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DOUBLE KOSZUL COMPLEX AND CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OF $\mathfrak{gl}(3|1)$

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Abstract. Let V be a super vector space with super dimension $(m|n)$. Manin introduced the Koszul complex associated to V, which is denoted K. There is another Koszul complex, denoted L. Our observation is that these two Koszul complexes can be combined into a double complex, which we call the double Koszul complex. By using the differential of this complex, we give a way to describe all irreducible representations of $\mathfrak{gl}(V)$ when V has super dimension $(3|1)$.

1. Introduction

Let V be a super vector space over a field k of characteristic 0. The super group $GL(V)$ of linear automorphisms of V is the subgroup of the semi-group $\mathrm{End}(V)$ of endomorphisms with invertible super determinant. In [12], Manin introduced the following Koszul complex K to define the super determinant. Its (k,l)-term is given by $K^{k,l} := \Lambda_k \otimes S_l^*$, where Λ_n and S_n are the n-th homogeneous components of the exterior and the symmetric tensor algebras on V. The differential $d_{k,l} : K^{k,l} \longrightarrow K^{k+1,l+1}$ is given by

$$d_{k,l}(h \otimes \varphi) = \sum_i h \wedge x_i \otimes \xi^i \cdot \varphi.$$

There is another Koszul complex associated to V, denoted by L. This complex was first defined by Priddy as a free resolution of k as a module over the symmetric tensor algebra of V; see [11]. Its (l,k)-term is given by $L^{l,k} := S_l \otimes \Lambda_k$ with differential $P_{l,k} : L^{l,k} \longrightarrow L^{l-1,k+1}$ given by

$$P_{l,k} : S_l \otimes \Lambda_k \hookrightarrow V^{\otimes l} \otimes V^{\otimes k} = V^{\otimes l-1} \otimes V^{\otimes k+1} \xrightarrow{X_{l-1} \otimes Y_{k+1}} S_{l-1} \otimes \Lambda_{k+1} \ ,$$

where X_l, Y_k are the symmetrizer and anti-symmetrizer operators. In [8], Kac proved that any finite-dimensional irreducible representation of the Lie super algebra $\mathfrak{gl}(V)$ is a quotient of the Kac module. He divided irreducible representations of $\mathfrak{gl}(V)$ into two classes, typical representations and atypical representations. By using the Kac module, Kac gave an explicit construction of all typical representations of $\mathfrak{gl}(V)$, and a character formula for all typical representations. In [15], Su

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and Zhang gave a character formula for all finite-dimensional irreducible representations of $\mathfrak{gl}(V)$. An explicit construction of atypical representations is however not known. The aim of this work is to give a combinatorial way to describe all irreducible representations in case the super dimension of V is $(3|1)$.

Our observation is that the two Koszul complexes above can be combined into a double complex which we call the double Koszul complex. We use the differential of this complex to describe all irreducible representations of $\mathfrak{gl}(V)$ when V has super dimension $(3|1)$.

The paper is organized as follows. Section 2 provides some background materials on the general linear super algebra needed for the rest of the paper. Section 3 introduces and studies the double Koszul complex. Section 4 uses the properties of the double Koszul complex to construct representations of the Lie super algebra $\mathfrak{gl}(V)$. Using the character formula of Su and Zhang in [15], we prove that the constructed representations furnish all irreducible representations of $\mathfrak{gl}(V)$.

2. Preliminaries

This section presents some results on the general linear Lie super algebras for later use. We shall work with a field k of characteristic 0. A super vector space is a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The spaces $V_{\bar{0}}, V_{\bar{1}}$ are called even and odd homogeneous components of V; their elements are called homogeneous. We denote the $\mathbb{Z}_2$-grade (or parity) of a homogeneous element a by $\hat{a}$. Assume $\dim V_{\bar{0}} = m, \dim V_{\bar{1}} = n$ and fix a homogeneous basis of V: $x_1, \ldots, x_m \in V_{\bar{0}}$, $x_{m+1}, \ldots, x_{m+n} \in V_{\bar{1}}$. For simplicity we denote the $\mathbb{Z}_2$-grade of x_i by $\hat{i}$. Thus $\hat{i} = \bar{0}$ if $1 \le i \le m$ and $\hat{i} = \bar{1}$ if $m+1 \le i \le m+n$.

A $\mathbb{Z}_2$-graded algebra A is called a super algebra. Similarly we have the notion of super Lie algebra L, where the super anti-commutativity and the super Leibniz rule read:

$$\begin{aligned} [a,b] &= (-1)^{\hat{a}\hat{b}}[b,a], \\ [a,[b,c]] &= [[a,b],c] + (-1)^{\hat{a}\hat{b}}[b,[a,c]]. \end{aligned}$$

Here we use the convention that $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$.

Given a super algebra A, the super commutator on A, defined by

$$[a,b] := ab - (-1)^{\hat{a}\hat{b}} ba,$$

makes A into a super Lie algebra, denoted by A^L.

2.1. **Super Lie algebras $\mathfrak{g} = \mathfrak{gl}(V)$.** Consider the algebra $\mathrm{End}(V)$ of linear endomorphisms of V. Fix a homogeneous basis of V as above. Every element of $\mathrm{End}(V)$ is given by a matrix of the form $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A, B, C, D are block matrices. The matrices of the form $\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right)$ define even maps $V \to V$ (i.e. maps that preserve the $\mathbb{Z}_2$-grading). The matrices of the form $\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right)$ define odd maps (i.e. maps that interchange the $\mathbb{Z}_2$-grading). An arbitrary map $V \to V$ is the sum of an even map with an odd map. This defines a $\mathbb{Z}_2$-grading on $\mathrm{End}(V)$ and makes $\mathrm{End}(V)$ a super algebra. The associated super Lie algebra $\mathrm{End}(V)^L$ is denoted by $\mathfrak{gl}(V)$.

2.2. **Representation of $\mathfrak{g} = \mathfrak{gl}(V)$.** Let W be a super vector space. A super representation ρ of $\mathfrak{g}$ in W is an even linear mapping $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(W)$ which preserves the super commutator, that is, a homomorphism of Lie super algebras. A super representation of $\mathfrak{g}$ is also called a $\mathfrak{g}$-module. A super representation is said to be irreducible if it has no proper non-zero subrepresentations. In order to construct all irreducible representations of $\mathfrak{g}$ we need the technique of induced representations, which we will now describe.

2.2.1. *Induced representations.* A pair $(\mathcal{U}(\mathfrak{g}), i)$, where $\mathcal{U}(\mathfrak{g})$ is an associative $\mathbb{Z}_2$-graded algebra and $i : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})^L$ is a homomorphism of Lie super algebras, is called a universal enveloping super algebra of $\mathfrak{g}$ if for any other pair $(\mathcal{U}', i')$, there is a unique homomorphism $\theta : \mathcal{U} \longrightarrow \mathcal{U}'$ such that $i' = \theta.i$. Thus, the concepts of "super representation of $\mathfrak{g}$", "$\mathfrak{g}$-module" and "left $\mathcal{U}(\mathfrak{g})$-module" are completely equivalent.

Let $\mathfrak{g}$ be a super Lie algebra, $\mathcal{U}(\mathfrak{g})$ be its universal enveloping super algebra, $\mathfrak{h}$ be a super Lie subalgebra of $\mathfrak{g}$, and V be an $\mathfrak{h}$-module. The $\mathbb{Z}_2$-graded space $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} V$ can be endowed with the structure of a $\mathfrak{g}$-module as follows: $g(u \otimes v) = gu \otimes v$ for $g \in \mathfrak{g}, u \in \mathcal{U}(\mathfrak{g}), v \in V$. The so-constructed $\mathfrak{g}$-module is said to be induced from the $\mathfrak{h}$-module V and is denoted by $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}} V$.

2.2.2. *Weights and roots of* $\mathfrak{g}$. The standard basis for $\mathfrak{g}$ consists of matrices $E_{ij} : i, j = 1, \ldots, m+n$, where E_{ij} is the matrix with 1 in the place (i, j) and 0 elsewhere. Consider the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ spanned by the elements $h_j := E_{jj} : i, j = 1, \ldots, m+n$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. The space $\mathfrak{h}^*$ dual to $\mathfrak{h}$ is spanned by $\epsilon_i : i = 1, \ldots, m+n$, where for $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$,

$$\epsilon_i : X \longmapsto A_{ii}, \text{ for } 1 \le i \le m, \quad \epsilon_j : X \longmapsto D_{jj}, \text{ for } m+1 \le j \le m+n.$$

The elements of $\mathfrak{h}^*$ are called the weights of $\mathfrak{g}$. Let $\lambda \in \mathfrak{h}^*$, $\lambda = \sum_{i+1}^{m} \lambda_i \epsilon_i - \sum_{j=m+1}^{m+n} \lambda_j \epsilon_j$. Then we write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m | \lambda_{m+1}, \ldots, \lambda_{m+n})$.

Definition 2.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m | \lambda_{m+1}, \ldots, \lambda_{m+n})$ be a weight.

(i) λ is called integral if $\lambda_i - \lambda_{i+1} \in \mathbb{Z}$ for all $i \neq m$.
(ii) λ is called dominant if $\lambda_i \ge \lambda_{i+1}$ for $1 \le i \le m$, and $\lambda_j \le \lambda_{j+1}$ for $m+1 \le j \le m+n-1$.
(iii) λ is called typical if $(\lambda_i + m + 1 - i) - (\lambda_{m+p} + p) \neq 0$ for all $1 \le i \le m, 1 \le p \le n$; otherwise it is called atypical.
(iiii) λ is called integrable if $\lambda_i \in \mathbb{Z}$ for all i.

Let $0 \neq \alpha \in \mathfrak{h}^*$. Set $\mathfrak{g}_\alpha := \{a \in \mathfrak{g} : [h, a] = \alpha(h)a, \forall h \in \mathfrak{h}\}$. If $\mathfrak{g}_\alpha \neq 0$, then α has the form $\epsilon_i - \epsilon_j : i \neq j$. It is called a root. We set $\Delta_0^+ = \{\epsilon_i - \epsilon_j : 1 \le i < j \le m \quad \text{or} \quad m+1 \le i < j \le m+n\}, \Delta_1^+ = \{\epsilon_i - \epsilon_j : 1 \le i \le m \quad \text{and} \quad m+1 \le j \le m+n\}$ and

$$\rho := (m, m-1, \ldots, 1 | 1, 2, \ldots, n) - \tfrac{m+n+1}{2}(1, 1, \ldots, 1 | 1, 1, \ldots, 1).$$

2.2.3. *Kac module.* For every integral dominant weight λ, we denote by $V^0(\lambda)$ the finite-dimensional irreducible $\mathfrak{g}_{\bar{0}}$-module with highest weight λ. $V^0(\lambda)$ is the $(\mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{+1})$-module with $\mathfrak{g}_{+1}$ acting by 0, where $\mathfrak{g}_{+1}$ is the set of matrices of the form $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. Set $\bar{V}(\lambda) := \mathrm{Ind}_{\mathfrak{g}_{\bar{0}} + \mathfrak{g}_{+1}}^{\mathfrak{g}} V^0(\lambda)$. $\bar{V}(\lambda)$ contains a unique maximal submodule

$M(\lambda)$, and we set

$$V(\lambda) := \bar{V}(\lambda)/M(\lambda).$$

Then $V(\lambda)$ is an irreducible representation with highest weight λ. The module $\bar{V}(\lambda)$ is called the generalized Verma module or Kac module [8]. Kac showed that the $V(\lambda)$'s furnish all irreducible $\mathfrak{g}$-modules of finite dimension.

If λ is a typical weight, then $M_\lambda = 0$; thus $V(\lambda) = \bar{V}(\lambda)$, and in this case $V(\lambda)$ is called typical. On the other hand, if λ is atypical, an explicit construction of $M(\lambda)$ is not known.

2.2.4. *Characters of representations.* Let V be a finite-dimensional irreducible $\mathfrak{g}$-module. For every element $\lambda \in \mathfrak{h}^*$, we define

$$V_\lambda := \{v \in V : \rho(h) = \lambda(h)v \quad \text{for all} \quad h \in \mathfrak{h}\}.$$

Then we have $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$. The character of V is $\mathrm{ch}(V) := \sum_{\lambda\in\mathfrak{h}^*}(\dim V_\lambda)e^\lambda$. The following formula for the character of typical irreducible modules is due to Kac [8]:

$$\mathrm{ch}(V) = \frac{L_1}{L_0}\sum_{w\in S_m\times S_n}\mathrm{sign}(w)e^{w(\lambda+\rho)}, \tag{1}$$

with $L_1 = \sum_{\alpha\in\Delta_1^+}(e^{\alpha/2} + e^{-\alpha/2})$ and $L_0 = \sum_{\beta\in\Delta_0^+}(e^{\beta/2} - e^{-\beta/2})$.

In [15], Su and Zhang gave a character formula for all finite-dimensional irreducible representations with any typical and atypical dominant integral weight λ. The formula is too complicated to recall here, but see below for a special case.

2.3. **Characters of irreducible representations of $\mathfrak{gl}(3|1)$.** In this section, we will recall formulas for the character of all typical and atypical finite-dimensional irreducible representations of $\mathfrak{gl}(3|1)$. See [15, Theorem 4.9].

In $\mathfrak{gl}(3|1)$, we have $\Delta_1^+ = \{\epsilon_1-\epsilon_4, \epsilon_2-\epsilon_4, \epsilon_3-\epsilon_4\}$, $\Delta_0^+ = \{\epsilon_1-\epsilon_2, \epsilon_1-\epsilon_3, \epsilon_2-\epsilon_3\}$, $\rho = (\frac{1}{2}, \frac{-1}{2}, \frac{-3}{2}|\frac{-3}{2})$.

Set $x_1 := e^{\epsilon_1}, x_2 := e^{\epsilon_2}, x_3 := e^{\epsilon_3}, y := e^{\epsilon_4}$, $R := (x_1+y)(x_2+y)(x_3+y), \Pi := (x_1-x_2)(x_2-x_3)(x_1-x_3)$,

$$a(t,u,v) := \det\begin{pmatrix} x_1^{t+2} & x_1^{u+1} & x_1^v \\ x_2^{t+2} & x_2^{u+1} & x_2^v \\ x_3^{t+2} & x_3^{u+1} & x_3^v \end{pmatrix}.$$

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3|\lambda_4)$ be a typical dominant integral weight. According to the character formula (1), we have

$$\mathrm{ch}(V(\lambda)) = \frac{R(x_1x_2x_3)^{\lambda_3-1}}{\Pi y^{\lambda_4}}a(\lambda_1-\lambda_3, \lambda_2-\lambda_3, 0).$$

Let λ be an atypical weight. Then there are three possibilities:

If $\lambda_1 + 2 = \lambda_4$, then

(2)

$$\mathrm{ch}(V(\lambda)) = \frac{R}{\Pi y^{\lambda_4}}\Big[\frac{x_1^{\lambda_1+2}}{x_1+y}(x_2^{\lambda_2}x_3^{\lambda_3-1} - x_2^{\lambda_3-1}x_3^{\lambda_2}) + \frac{x_2^{\lambda_1+2}}{x_2+y}(x_3^{\lambda_2}x_1^{\lambda_3-1} - x_3^{\lambda_3-1}x_1^{\lambda_2}) + \frac{x_3^{\lambda_1+2}}{x_3+y}(x_1^{\lambda_2}x_2^{\lambda_3-1} - x_1^{\lambda_3-1}x_2^{\lambda_2})\Big].$$

If $\lambda_2+1=\lambda_4$, then

$$\begin{aligned}(3)\quad \mathrm{ch}(V(\lambda)) = \frac{R}{\Pi y^{\lambda_4}}\Big[&\frac{x_1^{\lambda_2+1}}{x_1+y}(x_2^{\lambda_3-1}x_3^{\lambda_1+1}-x_2^{\lambda_1+1}x_3^{\lambda_3-1})\\ &+\frac{x_2^{\lambda_2+1}}{x_2+y}(x_3^{\lambda_3-1}x_1^{\lambda_1+1}-x_3^{\lambda_1+1}x_1^{\lambda_3-1})\\ &+\frac{x_3^{\lambda_2+1}}{x_3+y}(x_1^{\lambda_3-1}x_2^{\lambda_1+1}-x_1^{\lambda_1+1}x_2^{\lambda_3-1})\Big].\end{aligned}$$

If $\lambda_3=\lambda_4$, then

(4)

$$\begin{aligned}\mathrm{ch}(V(\lambda)) = \frac{R}{\Pi y^{\lambda_4}}\Big[&\frac{x_1^{\lambda_3}}{x_1+y}(x_2^{\lambda_1+1}x_3^{\lambda_2}-x_2^{\lambda_2}x_3^{\lambda_1+1})+\frac{x_2^{\lambda_3}}{x_2+y}(x_3^{\lambda_1+1}x_1^{\lambda_2}-x_3^{\lambda_2}x_1^{\lambda_1+1})\\ &+\frac{x_3^{\lambda_3}}{x_3+y}(x_1^{\lambda_1+1}x_2^{\lambda_2}-x_1^{\lambda_2}x_2^{\lambda_1+1})\Big].\end{aligned}$$

3. Double Koszul complexes

3.1. **The Koszul complex K.** In [12] Manin suggested the following construction to define the super determinant of a super matrix. Let V^* denote the vector space dual to V with the dual basis $\xi^1,\xi^2,\dots,\xi^d$, $\xi^i(x_j)=\delta^i_j$. The complex K has its (k,l)-term given by $K^{k,l}:=\Lambda_k\otimes S^*_l$, where Λ_k is the k-th homogeneous component of the exterior tensor algebra over V and S^*_l is the l-th homogeneous component of the symmetric tensor algebra over V^*. The differential $d_{k,l}:K^{k,l}\longrightarrow K^{k+1,l+1}$ is given by

$$d_{k,l}(h\otimes\varphi)=\sum_i h\wedge x_i\otimes\xi^i\cdot\varphi. \tag{5}$$

In fact, the construction above gives a series of complexes K_a:

$$K_a:\cdots\xrightarrow{d}\Lambda_k\otimes S^*_{k-a}\xrightarrow{d}\Lambda_{k+1}\otimes S^*_{k-a+1}\xrightarrow{d}\cdots.$$

Here for $k<0$ we define Λ_k and S_k to be 0. Thus each complex K_a is bounded from below.

It is easy to check that $d_{k,l}$ is $\mathfrak{gl}(V)$-equivariant; hence the homology groups of this complex are representations of $\mathfrak{gl}(V)$. On the other hand, one can show that the complex (K_a,d) is exact everywhere if $a\neq m-n$, and the complex (K_{m-n},d) is exact everywhere except at the term $\Lambda_m\otimes S^*_n$, where the homology group is one-dimensional. This homology group defines a one-dimensional representation of $\mathfrak{gl}(V)$. It turns out that elements of $\mathfrak{gl}(V)$ act on this representation by means of its super determinant.

Notice that there is another differential $\partial_{k,l}:K^{k+1,l+1}\longrightarrow K^{k,l}$, which is defined as follows:

$$\partial_{k,l}:\Lambda_{k+1}\otimes S_{l+1}\hookrightarrow V^{\otimes k+1}\otimes V^{\otimes l+1}\xrightarrow{(\mathrm{id}\otimes\mathrm{ev}_V\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau_{V\otimes V^*}\otimes\mathrm{id})}$$
$$V^{\otimes k}\otimes V^{*\otimes l}\xrightarrow{Y_k\otimes X^*_l}\Lambda_k\otimes S^*_l,$$

where

$$X_n:=\frac{1}{n!}\sum_{w\in\sigma_n}T_w,\ Y_n:=\frac{1}{n!}\sum_{w\in\sigma_n}(-1)^{l(w)}T_w,$$

$\tau(a \otimes \varphi) = (-1)^{\hat{a}.\hat{\varphi}} \varphi \otimes a$, $\mathrm{ev}(\varphi \otimes a) = \varphi(a)$, $a \in V, \varphi \in V^*$, and a and φ are homogeneous. One checks that on $K^{k,l}$,

$$lkd_{k-1,d-1}\partial_{k-1,l-1} + (l+1)(k+1)\partial_{k,l}d_{k,l} = (l-k-n+m)\mathrm{id}. \tag{6}$$

Since $(K_\bullet, d)$ is exact, $(K_\bullet, \partial)$ is also exact.

3.2. **The Koszul complex L.** There is another Koszul complex associated to V. This complex was first defined by Priddy as a free resolution of k as a module over the symmetric tensor algebra of V (see [11]). As in the case of the complex K, the complex L with $L^{p,r} := S_p \otimes \Lambda_r$ is defined as a series of complexes L_a,

$$L_a : \cdots \xrightarrow{P} S_p \otimes \Lambda_{a-p} \xrightarrow{P} S_{p-1} \otimes \Lambda_{a-p+1} \xrightarrow{P} \cdots$$

with differential $P_{p,r} : L^{p,r} \longrightarrow L^{p-1,r+1}$ given by

$$P_{p,r} : S_p \otimes \Lambda_r \hookrightarrow V^{\otimes p} \otimes V^{\otimes r} \xrightarrow{X_{p-1} \otimes Y_{r+1}} S_{p-1} \otimes \Lambda_{r+1} \ .$$

The complexes $(L_\bullet, P)$ are exact, except for $a = 0$.

We also have another differential $Q_{p,r} : L^{p-1,r+1} \longrightarrow L^{p,r}$, given by

$$Q_{p,r} : S_{p-1} \otimes \Lambda_{r+1} \hookrightarrow V^{\otimes p-1} \otimes V^{\otimes r+1} = V^{\otimes p} \otimes V^{\otimes r} \xrightarrow{X_p \otimes Y_r} S_p \otimes \Lambda_r \ .$$

One checks that on $L^{p,r}$,

$$r(p+1)PQ + p(r+1)QP = (p+r)\mathrm{id}. \tag{7}$$

Consequently the complexes $(L_\bullet, Q)$ are exact too.

3.3. **The double Koszul complex.** The main observation of this work is the fact that the two Koszul complexes mentioned in the previous section can be combined into a double complex, which we call the double Koszul complex. In this section we describe this complex. An application to the construction of irreducible representations of the super Lie algebra $\mathfrak{gl}(3|1)$ will be given in the next section.

For simplicity we shall use the dot "$\cdot$" to denote the tensor product. Fix an integer $a \geq 1$. We arrange the Koszul complexes $K_{-a}, K_{-a-1}, K_{-a-2}, \ldots$ as in the diagram below:

$$\begin{array}{lccccccccc}
K_{-a}: 0 & \longrightarrow & S_a^* & \xrightarrow{d_{0,a}} & \Lambda_1 \cdot S_{a+1}^* & \xrightarrow{d_{1,a+1}} & \Lambda_2 \cdot S_{a+2}^* & \xrightarrow{d_{2,a+2}} & \Lambda_3 \cdot S_{a+3}^* & \longrightarrow \cdots \\
K_{-a-1}: & 0 & \longrightarrow & S_{a+1}^* & \xrightarrow{d_{0,a+1}} & \Lambda_1 \cdot S_{a+2}^* & \xrightarrow{d_{1,a+2}} & \Lambda_2 \cdot S_{a+3}^* & \longrightarrow & \cdots \\
K_{-a-2}: & & & 0 & \longrightarrow & S_{a+2}^* & \xrightarrow{d_{0,a+2}} & \Lambda_1 \cdot S_{a+3}^* & \longrightarrow & \cdots
\end{array}$$

To get the entries on a column into a complex we tensor each complex K_{-a-i} with S_i; i.e. the complex K_{-a-1} is tensored with S_1, the complex K_{-a-2} is tensored

with S_2, etc. Then each column can be interpreted as the complexes L_j tensored with S^*_{a+j}. Thus we have the following diagram, where all rows are the Koszul complex $K_\bullet$ tensored with $S_\bullet$ and the columns are the Koszul complex $L_\bullet$ tensored with $S^*_\bullet$:

(8)
$$\begin{tikzcd}
 & 0 & 0 & 0 & 0 & \\
0 \arrow[r] & S^*_a \arrow[r, "d"] \arrow[u] & \Lambda_1 \cdot S^*_{a+1} \arrow[r, "d"] \arrow[u] & \Lambda_2 \cdot S^*_{a+2} \arrow[r, "d"] \arrow[u] & \Lambda_3 \cdot S^*_{a+3} \arrow[r, "d"] \arrow[u] & \cdots \\
 & 0 \arrow[r] \arrow[u] & S_1 \cdot S^*_{a+1} \arrow[r, "d"] \arrow[u, "P"] & S_1 \cdot \Lambda_1 \cdot S^*_{a+2} \arrow[r, "d"] \arrow[u, "P"] & S_1 \cdot \Lambda_2 \cdot S^*_{a+3} \arrow[r, "d"] \arrow[u, "P"] & \cdots \\
 & & 0 \arrow[r] \arrow[u] & S_2 \cdot S^*_{a+2} \arrow[r, "d"] \arrow[u, "P"] & S_2 \cdot \Lambda_1 \cdot S^*_{a+3} \arrow[r, "d"] \arrow[u, "P"] & \cdots \\
 & & & 0 \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

A general square in diagram (8) has the form

(9)
$$\begin{tikzcd}
S_i \cdot \Lambda_k \cdot S^*_l \arrow[r, "\mathrm{id}\otimes d"] & S_i \cdot \Lambda_{k+1} \cdot S^*_{l+1} \\
S_{i+1} \cdot \Lambda_{k-1} \cdot S^*_l \arrow[r, "\mathrm{id}\otimes d"'] \arrow[u, "P\otimes\mathrm{id}"] & S_{i+1} \cdot \Lambda_k \cdot S^*_{l+1} \arrow[u, "P\otimes\mathrm{id}"']
\end{tikzcd} \qquad \text{with} \quad l = i+k+a.$$

For convenience, we denote $d := \mathrm{id} \otimes d, P := P \otimes \mathrm{id}$. It is easy to show that $Pd = dP$ for all the above squares.

We also have an exact double Koszul complex with d, P replaced by ∂, Q:

(10)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 & S^*_a \arrow[l, dashed, "\partial"'] \arrow[d, "Q"] & \Lambda_1 \cdot S^*_{a+1} \arrow[l, dashed, "\partial"'] \arrow[d, "Q"] & \Lambda_2 \cdot S^*_{a+2} \arrow[l, dashed, "\partial"'] \arrow[d, "Q"] & \Lambda_3 \cdot S^*_{a+3} \arrow[l, dashed, "\partial"'] \arrow[d, "Q"] & [d]^Q \cdots \arrow[l, dashed, "\partial"'] \\
 & 0 & S_1 \cdot S^*_{a+1} \arrow[l, dashed, "\partial"'] \arrow[d, "Q"] & S_1 \cdot \Lambda_1 \cdot S^*_{a+2} \arrow[l, dashed, "\partial"'] \arrow[d, "Q"] & S_1 \cdot \Lambda_2 \cdot S^*_{a+3} \arrow[l, dashed, "\partial"'] \arrow[d, "Q"] & [d]^Q \cdots \arrow[l, dashed, "\partial"'] \\
 & & 0 & S_2 \cdot S^*_{a+2} \arrow[l, dashed, "\partial"'] \arrow[d] & S_2 \cdot \Lambda_1 \cdot S^*_{a+3} \arrow[l, dashed, "\partial"'] \arrow[d] & \cdots \arrow[l, dashed, "\partial"'] \\
 & & & 0 & \vdots &
\end{tikzcd}$$

The commutativity of this diagram is easy to check.

3.4. **Some remarks on the structure of the double complex.** In this subsection we study some maps obtained from the differentials of the double Koszul complex. From now on, we only consider the case $(m|n) = (3|1)$.

We put the two diagrams (8) and (10) into one:

(11)
$$\begin{tikzcd}
S_{i-1} \cdot S^*_{a+i-1} \arrow[r, hook, shift left, "d_{0,a+i-1}"] & S_{i-1} \cdot \Lambda_1 \cdot S^*_{a+i} \arrow[l, two heads, shift left, dashed, "\partial_{0,a+i-1}"] \arrow[r, shift left, "d_{1,a+i}"] \arrow[d, hook, "P"'] \arrow[d, two heads, shift left, "Q"] & S_{i-1} \cdot \Lambda_2 \cdot S_{a+i+1^*} \arrow[l, two heads, shift left, dashed, "\partial_{1,a+i}"] \arrow[r, shift left, "d_{2,a+i+1}"] \arrow[d, hook, "P"'] \arrow[d, two heads, shift left, "Q"] & \cdots \arrow[l, two heads, shift left, dashed, "\partial_{2,a+i+1}"] \\
 & S_i \cdot S^*_{a+i} \arrow[r, hook, shift left, "d_{0,a+i}"] & S_i \cdot \Lambda_1 . S^*_{a+i+1} \arrow[l, two heads, shift left, dashed, "\partial_{0,a+i}"] \arrow[r, shift left, "d_{1,a+i+1}"] \arrow[d, hook, "P"'] \arrow[d, two heads, shift left, "Q"] & S_i \cdot \Lambda_2 \cdot S^*_{a+i+2} \arrow[l, two heads, shift left, dashed, "\partial_{1,a+i+1}"] \\
 & & S_{i+1} \cdot S^*_{a+i+1} \arrow[r, hook, shift left, "d_{0,a+i+1}"] & S_{i+1} \cdot \Lambda_1 \cdot S^*_{a+i+2} \arrow[l, two heads, shift left, dashed, "\partial_{0,a+i+1}"] \arrow[u, hook, "P"'] \arrow[u, two heads, shift left, "Q"]
\end{tikzcd}$$

Proposition 3.1. *The composed map* $\partial PQd : S_i \cdot S^*_{a+i} \longrightarrow S_i \cdot S^*_{a+i}$ *in diagram* (11) *is an isomorphism for all* $i \geq 0$. *Consequently* $S_i \cdot S^*_{a+i}$ *is isomorphic to a direct summand of* $S_{i+1} \cdot S^*_{a+i+1}$.

Proof. According to formulas (6) and (7) and the commutativity between d, P and ∂, Q, we have

$$\begin{aligned}\partial PQd &= \partial d - \frac{2i}{i+1}\partial QPd \\ &= \partial d - \frac{i}{i+1}QP + \frac{i(a+i)}{(i+1)(a+i+1)}Qd\partial P \\ &= \Big[\frac{(a+i+2)}{(a+i+1)} - \frac{i}{i+1}\Big]\mathrm{id} + \frac{i(a+i)}{(i+1)(a+i+1)}Qd\partial P.\end{aligned}$$

We will use induction on i to prove that $\partial PQd : S_i \cdot S^*_{a+i} \longrightarrow S_i \cdot S^*_{a+i}$ is diagonalizable with the set of eigenvalues

$$A_i := \Big\{\frac{(a+i+3-j)j}{(i+1)(a+i+1)}, j = 1, 2, \ldots, i+1\Big\}.$$

For $i = 0$ the claim follows from the equation above. Assume that the proposition is true for $i - 1$.

By assumption, $\partial PQd : S_{i-1} \cdot S^*_{a+i-1} \longrightarrow S_{i-1} \cdot S^*_{a+i-1}$ is diagonalizable with the set of eigenvalues A_{i-1}; hence $Qd\partial P : S_i \cdot S^*_{a+i} \longrightarrow S_i \cdot S^*_{a+i}$ is diagonalizable with the set of eigenvalues $A_{i-1} \cup \{0\}$. Thus it is easy to see that $\partial PQd : S_i \cdot S^*_{a+i} \longrightarrow S_i \cdot S^*_{a+i}$ is diagonalizable with A_i the set of eigenvalues. $\square$

Consider the diagram in (8) as an exact sequence of horizontal complexes (except for the first column) and split it into short exact sequences:
(12)

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ker}P_{i,k} \cdot S^*_{i+k+a} \arrow[r, "d'_{k,i+k+a}"] \arrow[d, shift left, "Q"] & \mathrm{Ker}P_{i,k+1} \cdot S^*_{i+k+a+1} \arrow[r, "d'_{k+1,i+k+a+1}"] \arrow[d, shift left, "Q"] & \mathrm{Ker}P_{i,k+2} \cdot S^*_{i+k+a+2} \arrow[r] \arrow[d, shift left, "Q"] & \cdots \\
\cdots \arrow[r] & S_{i+1} \cdot \Lambda_{k-1} \cdot S^*_{i+k+a} \arrow[r, "d_{k-1,i+k+a}"] \arrow[u, shift left, "P_{i+1,k-1}"] \arrow[d, shift left, "Q"] & S_{i+1} \cdot \Lambda_k \cdot S^*_{i+k+a+1} \arrow[r, "d_{k,i+k+a+1}"] \arrow[u, shift left, "P_{i+1,k}"] \arrow[d, shift left, "Q"] & S_{i+1} \cdot \Lambda_{k+1} \cdot S^*_{i+k+a+2} \arrow[r] \arrow[u, shift left, "P_{i+1,k+1}"] \arrow[d, shift left, "Q"] & \cdots \\
\cdots \arrow[r] & \mathrm{Ker}P_{i+1,k-1} \cdot S^*_{i+k+a} \arrow[r, "d'_{k-1,i+k+a}"] \arrow[u, shift left, "i"] & \mathrm{Ker}P_{i+1,k} \cdot S^*_{i+k+a+1} \arrow[r, "d'_{k,i+k+a+1}"] \arrow[u, shift left, "i"] & \mathrm{Ker}P_{i+1,k+1} \cdot S^*_{i+k+a+2} \arrow[r] \arrow[u, shift left, "i"] & \cdots
\end{tikzcd}$$

where $d'_{k,i+k+a} := d_{k,i+k+a}\big|_{\mathrm{Ker}P_{i,k} \cdot S^*_{i+k+a}}$, $\mathrm{Ker}P_{i,j} = \mathrm{Im}P_{i+1,j-1}$ for all $i \geq 0$. The differentials ∂ however do not restrict to differentials on the first and the third

horizontal complexes. Consider the following part of (12) for $i, k \geq 1$:

$$
\begin{tikzcd}
S_{i-1}\cdot\Lambda_{k+1}\cdot S^*_{a+i+k} \arrow[r, shift left] \arrow[d, shift left] & S_{i-1}\cdot\Lambda_{k+2}\cdot S_{a+i+k+1} \arrow[l, shift left] \arrow[d, shift left] & \\
\mathrm{Ker}P_{i-,k+1}\cdot S^*_{a+i+k} \arrow[r, shift left] \arrow[d, shift left, "Q"'] & \mathrm{Ker}P_{i-1,k+2}\cdot S^*_{a+i+k+1} \arrow[l, shift left] \arrow[d, shift left] & \\
S_i\cdot\Lambda_k\cdot S^*_{a+i+k} \arrow[r, shift left, "\partial", "d"'] \arrow[u, shift left, "P"'] & S_i\cdot\Lambda_{k+1}\cdot S^*_{a+i+k+1} \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & \cdot S_i\cdot\Lambda_{k+2}\cdot S^*_{a+i+k+2} \arrow[l, shift left] \arrow[d, shift left] \\
 & \mathrm{Ker}P_{i,k+1}\cdot S^*_{a+i+k+1} \arrow[r, shift left] \arrow[d, shift left, "Q"'] & \mathrm{Ker}P_{i,k+2}\cdot S^*_{a+i+k+2} \arrow[l, shift left] \arrow[d, shift left] \\
 & S_{i+1}\cdot\Lambda_k\cdot S^*_{a+i+k+1} \arrow[r, shift left, "\partial", "d"'] \arrow[u, shift left, "P"'] & S_{i+1}\cdot\Lambda_{k+1}\cdot S^*_{a+i+k+2} \arrow[l, shift left]
\end{tikzcd}
\tag{13}
$$

Proposition 3.2. *The composed map*

$$P\partial dQ : \mathrm{Ker}P_{i,k+1}\cdot S^*_{a+i+k+1} \longrightarrow \mathrm{Ker}P_{i,k+1}\cdot S^*_{a+i+k+1}$$

(for $i \geq 0, k \geq 1$) in the diagram (13) *is an isomorphism. Consequently* $\mathrm{Ker}P_{i,k+1}\cdot S^*_{a+i+k+1}$ *is isomorphic to a direct summand of* $S_{i+1}\cdot \mathrm{Im}d_{k,a+i+k+1}$.

Proof. By using the method of induction, we will prove that

$$P\partial dQ : \mathrm{Ker}P_{i,k+1}\cdot S^*_{a+i+k+1} \longrightarrow \mathrm{Ker}P_{i,k+1}\cdot S^*_{a+i+k+1}$$

is diagonalizable with the set of eigenvalues

$$A_i := \Big\{\frac{(a+k+2i+4-j)j}{(i+1)(k+1)^2(a+i+k+2)}, j = 1, 2, \ldots, i+1, i+k+1\Big\}.$$

For $i = 0$, consider the following part of (13):

$$
\begin{tikzcd}
\Lambda_k\cdot S^*_{a+k} \arrow[r, shift left, "\partial", "d"'] & \Lambda_{k+1}\cdot S^*_{a+k+1} \arrow[l, shift left] \arrow[r, shift left, "\partial", "d"'] \arrow[d, shift left, "Q"'] & \cdot\Lambda_{k+2}\cdot S^*_{a+k+2} \arrow[l, shift left] \arrow[d, shift left] \\
\cdots S_1\cdot\Lambda_{k-1}\cdot S^*_{a+k} \arrow[r, shift left, "\partial", "d"'] \arrow[u, shift left, "P"'] & S_1\cdot\Lambda_k\cdot S^*_{a+k+1} \arrow[l, shift left] \arrow[r, shift left, "\partial", "d"'] \arrow[u, shift left, "P"'] & S_1\cdot\Lambda_{k+1}\cdot S^*_{a+k+2}. \arrow[l, shift left]
\end{tikzcd}
$$

For the composed map $P\partial DQ : \Lambda_{k+1}\cdot S^*_{a+k+1} \longrightarrow \Lambda_{k+1}\cdot S^*_{a+k+1}$, by means of formulas (6) and (7) we have

$$
\begin{aligned}
P\partial dQ &= P\frac{[(a+3)-k(a+k+1)d\partial]}{(k+1)(a+k+2)}Q \\
&= \frac{(a+3)}{(k+1)(a+k+2)}\mathrm{id} - \frac{k(a+k+1)}{(k+1)(a+k+2)}d\partial.
\end{aligned}
$$

We have that $d\partial$ is diagonalizable with eigenvalues 0 and $\frac{a+2}{(k+1)(a+k+1)}$; hence $P\partial dQ$ is diagonalizable with the set of eigenvalues

$$A_0 := \Big\{\frac{(a+3)}{(k+1)(a+k+2)}, \frac{(a+k+3)}{(k+1)^2(a+k+2)}\Big\}.$$

For $i = 1$, consider in (13) the map

$$P\partial dQ : \mathrm{Ker}P_{1,k+1} \cdot S^*_{a+k+2} \longrightarrow \mathrm{Ker}P_{1,k+1} \cdot S^*_{a+k+2}.$$

On $S_i \cdot \Lambda_{k+1} \cdot S^*_{a+i+k+1}$, we have

$$\begin{aligned} P\partial dQ &= P\frac{[(a+4) - k(a+k+2)d\partial]}{(k+1)(a+k+3)}Q \\ &= \frac{(a+4)}{(k+1)(a+k+3)}PQ - \frac{k(a+k+2)}{(k+1)(a+k+3)}Pd\partial Q. \end{aligned}$$

We have $Pd\partial Q = dPQ\partial$ and this operator can be restricted to $\mathrm{Ker}P_{1,k+1} \cdot S^*_{a+k+2}$. We compute the eigenvalue of this operator. First we have

$$dPQ\partial = d[\frac{(k+1) - (k+1)QP}{2k}]\partial = \frac{k+1}{2k}d\partial - \frac{k+1}{2k}dQP\partial.$$

Notice that $dQP\partial$ is an endomorphism of $\mathrm{Im}d \subset S_i \cdot \Lambda_{k+1} \cdot S^*_{a+i+k+1}$; on this space $d\partial$ operates by multiplication with $\frac{a+3}{(k+1)(a+k+2)}$. On the other hand, from above we know that the eigenvalues of $P\partial dQ$ form the set A_0. Thus $dQP\partial$ is diagonalizable with eigenvalues $A_0 \cup \{0\}$. Consequently $dPQ\partial$ is diagonalizable with eigenvalues

$$\Big\{\frac{a+3}{2k(a+k+2)}, \frac{a+2}{2(k+1)(a+k+2)}, 0\Big\}.$$

On the other hand, the restriction of PQ to $\mathrm{Ker}P_{1,k+1} \cdot S^*_{a+k+2}$ is the multiplication with $\frac{k+2}{2(k+1)}$. Therefore, the eigenvalues of $P\partial dQ$ are

$$A_1 := \Big\{\frac{(a+4)(k+2)}{2(k+1)^2(a+k+3)}, \frac{(a+k+5)}{2(k+1)^2(a+k+3)}, \frac{2(a+k+4)}{2(k+1)^2(a+k+3)}\Big\}.$$

In general, we consider the composed map

$$P\partial dQ : \mathrm{Ker}P_{i,k+1} \cdot S^*_{a+i+k+1} \longrightarrow \mathrm{Ker}P_{i,k+1} \cdot S^*_{a+i+k+1} \text{ in the diagram (13).}$$

We have

$$\begin{aligned} P\partial dQ &= P[\frac{(a+i+3) - k(a+i+k+1)d\partial}{(k+1)(a+i+k+2)}]Q \\ &= \frac{(a+i+3)}{(k+1)(a+i+k+2)}PQ - \frac{k(a+i+k+1)}{(k+1)(a+i+k+2)}dPQ\partial \\ &= \frac{(a+i+3)(i+k+1)}{(k+1)^2(i+1)(a+i+k+2)}\mathrm{id} - \frac{k(a+i+k+1)}{(k+1)(a+i+k+2)}d[\frac{(i+k) - i(k+1)QP}{k(i+1)}]\partial \\ &= \frac{(a+i+3)(i+k+1)}{(k+1)^2(i+1)(a+i+k+2)}\mathrm{id} - \frac{(i+k)(a+i+k+1)}{(k+1)(i+1)(a+i+k+2)}d\partial \\ &\quad + \frac{i(a+i+k+1)}{(i+1)(a+i+k+2)}dQP\partial. \end{aligned}$$

Similar arguments show that $dQP\partial : \mathrm{Ker}P_{i,k+1} \cdot S^*_{a+i+k+1} \longrightarrow \mathrm{Ker}P_{i,k+1} \cdot S^*_{a+i+k+1}$ is diagonalizable with the set of eigenvalues being $A_{i-1} \cup \{0\}$. Thus the composed map $P\partial dQ : \mathrm{Ker}P_{i,k+1} \cdot S^*_{a+i+k+1} \longrightarrow \mathrm{Ker}P_{i,k+1} \cdot S^*_{a+i+k+1}$ is diagonalizable with A_i the set of eigenvalues, hence is an isomorphism. $\square$

4. Construction of irreducible representations of $\mathfrak{gl}(V)$

Let V be a super vector space with super dimension $(3|1)$. In this section, using the double Koszul complex, we will construct all irreducible representations of this super algebra. To show that the representations obtained are in fact irreducible we compute their characters.

4.1. Combinatorial construction of irreducible representations of $\mathfrak{gl}(V)$.

In this section, we will compute the character of the duals of irreducible direct summands of the power of the fundamental representation V. By the combinatorial method, we have

$$V^{\otimes k} = \bigoplus_{\lambda\in\Gamma_{3,1}} I_\lambda^{\oplus C_\lambda},$$

where I_λ are simple and $\Gamma_{3,1}$ is the set of partitions with $\lambda_4 \le 1$. Since the character of V is $x_1+x_2+x_3-y$, using the determinant formula (3.5) of [10], we can compute the character of I_λ for all $\lambda \in \Gamma_{3,1}$.

If $\lambda \in \Gamma_{3,1}$ and $\lambda_3 \ge 1$, we have

$$\mathrm{ch}(I_{\lambda_1,\lambda_2,\lambda_3,1_4^\lambda}) = \frac{R(x_1x_2x_3)^{\lambda_3-1}}{\Pi y^{\lambda_4}} a(\lambda_1-\lambda_3,\lambda_2-\lambda_3,0); \tag{14}$$

hence

$$\mathrm{ch}(I^*_{\lambda_1,\lambda_2,\lambda_3,1_4^\lambda}) = \frac{R(x_1x_2x_3)^{-\lambda_1}}{\Pi y^{\lambda_4+3}} a(\lambda_1-\lambda_3,\lambda_1-\lambda_2,0).$$

Thus $I^*_{\lambda_1,\lambda_2,\lambda_3,1_4^\lambda}$ has highest weight $(-\lambda_3+1,-\lambda_2+1,-\lambda_1+1|\lambda_4+3)$. Therefore we have

$$\mathrm{ch}(I_{\lambda_1,\lambda_2,\lambda_3,1_4^\lambda}) = \mathrm{ch}(V(\lambda_1,\lambda_2,\lambda_3|-\lambda_4)),$$
$$\mathrm{ch}(I^*_{\lambda_1,\lambda_2,\lambda_3,1_4^\lambda}) = \mathrm{ch}(V(-\lambda_3+1,-\lambda_2+1,-\lambda_1+1|\lambda_4+3)).$$

Now,

$$\mathrm{ch}(I_{\lambda_1,\lambda_2,0,0}) = \frac{R}{\Pi}\Big[\frac{x_2^{\lambda_1+1}x_3^{\lambda_2}-x_2^{\lambda_2}x_3^{\lambda_1+1}}{x_1+y} + \frac{x_3^{\lambda_1+1}x_1^{\lambda_2}-x_3^{\lambda_2}x_1^{\lambda_1+1}}{x_2+y} + \frac{x_1^{\lambda_1+1}x_2^{\lambda_2}-x_1^{\lambda_2}x_2^{\lambda_1+1}}{x_3+y}\Big];$$

hence

$$\mathrm{ch}(I^*_{\lambda_1,\lambda_2,0,0}) = \frac{R}{\Pi y^2}\Big[\frac{x_1^2}{x_1+y}(x_2^{-\lambda_2+1}x_3^{-\lambda_1}-x_2^{-\lambda_1}x_3^{-\lambda_2+1}) + \frac{x_2^2}{x_2+y}(x_3^{-\lambda_2+1}x_1^{-\lambda_1}-x_3^{-\lambda_1}x_1^{-\lambda_2+1}) + \frac{x_3^2}{x_1+y}(x_1^{-\lambda_2+1}x_2^{-\lambda_1}-x_1^{-\lambda_1}x_2^{-\lambda_2+1})\Big].$$

Therefore $I^*_{\lambda_1,\lambda_2,0,0}$ has highest weight $(0,-\lambda_2+1,-\lambda_1+1|2)$. Thus we have

$$\mathrm{ch}(I_{\lambda_1,\lambda_2,0,0}) = \mathrm{ch}(V(\lambda_1,\lambda_2,0|0)),$$
$$\mathrm{ch}(I^*_{\lambda_1,\lambda_2,0,0}) = \mathrm{ch}(V(0,-\lambda_2+1,-\lambda_1+1|2)).$$

Further we have

$$\mathrm{ch}(I_{\lambda_1,0,0,0}) = \frac{1}{\Pi}\Big[x_2^{\lambda_1+1}(x_2+y)(x_3-x_1)+x_3^{\lambda_1+1}(x_3+y)(x_1-x_2) + x_1^{\lambda_1+1}(x_1+y)(x_2-x_3)\Big], \tag{15}$$

and hence

$$\begin{aligned}\mathrm{ch}(I^*_{\lambda_1,0,0,0}) = \frac{1}{\Pi y}\Big[&x_1^2(-x_2^{-\lambda_1+1}x_3+x_2x_3^{-\lambda_1+1})+x_2^2(-x_3^{-\lambda_1+1}x_1+x_3x_1^{-\lambda_1+1})\\ &+x_3^2(-x_1^{-\lambda_1+1}x_2+x_1x_2^{-\lambda_1+1})+x_1^2y(-x_2^{-\lambda_1}x_3+x_2x_3^{-\lambda_1})\\ &+x^2y(-x_3^{-\lambda_1}x_1+x_3x_1^{-\lambda_1})+x_3^2y(-x_1^{-\lambda_1}x_2+x_1x_2^{-\lambda_1})\Big].\end{aligned}$$

Therefore $I^*_{\lambda_1,0,0,0}$ has highest weight $(0,0,-\lambda_1+1|1)$. Thus we have

$$\mathrm{ch}(I_{\lambda_1,0,0,0}) = \mathrm{ch}(V(\lambda_1,0,0|0)),$$
$$\mathrm{ch}(I^*_{\lambda_1,0,0,0}) = \mathrm{ch}(V(0,0,-\lambda_1+1|1)).$$

4.2. **Construct representations by using the Koszul complex K.** Consider complexes K_a, with $a := k-l \neq 2$:

$$K_a : \cdots \longrightarrow \Lambda_k.S_l^* \longrightarrow \Lambda_{k+1}.S_{l+1}^* \longrightarrow \Lambda_{k+2}.S_{l+2}^* \longrightarrow \cdots .$$

By using the exactness property of the Koszul complex K, we will construct a class of irreducible representations of $\mathfrak{gl}(3|1)$. According to (6) we have

$$\Lambda_k.S_l^* \cong \mathrm{Im}d_{k-1,l-1} \oplus \mathrm{Im}d_{k,l}. \tag{16}$$

Consequently, we have (see [3])

Proposition 4.1. *The module* $\mathrm{Im}d_{k+1,l+1}$ *is simple for all pairs* (k,l) *with* $l,k \geq 1, k-l \neq 2$.

Using induction, we find that

$$\mathrm{ch}(\mathrm{Im}d_{k,l}) = \frac{Ry^{k-3}}{\Pi(x_1x_2x_3)^l}a(l,l,0). \tag{17}$$

Set $M^{m,p} := \mathrm{Im}d_{m+2,m+p}.H_{3,1}^{\otimes m-1}$ with $H_{3,1} := \mathrm{Ker}d_{3,1}/\mathrm{Im}d_{2,0}$. We have

$$\mathrm{ch}(M^{m,p}) = \frac{R}{\Pi(x_1x_2x_3)^{p+1}}a(m+p,m+p,0) = \mathrm{ch}(V(m,m,-p|0)).$$

Hence

$$\mathrm{ch}(M^{m,p})^* = \frac{R(x_1x_2x_3)^{-m}}{\Pi y^3}a(m+p,0,0) = \mathrm{ch}V(p+1,-m+1,-m+1|3).$$

Therefore M is isomorphic to $V(m,m,-p|0)$, and M^* is isomorphic to $V(p+1, -m+1,-m+1|3)$.

Thus every irreducible representation with highest weight in the set

$$\{(m,n,p|-q),(-p,-n,-m|q) : (m,n,p,q) \in \Gamma_{3,1}\} \cup \{(m,m,-p|0),(p,-m,-m|0) : m,p \geq 1\}$$

is constructed.

It remains to construct representations with highest weights in the set

$$\{(n,0,-p|0) : n,p \geq 1\} \cup \{(m+a,m,-p|,0) : m,a,p \geq 1\}.$$

4.3. **Construct representations by using the double Koszul complex.** According to Prop. 3.1, there exists Y such that $S_n \cdot S_p^* = S_{n-1} \cdot S_{p-1}^* \oplus Y$. It is easy to compute for $n, p \geq 1$ that

$$\mathrm{ch}(Y) = \frac{(x_1x_2x_3)R}{\Pi y}\Big[\frac{x_2^{-p-1}x_3^n - x_2^n x_3^{-p-1}}{x_1+y} + \frac{x_3^{-p-1}x_1^n - x_3^n x_1^{-p-1}}{x_2+y} + \frac{x_1^{-p-1}x_2^n - x_1^n x_2^{-p-1}}{x_3+y}\Big]. \tag{18}$$

Hence, Y has highest weight $(n, 0, -p+1|1)$.

Next, we will construct representations having highest weights in the set $\{\lambda = (m+t, m, -p, 0) : m, p, t \geq 1\}$. According to Proposition 3.2, we have

$$S_1 \cdot \mathrm{Im} d_{2,m+1} = \Lambda_3 \cdot S_{m+1}^* \oplus Z_1;$$

hence

$$\mathrm{ch}(Z_1) = \frac{R}{\Pi y (x_1x_2x_3)^{m+1}} a(m+2, m+1, 0) = \mathrm{ch}(V(2, 1, -m+1|1)). \tag{19}$$

Therefore Z_1 is isomorphic to $V(2, 1, -m+1|1)$.

In general, according to Proposition 3.2, we have $\mathrm{Im}(\mathrm{id}_{I_{k,0,0,0}} \cdot d_{l,m}) = \mathrm{Ker} P_{k,l} \cdot S_m^* \oplus Z_k$, where $\mathrm{Ker} P_{k,l} \cong I_{k,1^l}$. Therefore $\mathrm{ch}(Z_k) = \mathrm{ch}[\mathrm{Im}(\mathrm{id}_{I_{k,0,0,0}} \cdot d_{l,m})] - \mathrm{ch}(I_{k,1^l} \cdot S_m^*)$.

According to (14), (15) and (17), we have

$$\mathrm{ch} Z_k = \frac{R(x_1x_2x_3)^{-m} y^{l-3}}{\Pi} a(k+m, m-1, 0). \tag{20}$$

Set $M := Z_k \cdot I_{1,1,1,-1}^{\otimes(l-2)}$. Then

$$\mathrm{ch}(M) = \frac{R(x_1x_2x_3)^{-p}}{\Pi y} a(m+p+t-1, m+p-1, 0) = \mathrm{ch}(V(m+t, m, -p+1|1)), \tag{21}$$

where $t := l-1, p := -m-2+l$. According to (21), we have

$$\mathrm{ch}(M^*) = \frac{R(x_1x_2x_3)^{-m-a}}{\Pi y^2} a(m+t+p-1, t, 0) = \mathrm{ch}(V(p+1, -m+1, -m-t+1|3)). \tag{22}$$

Therefore M is isomorphic to $V(m+t, m, -p+1|1)$ and M^* is isomorphic to $V(p+1, -m+1, -m-t+1|3)$.

Thus, for any integrable dominant weight $\lambda = (\lambda_1, \lambda_2, \lambda_3|\lambda_4)$, we have constructed a representation which has highest weight λ and has character equal to the character of the irreducible representation with highest weight λ.

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