbb{Z})$ be the set of all $p$-summable sequences on $\mathbb{Z}$ with the standard norm $\| \cdot \|_p$. An infinite matrix $A := (a(j,j'))_{j,j' \in \mathbb{Z}} \in \mathcal{C}$ defines a bounded linear operator on $\ell^p, 1 \leq p \leq \infty$, in the sense that

$$(1.1) \quad Ac = \left( \sum_{j' \in \mathbb{Z}} a(j,j') c(j') \right)_{j \in \mathbb{Z}},$$

where $c = (c(j))_{j \in \mathbb{Z}} \in \ell^p$. Given a summable sequence $h = (h(j))_{j \in \mathbb{Z}} \in \ell^1$, define the convolution operator $C_h$ on $\ell^p, 1 \leq p \leq \infty$, by

$$(1.2) \quad C_h : \ell^p \ni (b(j))_{j \in \mathbb{Z}} \mapsto \left( \sum_{k \in \mathbb{Z}} h(j-k)b(k) \right)_{j \in \mathbb{Z}} \in \ell^p.$$ 

Observe that the linear operator associated with an infinite matrix $A \in \mathcal{C}$ is dominated by a convolution operator in the sense that

$$(1.3) \quad \|(Ac)(j)\| \leq (C_h|c|(j) := \sum_{j' \in \mathbb{Z}} h(j-j')|c(j')|, \quad j \in \mathbb{Z},$$

for any sequence $c = (c(j))_{j \in \mathbb{Z}} \in \ell^p, 1 \leq p \leq \infty$, where $|c| = (|c(j)|)_{j \in \mathbb{Z}}$ and the sequence $(\sup_{j - j' = k} |a(j,j')|)_{k \in \mathbb{Z}}$ can be chosen to be the sequence $h = (h(j))_{j \in \mathbb{Z}}$ in (1.2). So infinite matrices in the set $\mathcal{C}$ are said to be convolution-dominated.

Convolution-dominated infinite matrices were introduced by Gohberg, Kaashoek, and Woerdeman [12] as a generalization of Toeplitz matrices. They showed that the class $\mathcal{C}$ equipped with the standard matrix multiplication and the above norm $\| \cdot \|_C$
is an inverse-closed Banach subalgebra of $\mathcal{B}(\ell^p)$ for $p = 2$. Here $\mathcal{B}(\ell^p)$, $1 \leq p \leq \infty$, is the space of all bounded linear operators on $\ell^p$ with the standard operator norm, and a subalgebra $\mathcal{A}$ of a Banach algebra $\mathcal{B}$ is said to be inverse-closed if when an operator $T \in \mathcal{A}$ has an inverse $T^{-1}$ in $\mathcal{B}$, then $T^{-1} \in \mathcal{A}$ ([17][11][21]). The inverse-closed property for convolution-dominated infinite matrices was rediscovered by Sjöstrand [25] with a completely different proof and an application to a deep theorem about pseudodifferential operators. Recently Shin and Sun [23] generalized Gohberg, Kaashoek and Woerdeman’s result and proved that the class of convolution-dominated infinite matrices will play important roles in the further study of those topics.

Convolutedominated infinite matrices arise and have been used in the study of spline approximation ([8][9]), wavelets and affine frames ([6][18]), Gabor frames and non-uniform sampling ([3][14][15][26]), and pseudo-differential operators ([13][16][24][25] and the references therein). Examples of convolution-dominated infinite matrices include the infinite matrix $(a(\xi - \eta))_{\xi,\eta \in \mathbb{Z}}$ associated with convolution operators and the infinite matrix $(a(j - j')e^{-2\pi \sqrt{-1}\theta j' (j-j')})_{j,j' \in \mathbb{Z}}$ associated with twisted convolution operators, where $\theta \in \mathbb{R}$ and the sequence $a = (a(j))_{j \in \mathbb{Z}}$ satisfies $\sum_{j \in \mathbb{Z}} |a(j)| < \infty$ ([1][14][19][27][29]).

A convolution-dominated infinite matrix $A$ is said to have $\ell^p$-stability if there are two positive constants $C_1$ and $C_2$ such that

$$C_1 \|c\|_p \leq \|Ac\|_p \leq C_2 \|c\|_p \quad \text{for all} \quad c \in \ell^p.$$  

The $\ell^p$-stability is one of basic assumptions for infinite matrices arising in the study of spline approximation, Gabor time-frequency analysis, nonuniform sampling, and algebra of pseudo-differential operators; see [1][3][6][8][9][10][14][13][16][18][19][23][24][25][26][27][29] and the references therein. Practical criteria for the $\ell^p$-stability of a convolution-dominated infinite matrix will play important roles in the further study of those topics.

However, up to the knowledge of the author, little is known about practical criteria for the $\ell^p$-stability of an infinite matrix. For an infinite matrix $A = (a(j - j'))_{j,j' \in \mathbb{Z}}$ associated with convolution operators, there is a very useful criterion for its $\ell^p$-stability. It states that $A$ has $\ell^p$-stability if and only if the Fourier series $\hat{a}(\xi) := \sum_{j \in \mathbb{Z}} a(j)e^{-ij\xi}$ of the generating sequence $a = (a(j))_{j \in \mathbb{Z}} \in \ell^p$ does not vanish on the real line, i.e.,

$$\hat{a}(\xi) \neq 0 \quad \text{for all} \quad \xi \in \mathbb{R}.$$  

Applying this criterion for the $\ell^p$-stability, one concludes that the spectrum $\sigma_p(C_a)$ of the convolution operator $C_a$ as an operator on $\ell^p$ is independent of $1 \leq p \leq \infty$, i.e.,

$$\sigma_p(C_a) = \sigma_q(C_a) \quad \text{for all} \quad 1 \leq p, q \leq \infty;$$  

see [4][17][22][23] and the references therein for the discussion on spectrum of various convolution operators. Applying the above criterion again, together with the classical Wiener’s lemma ([29]), it follows that the inverse of an $\ell^p$-stable convolution operator $C_a$ is a convolution operator $C_b$ associated with another summable sequence $b$. 
For a convolution-dominated infinite matrix $A = (a(j, j'))_{j, j' \in \mathbb{Z}}$, a popular sufficient condition for its $\ell^1$-stability and $\ell^\infty$-stability is that $A$ is diagonal-dominated, i.e.,

$$
(1.7) \quad \inf_{j \in \mathbb{Z}} \left( \sum_{j' \neq j} |a(j, j')|, \sum_{j' \neq j} |a(j', j)| \right) > 0.
$$

In this paper, we provide a practical criterion for the $\ell^p$-stability of convolution-dominated infinite matrices. We show that a convolution-dominated infinite matrix $A$ has $\ell^p$-stability if and only if it has certain “diagonal-blocks-dominated” property (see Theorem 2.1 for the precise statement).

2. MAIN THEOREM

To state our criterion for the $\ell^p$-stability of convolution-dominated infinite matrices, we introduce two concepts. Given an infinite matrix $A$, define the truncation matrices $A_s$, $s \geq 0$, by

$$
A_s = (a(i, j)\chi(-s, s)(i - j))_{i, j \in \mathbb{Z}},
$$

where $\chi_E$ is the characteristic function on a set $E$. Given $y \in \mathbb{R}$ and $1 \leq N \in \mathbb{Z}$, define the operator $\chi_y^N$ on $\ell^p$ by

$$
\chi_y^N: \ell^p \ni (c(j))_{j \in \mathbb{Z}} \mapsto (c(j)\chi_{(-N,N)}(j - y))_{j \in \mathbb{Z}} \in \ell^p.
$$

The operator $\chi_y^N$ is a diagonal matrix $\text{diag}(\chi_{(-N,N)}(j - y))_{j \in \mathbb{Z}}$.

**Theorem 2.1.** Let $1 \leq p \leq \infty$, and let $A$ be a convolution-dominated infinite matrix in the class $\mathcal{C}$. Then the following statements are equivalent:

(i) The infinite matrix $A$ has $\ell^p$-stability.

(ii) There exist a positive constant $C_0$ and a positive integer $N_0$ such that

$$
(2.1) \quad \|\chi_N^N A \chi_N^N c\|_p \geq C_0 \|\chi_N^N c\|_p, \quad c \in \ell^p,
$$

hold for all integers $N \geq N_0$ and $n \in N\mathbb{Z}$.

(iii) There exist a positive integer $N_0$ and a positive constant $\alpha$ satisfying

$$
(2.2) \quad \alpha > 2(5 + 2^{1-p})^{1/p} \inf_{0 \leq s \leq N_0} \left( \|A - A_s\|c + \frac{s}{N_0} \|A\|c \right)
$$

such that

$$
(2.3) \quad \|\chi_N^{N_0} A \chi_N^{N_0} c\|_p \geq \alpha \|\chi_N^{N_0} c\|_p, \quad c \in \ell^p,
$$

hold for all $n \in N_0\mathbb{Z}$.

Taking $N_0 = 1$ in (2.2) and (2.3), we obtain a sufficient condition (2.4), which is a strong version of the diagonal-domination condition (1.7), for the $\ell^\infty$-stability of a convolution-dominated infinite matrix.

**Corollary 2.2.** Let $A = (a(j, j'))_{j, j' \in \mathbb{Z}}$ be a convolution-dominated infinite matrix in the class $\mathcal{C}$. If

$$
(2.4) \quad \inf_{j \in \mathbb{Z}} \left( \max_{j' \neq j} |a(j, j')| - 2 \sup_{0 \neq k \in \mathbb{Z}} \sum_{j' = k} \sum_{j' = k} \sum_{j' = k} |a(j, j')| \right) > 0,
$$

then $A$ has $\ell^\infty$-stability.
We say that an infinite matrix $A = (a(i, j))_{i, j \in \mathbb{Z}}$ is a band matrix if $a(i, j) = 0$ for all $i, j \in \mathbb{Z}$ satisfying $j > i + k$ or $j < i - k$. The quantity $2k + 1$ is the bandwidth of the matrix $A$. For a band matrix $A$ with bandwidth $2k + 1$, $A - A_s$ is the zero matrix if $s > k$. Therefore for $N > k$,

$$
\inf_{0 \leq s \leq N} \left( \|A - A_s\|_c + \frac{s}{N} \|A\|_c \right) \leq \frac{k}{N} \|A\|_c.
$$

This, together with Theorem 2.1, gives the following sufficient condition for a band matrix to have $\ell^p$-stability.

**Corollary 2.3.** Let $1 \leq p \leq \infty$, and let $A$ be a convolution-dominated band matrix in the class $\mathcal{C}$ with bandwidth $2k + 1$. If there exists an integer $N_0 > k$ such that

$$
\|A\chi_n N_0 c\|_p \geq \alpha \|\chi_n N_0 c\|_p, \quad c \in \ell^p,
$$

holds for some constant $\alpha$ strictly larger than $2(5 + 2^{1-p})^{1/p}k\|A\|_C/N_0$, then $A$ has $\ell^p$-stability.

If we further assume that the infinite matrix $A$ in Corollary 2.3 has the form $A = (a(j - j'))_{j, j' \in \mathbb{Z}}$ for some finite sequence $a = (a(j))_{j \in \mathbb{Z}}$ satisfying $a(j) = 0$ for $|j| > k$, then $\|A\|_C = \sum_{|j| \leq k} |a(j)|$ and the condition (2.5) can reformulated as follows:

$$
\|\tilde{A}_{N_0} c\|_p \geq \frac{\gamma k}{N_0} \left( \sum_{|j| \leq k} |a(j)| \right) \|c\|_p, \quad c \in \mathbb{R}^{2N_0 + 1},
$$

holds for some $\gamma > 2(5 + 2^{1-p})^{1/p}$, where

$$
\tilde{A}_{N_0} = (a(j - j'))_{-N_0 - k \leq j \leq N_0 + k, -N_0 \leq j' \leq N_0}
$$

and

$$
\|c\|_p = \left\{ \begin{array}{ll}
(\sum_{j = -k_1}^{k_2} |c(j)|^p)^{1/p} & \text{if } 1 \leq p < \infty, \\
\sup_{-k_1 \leq j \leq k_2} |c(j)| & \text{if } p = \infty,
\end{array} \right.
$$

for $c = (c(-k_1), \ldots, c(0), \ldots, c(k_2))^T \in \mathbb{R}^{k_1 + k_2 + 1}$. As a conclusion from (2.6) and (2.7), we see that if $A = (a(j - j'))_{j, j' \in \mathbb{Z}}$ does not have $\ell^p$-stability, then for any large integer $N$,

$$
\inf_{0 \neq c \in \mathbb{R}^{N+1}} \frac{\|\tilde{A}_N c\|_p}{\|c\|_p} \leq \frac{2(5 + 2^{1-p})^{1/p}k}{N} \left( \sum_{|j| \leq k} |a(j)| \right).
$$

For the special case $p = 2$, the above inequality (2.8) can be interpreted as the minimal eigenvalue of $(\tilde{A}_N)^T \tilde{A}_N$ is less than or equal to $\frac{\sqrt{2\pi k}}{N}^2 (\sum_{|j| \leq k} |a(j)|)^2$, and it can also be rewritten as

$$
\inf_{0 \neq c \in \Pi_N} \frac{\left( \int_{-\pi}^{\pi} |\hat{a}(\xi)|^2 |P_N(\xi)|^2 d\xi \right)^{1/2}}{\left( \int_{-\pi}^{\pi} |P_N(\xi)|^2 d\xi \right)^{1/2}} \leq \frac{\sqrt{2\pi k}}{N} \left( \sum_{|j| \leq k} |a(j)| \right),
$$

where $\hat{a}(\xi) = \sum_{j \in \mathbb{Z}} a(j) e^{-ij\xi}$ and $\Pi_N$ is the set of all trigonometrical polynomials of degree at most $N$.

If the sequence $a = (a(j))_{j \in \mathbb{Z}}$ satisfies $a(0) = 1, a(-1) = -1$, and $a(j) = 0$ otherwise, then the bandwidth of the infinite matrix $A = (a(j - j'))_{j, j' \in \mathbb{Z}}$ is equal to 3, the norm $\|A\|_C$ of the associated infinite matrix $A$ is equal to 2,
\( \tilde{A}_N = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \),

and

\[
\inf_{0 \neq c \in \mathbb{R}^{2N+1}} \frac{\|\tilde{A}_N c\|_p}{\|c\|_p} \geq \frac{1}{N+1},
\]

where the last inequality holds since the matrix

\[
\tilde{B}_N := \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}
\]

is a left inverse of the matrix \( \tilde{A}_N \). Therefore the order \( N^{-1} \) in (2.8) cannot be improved in general, but the author believes that the bound constant \( 2(5+2^{1-p})^{1/p} \) in (2.2) and (2.8) is not optimal and could be improved.

3. Proof

We say that a discrete subset \( \Lambda \) of \( \mathbb{R}^d \) is relatively-separated if

\[
R(\Lambda) := \sup_{x \in \mathbb{R}^d} \sum_{\lambda \in \Lambda} \chi_{\lambda+[-1/2,1/2]^d}(x) < \infty
\]

(12.23 [27]). Clearly, the set \( \mathbb{Z} \) of all integers is a relatively-separated subset of \( \mathbb{R} \) with

\[
R(\mathbb{Z}) = 1.
\]

Given a discrete set \( \Lambda \), let \( \ell^p(\Lambda) \) be the set of all \( p \)-summable sequences on the set \( \Lambda \) with standard norm \( \| \cdot \|_{\ell^p(\Lambda)} \) or \( \| \cdot \|_p \) for brevity.

Given two relatively-separated subsets \( \Lambda \) and \( \Lambda' \) of \( \mathbb{R}^d \), define

\[
C(\Lambda, \Lambda') = \left\{ A := (a(\lambda, \lambda'))_{\lambda \in \Lambda, \lambda' \in \Lambda'} \mid \|A\|_{C(\Lambda, \Lambda')} < \infty \right\},
\]

where

\[
\|A\|_{C(\Lambda, \Lambda')} = \sum_{k \in \mathbb{Z}^d} \sup_{\lambda \in \Lambda, \lambda' \in \Lambda'} |a(\lambda, \lambda')| \chi_{k+[-1/2,1/2]^d}(\lambda - \lambda').
\]

It is obvious that

\[
C(\mathbb{Z}, \mathbb{Z}) = C.
\]
Given an infinite matrix \( A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} \), define its truncation matrices \( A_s, s \geq 0 \), by
\[
A_s = \left( a(\lambda, \lambda') \chi_{(-s,s]}(\lambda - \lambda') \right)_{\lambda, \lambda' \in \Lambda}.
\]

For any \( y \in \mathbb{R}^d \) and a positive integer \( N \), define the operator \( \chi_N^y \) on \( \ell^p(\Lambda) \) by
\[
\chi_N^y : \ell^p(\Lambda) \ni (c(\lambda))_{\lambda \in \Lambda} \mapsto (c(\lambda) \chi_{(-N,N)}(\lambda - y))_{\lambda \in \Lambda} \in \ell^p(\Lambda).
\]

In this section, we establish the following criterion for the \( \ell^p \)-stability of infinite matrices in the class \( \mathcal{C}(\Lambda, \Lambda') \), which is a slight generalization of Theorem 2.1 by [2, 28, 23].

**Theorem 3.1.** Let \( 1 \leq p \leq \infty \), the subsets \( \Lambda, \Lambda' \) of \( \mathbb{R}^d \) be relatively-separated, and the infinite matrix \( A \) belong to \( \mathcal{C}(\Lambda, \Lambda') \). Then the following statements are equivalent to each other:

(i) The infinite matrix \( A \) has \( \ell^p \)-stability, i.e., there exist positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 \|c\|_{\ell^p(\Lambda')} \leq \|Ac\|_{\ell^p(\Lambda)} \leq C_2 \|c\|_{\ell^p(\Lambda')} \quad \text{for all} \ c \in \ell^p(\Lambda').
\]

(ii) There exist a positive constant \( C_0 \) and a positive integer \( N_0 \) such that
\[
\|\chi_n^N A_n^N c\|_{\ell^p(\Lambda)} \geq C_0 \|\chi_n^N c\|_{\ell^p(\Lambda')} \quad \text{for all} \ c \in \ell^p(\Lambda'),
\]
where \( N_0 \leq N \in \mathbb{Z} \) and \( n \in N\mathbb{Z}^d \).

(iii) There exist a positive integer \( N_0 \) and a positive constant \( \alpha \) satisfying
\[
\alpha > 2(5 + 2^{1-p})^d/p R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N_0} \left( \|A - A_s\|_{\mathcal{C}(\Lambda, \Lambda')} + \frac{ds}{N_0} \|A\|_{\mathcal{C}(\Lambda, \Lambda')} \right)
\]
such that
\[
\|\chi_n^{2N_0} A_n^{N_0} c\|_{\ell^p(\Lambda)} \geq \alpha \|\chi_n^{N_0} c\|_{\ell^p(\Lambda')}
\]
hold for all \( c \in \ell^p(\Lambda') \) and \( n \in N_0\mathbb{Z} \).

Using the above theorem, we obtain the following equivalence of \( \ell^p \)-stability for infinite matrices having certain off-diagonal decay, which is established in [2, 28, 23] for \( \gamma > d(d + 1), \gamma > 0 \), and \( \gamma \geq 0 \) respectively.

**Corollary 3.2.** Let \( \Lambda, \Lambda' \) be relatively-separated subsets of \( \mathbb{R}^d \), and let \( A = (a(\lambda, \lambda'))_{\lambda, \lambda' \in \Lambda} \) satisfy
\[
\|A\|_{\mathcal{C}(\Lambda, \Lambda')} = \sum_{k \in \mathbb{Z}^d} (1 + |k|)^{\gamma} \sup_{\lambda, \lambda' \in \Lambda} |a(\lambda, \lambda')| \chi_{k+[-1/2,1/2]^d}(\lambda - \lambda') < \infty,
\]
where \( \gamma > 0 \). Then the \( \ell^p \)-stability of the infinite matrix \( A \) are equivalent to each other for different \( 1 \leq p \leq \infty \).

**Proof.** Let \( 1 \leq p \leq \infty \) and let \( A \) have \( \ell^p \)-stability. Then by Theorem 3.1 there exists a positive constant \( C_0 \) and a positive integer \( N_0 \) such that
\[
\|\chi_n^{2N_0} A_n^{N_0} c\|_{\ell^p(\Lambda)} \geq C_0 \|\chi_n^{N_0} c\|_{\ell^p(\Lambda')} \quad \text{for all} \ c \in \ell^p(\Lambda'),
\]
where \( N_0 \leq N \in \mathbb{Z} \) and \( n \in N\mathbb{Z}^d \). From the equivalence of different norms on a finite-dimensional space, we have that
\[
\left( (2N)^d R(\Lambda)^{\min(1/q-1/p,0)} \right) \|\chi_n^N c\|_{\ell^p(\Lambda)} \leq \|\chi_n^N c\|_{\ell^p(\Lambda)} \leq \left( (2N)^d R(\Lambda)^{\max(1/q-1/p,0)} \right) \|\chi_n^N c\|_{\ell^p(\Lambda)} \quad \text{for all} \ c \in \ell^p(\Lambda),
\]
where $1 \leq p, q \leq \infty, 1 \leq N \in \mathbb{Z}$ and $n \in N\mathbb{Z}^d$. Therefore for $1 \leq q \leq \infty,$

$$\|\chi_n^N A \chi_n^N c\|_{\ell^q(\Lambda)} \geq C_0(2N)^{-d(1/p-1/q)} R(\Lambda')^{\min(1/p-1/q,0)}$$

\times R(\Lambda)^{-\max(1/p-1/q,0)} \|\chi_n^N c\|_{\ell^q(\Lambda')} \quad \text{for all } c \in \ell^q(\Lambda'),$$

(3.10)

where $N_0 \leq N \in \mathbb{Z}$ and $n \in N\mathbb{Z}^d$. We notice that

$$\inf_{0 \leq s \leq N} (\|A - A_s\|_{C(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{C(\Lambda,\Lambda')}) \leq \|A\|_{C(\Lambda,\Lambda')} \inf_{0 \leq s \leq N} (s^q + \frac{ds}{N})$$

\leq (d + 1) \|A\|_{C(\Lambda,\Lambda')} N^{-\gamma/(1+\gamma)}.$$

Thus for $1 \leq q \leq \infty$ with $d[1/p - 1/q] < \gamma/(1+\gamma)$, it follows from (3.10) and (3.11) that there exists a sufficiently large integer $N_0$ such that

$$\|\chi_n^N A \chi_n^N c\|_{\ell^q(\Lambda)} \geq \alpha \|\chi_n^N c\|_{\ell^q(\Lambda')}$$

hold for all $c \in \ell^q(\Lambda'), N \geq N_0$ and $n \in N\mathbb{Z}^d$, where $\alpha$ is a positive constant larger than $2(5 + 2^{-1-q})d/q R(\Lambda')^{1/q} R(\Lambda')^{-1/q} \inf_{0 \leq s \leq N_0} (\|A - A_s\|_{C(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{C(\Lambda,\Lambda')}).$

Then by Theorem 3.1, the infinite matrix $A$ has $\ell^q$-stability for all $1 \leq q \leq \infty$ with $d[1/q - 1/p] < \gamma/(1+\gamma)$. Applying the above trick repeatedly, we prove the $\ell^q$-stability of the infinite matrix $A$ for any $1 \leq q \leq \infty$. □

To prove Theorem 3.1 we first recall some basic properties for infinite matrices $A$ in the class $C(\Lambda,\Lambda')$ and its truncation matrices $A_s, s \geq 0$.

**Lemma 3.3 [23].** Let $1 \leq p \leq \infty$, the subsets $\Lambda, \Lambda'$ of $\mathbb{R}^d$ be relatively-separated, $A$ be an infinite matrix in the class $C(\Lambda,\Lambda')$, and $A_s, s \geq 0$, be the truncation matrices of $A$. Then

$$\|Ac\|_{\ell^p(\Lambda)} \leq R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A\|_{C(\Lambda,\Lambda')} \|c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(\Lambda'),$$

(3.13)

$$\lim_{s \to +\infty} \|A - A_s\|_{C(\Lambda,\Lambda')} = 0,$$

(3.14)

$$\lim_{N \to +\infty} \inf_{0 \leq s \leq N} (\|A - A_s\|_{C(\Lambda,\Lambda')} + \frac{ds}{N} \|A\|_{C(\Lambda,\Lambda')}) = 0,$$

(3.15)

and

$$\|A_s\|c \leq \|A\|c \quad \text{for all } s \geq 0.$$  

(3.16)

Let $\psi_0(x_1, \ldots, x_d) = \prod_{i=1}^d \max(\min(2 - 2|x_i|, 1), 0)$ be a cut-off function on $\mathbb{R}^d$. Then

$$0 \leq \chi_{[-1/2,1/2]}^d(x) \leq \psi_0(x) \leq \chi_{(-1,1)^d}(x) \leq 1 \quad \text{for all } x \in \mathbb{R}^d,$$

(3.17)

and

$$|\psi_0(x) - \psi_0(y)| \leq 2d \|x - y\|_\infty \quad \text{for all } x, y \in \mathbb{R},$$

(3.18)

where $\|x\|_\infty = \max_{1 \leq i \leq d} |x_i|$ for $x = (x_1, \ldots, x_d)$. Define the multiplication operator $\Psi_n^\Lambda$ on $\ell^p(\Lambda)$ by

$$\Psi_n^\Lambda : \ell^p(\Lambda) \ni (c(\lambda))_{\lambda \in \Lambda} \mapsto \left(\psi_0\left(\frac{\lambda - n}{N}\right) c(\lambda)\right)_{\lambda \in \Lambda} \in \ell^p(\Lambda).$$

(3.19)

Applying (3.17) and (3.18) for the cut-off function $\psi_0$, we obtain the following properties for the multiplication operators $\Psi_n^\Lambda, n \in N\mathbb{Z}.$
Lemma 3.4. Let $1 \leq N \in \mathbb{Z}$, $A$ be a relatively-separated subset of $\mathbb{R}^d$, and the multiplication operators $\Psi^N_n, n \in N\mathbb{Z}^d$, be as in (3.19). Then

\begin{equation}
\|\Psi^N_n c\|_{\ell^p(A)} \leq \|\chi^N_n c\|_{\ell^p(A)} \quad \text{for all } c \in \ell^p(A),
\end{equation}

where $1 \leq p \leq \infty$.

\begin{equation}
\|c\|_{\ell^p(A)} \leq \left( \sum_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^p(A)}^p \right)^{1/p} \leq 2^{d/p}\|c\|_{\ell^p(A)} \quad \text{for all } c \in \ell^p(A),
\end{equation}

(3.22)

\begin{equation}
4^{d/p}\|c\|_{\ell^p(A)} \leq \left( \sum_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^p(A)}^p \right)^{1/p} \leq (5 + 2^{1-p})^{d/p}\|c\|_{\ell^p(A)} \quad \text{for all } c \in \ell^p(A),
\end{equation}

where $1 \leq p < \infty$, and

\begin{equation}
\|c\|_{\ell^\infty(A)} = \sup_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^\infty(A)} = \sup_{n \in N\mathbb{Z}^d} \|\Psi^N_n c\|_{\ell^\infty(A)} \quad \text{for all } c \in \ell^\infty(A).
\end{equation}

To prove Theorem 2.1, we also need the following result.

Lemma 3.5. Let $N \geq 1$, the subsets $\Lambda, \Lambda'$ of $\mathbb{R}^d$ be relatively-separated, $A$ be an infinite matrix in the class $C(\Lambda, \Lambda')$, $A_N$ be the truncation matrix of $A$, and $\Psi^N_n, n \in N\mathbb{Z}^d$, be the multiplication operators in (3.19). Then

\begin{equation}
\|\Psi^N_n A_N - A_N \Psi^N_n\|_{C(\Lambda, \Lambda')} \leq \inf_{0 \leq s \leq N} \left( \|A_N - A_s\|_{C(\Lambda, \Lambda')} + \frac{2ds}{N} \|A_s\|_{C(\Lambda, \Lambda')} \right).
\end{equation}

Now we start to prove Theorem 3.1.

Proof of Theorem 3.1

(i) $\implies$ (ii): By the $\ell^p$-stability of the infinite matrix $A$, there exists a positive constant $C_0$ (independent of $n \in N\mathbb{Z}^d$ and $1 \leq N \in \mathbb{Z}$) such that

\begin{equation}
\|A\chi^N_n c\|_{\ell^p(\Lambda')} \geq C_0 \|\chi^N_n c\|_{\ell^p(\Lambda')} \quad \text{for all } c \in \ell^p(A'),
\end{equation}

where $n \in N\mathbb{Z}^d$ and $N \geq 1$. Noting that

\begin{equation}
\chi^N_n A_N \Psi^N_n = A_N \Psi^N_n
\end{equation}

and applying (3.13) yield

\begin{equation}
\|A\chi^N_n c - \chi^N_n A_N \Psi^N_n\|_{\ell^p(\Lambda')} = \|(I - \chi^N_n)(A - A_N)\chi^N_n c\|_{\ell^p(\Lambda')} \leq R(\Lambda)^{1/p}R(\Lambda')^{1-1/p}\|A - A_N\|_{C(\Lambda, \Lambda')} \|\chi^N_n c\|_{\ell^p(\Lambda')}.
\end{equation}

(3.27)

where $I$ is the identity operator. Combining the estimates in (3.25) and (3.27) proves that

\begin{equation}
\|\chi^N_n^{2N} A_N \Psi^N_n c\|_{\ell^p(\Lambda')} \geq (C_0 - R(\Lambda)^{1/p}R(\Lambda')^{1-1/p})\|A - A_N\|_{C(\Lambda, \Lambda')} \|\chi^N_n c\|_{\ell^p(\Lambda')}.
\end{equation}

(3.28)

hold for all $c \in \ell^p(A')$, where $n \in N\mathbb{Z}^d$ and $N \geq 1$. The conclusion (ii) then follows from (3.14) and (3.28).

(ii) $\implies$ (iii): The implication follows from (3.15).

(iii) $\implies$ (i): Let $1 \leq p < \infty$. Take any $n \in N_0\mathbb{Z}^d$ and $c \in \ell^p(A')$. By the assumption (iii) for the infinite matrix $A$,

\begin{equation}
\|\chi^N_n^{2N_0} A \Psi^N_{N_0} c\|_{\ell^p(\Lambda')} = \|\chi^N_n^{2N_0} A_N \Psi^N_{N_0} c\|_{\ell^p(\Lambda')} \geq \alpha \|\Psi^N_{N_0} c\|_{\ell^p(\Lambda')}.
\end{equation}

(3.29)
This together with (3.13) and (3.26) implies that
\[
\|A_{N_0} \Psi_n^{N_0} c\|_{\ell^p(\Lambda)}
\]
\[
= \|\chi_1^{2N_0}(A_{N_0} - A + A) \Psi_n^{N_0} c\|_{\ell^p(\Lambda)}
\]
\[
\geq \|\chi_r^{2N_0} A_{N_0} \Psi_n^{N_0} c\|_{\ell^p(\Lambda)} - \|\chi_r^{2N_0} (A_{N_0} - A) \Psi_n^{N_0} c\|_{\ell^p(\Lambda)}
\]
\[
(3.30) \quad \geq (\alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A - A_{N_0}\|_{\ell^p(\Lambda,\Lambda')}) \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}. 
\]

From (3.13) and (3.21) it follows that
\[
\| (\Psi_n^{N_0} A_{N_0} - A_{N_0} \Psi_n^{N_0}) c\|_{\ell^p(\Lambda)}
\]
\[
= \| (\Psi_n^{N_0} A_{N_0} - A_{N_0} \Psi_n^{N_0}) \Psi_n^{N_0} c\|_{\ell^p(\Lambda)}
\]
\[
\leq R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|\Psi_n^{N_0} A_{N_0} - A_{N_0} \Psi_n^{N_0}\|_{\ell^p(\Lambda,\Lambda')} \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}
\]
\[
\leq R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}
\]
\[
\times \inf_{0 \leq s \leq N_0} \left( \|A_{N_0} - A_s\|_{\ell^p(\Lambda,\Lambda')} + \frac{2d_s}{N_0} \|A_{N_0}\|_{\ell^p(\Lambda,\Lambda')} \right) \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}.
\]

Combining (3.21), (3.22), (3.30) and (3.31), we get
\[
2^{d/p} \|A_{N_0} c\|_{\ell^p(\Lambda)} \geq \left( \sum_{n \in N_0} \|\Psi_n^{N_0} A_{N_0} c\|_{\ell^p(\Lambda)}^p \right)^{1/p}
\]
\[
\geq \left( \alpha - R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A - A_{N_0}\|_{\ell^p(\Lambda,\Lambda')} \right) \left( \sum_{n \in N_0} \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}^p \right)^{1/p}
\]
\[
- R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N_0} \left( \|A_{N_0} - A_s\|_{\ell^p(\Lambda,\Lambda')} + \frac{2d_s}{N_0} \|A_{N_0}\|_{\ell^p(\Lambda,\Lambda')} \right) \|\Psi_n^{N_0} c\|_{\ell^p(\Lambda')}.
\]

Therefore
\[
\|A c\|_{\ell^p(\Lambda)} \geq \|A_{N_0} c\|_{\ell^p(\Lambda)} - \|A - A_{N_0}\|_{\ell^p(\Lambda,\Lambda')} \|A - A_{N_0}\|_{\ell^p(\Lambda,\Lambda')}.
\]
\[
\geq 2^{-d/p} \left( \alpha - (1 + 2^{d/p}) R(\Lambda)^{1/p} R(\Lambda')^{1-1/p} \|A - A_{N_0}\|_{\ell^p(\Lambda,\Lambda')} \right.
\]
\[
- (5 + 2^{1-p})^{d/p} R(\Lambda)^{1/p} R(\Lambda')^{1-1/p}
\]
\[
\times \inf_{0 \leq s \leq N_0} \left( \|A_{N_0} - A_s\|_{\ell^p(\Lambda,\Lambda')} + \frac{2d_s}{N_0} \|A_{N_0}\|_{\ell^p(\Lambda,\Lambda')} \right) \|c\|_{\ell^p(\Lambda')}.
\]
\[
\geq 2^{-d/p} \left( \alpha - 2(5 + 2^{1-p})^{1/p} R(\Lambda)^{1/p} \right.
\]
\[
\times R(\Lambda')^{1-1/p} \inf_{0 \leq s \leq N_0} \left( \|A - A_s\|_{\ell^p(\Lambda,\Lambda')} + \frac{d_s}{N_0} \|A\|_{\ell^p(\Lambda,\Lambda')} \right) \|c\|_{\ell^p(\Lambda')},
\]
and the conclusion (i) for $1 \leq p < \infty$ follows.

The conclusion (i) for $p = \infty$ can be proved by a similar argument. We omit the details here. \qed
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References


STABILITY CRITERION FOR INFINITE MATRICES


