LINEAR SERIES ON RIBBONS

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Abstract. A ribbon is a double structure on $\mathbb{P}^1$. The geometry of a ribbon is closely related to that of a smooth curve. In this paper we consider linear series on ribbons. Our main result is an explicit determinantal description for the locus $W_{2n}^r$ of degree $2n$ line bundles with at least $(r + 1)$-dimensional sections on a ribbon. We also discuss some results of Clifford and Brill-Noether type.

1. Introduction

In this section, we recall some basic theory of ribbons. In the literature a ribbon is also called a Fossum-Ferrand doubling structure. Here we will mainly follow Bayer-Eisenbud \cite{BE} for the related terminology. Many results and much of the notation below come from their paper.

We work over an algebraically closed field $k$ of characteristic 0. A ribbon on $\mathbb{P}^1$ is a scheme $C$ equipped with an isomorphism $\mathbb{P}^1 \to C_{\text{red}}$, such that the ideal sheaf $\mathcal{L}$ of $\mathbb{P}^1$ in $C$ satisfies $\mathcal{L}^2 = 0$.

Because of this condition, $\mathcal{L}$ can be regarded as a line bundle on $\mathbb{P}^1$. It is called the conormal bundle of $\mathbb{P}^1$ in $C$. There is a short exact sequence called the conormal sequence:

$$0 \to \mathcal{L} \to \mathcal{O}_C \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$

Define the arithmetic genus $g$ of a ribbon $C$ as

$$g = 1 - \chi(\mathcal{O}_C).$$

From the conormal sequence, we see that $C$ has genus $g$ if and only if the conormal bundle $\mathcal{L}$ on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-g - 1)$.

There is another short exact sequence called the restricted cotangent sequence:

$$0 \to \mathcal{L} \to \Omega_{C|\mathbb{P}^1} \to \Omega_{\mathbb{P}^1} \to 0.$$

This restricted cotangent sequence defines the extension class of $C$:

$$e_c \in \text{Ext}^1_{\mathbb{P}^1}(\Omega_{\mathbb{P}^1}, \mathcal{L}).$$

We will say that two ribbons are isomorphic over $\mathbb{P}^1$ if there is an isomorphism between them that extends the identity on $\mathbb{P}^1$. A ribbon $C$ is split if the inclusion $\mathbb{P}^1 \hookrightarrow C$ admits a section. Such a section is a scheme-theoretically degree two map from $C$ to $\mathbb{P}^1$. We also call $C$ hyperelliptic if it is split.

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Theorem 1.1. Given any line bundle $\mathcal{L}$ on $\mathbb{P}^1$ and any class $e \in \operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{L})$, there is a unique ribbon $C$ on $\mathbb{P}^1$ with $e_C = e$. If there is another class $e' \in \operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, \mathcal{L})$ corresponding to a ribbon $C'$, then $C \cong C'$ if and only if $e = ae'$ for some $a \in k^*$. A hyperelliptic ribbon corresponds to the split extension. The set of nonhyperelliptic ribbons of genus $g$, up to isomorphism over $\mathbb{P}^1$, is the set
\[ \mathbb{P}^{g-3} = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(g - 3))). \]

Proof. The above results are essentially from [BE, Thm. 1.2, 2.1].

There is an explicit way to write down the structure of a ribbon by a gluing method; cf. [BE Sec. 3]. Define two open sets
\[ u_1 = \operatorname{Spec} k[s], \quad u_2 = \operatorname{Spec} k[t] \]
that cover $\mathbb{P}^1$ via the identification $s^{-1} = t$ on $u_1 \cap u_2$.

If $C$ is a genus $g$ ribbon on $\mathbb{P}^1$, we can write
\[ U_1 := C|_{u_1} \cong \operatorname{Spec} k[s, \epsilon]/\epsilon^2, \]
\[ U_2 := C|_{u_2} \cong \operatorname{Spec} k[t, \eta]/\eta^2. \]

$C$ may be specified by a gluing isomorphism between $U_1$ and $U_2$ over $u_1 \cap u_2$. The ideal sheaf $\mathcal{I} \cong \mathcal{O}_{\mathbb{P}^1}(-g - 1)$ of $\mathbb{P}^1$ in $C$ is generated by $\epsilon$ on $u_1$ and by $\eta$ on $u_2$. So we can further write
\[ \epsilon = t^{-g-1}\eta, \]
\[ s^{-1} = t + F(t)\eta \]
on $u_1 \cap u_2$, with $F(t) \in k[t, t^{-1}]$. Conversely, any such gluing data can determine a ribbon of genus $g$ on $\mathbb{P}^1$.

If we change the coordinates with
\[ s' = s + p(s)\epsilon, \quad t = t' + q(t)\eta \]
on $U_1$ and $U_2$ with polynomials $p(s), q(t)$, then we get
\[ s'^{-1} = s^{-1} - s^{-2}p(s) \]
\[ = t' + \left( F(t) - t^{-g+1}p(t^{-1}) - F(t)\eta + q(t)\right)\eta \]
\[ = t + F(t)\eta - (t + F(t)\eta)^2p(t^{-1})t^{-g-1}\eta \]
\[ = t + F(t)\eta - t^{1-g}p(t^{-1})\eta \]
\[ = t' + \left( F(t) + q(t) - t^{1-g}p(t^{-1})\right)\eta \]
\[ = t' + \left( F(t') + q(t') - t^{1-g}p(t'^{-1})\right)\eta. \]
The fact that $t\eta = t'\eta$ is used in the last step. If we multiply $s$ or $t$ by a scalar, $F$ will also be multiplied by the same scalar. Therefore, $F$ can be determined as an element of the projective space of lines in the quotient
\[ k[t, t^{-1}]/(k[t] + t^{-g+1}k[t^{-1}]). \]
From now on, we shall write $F$ as
\[ F = \sum_{i=1}^{g-2} F_i t^{-i}. \]
A line bundle $L$ is defined as

$$\deg L := \chi(L) - \chi(\mathcal{O}_C).$$

**Proposition 1.2.** If $L|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(n)$, then $\deg L = 2n$. The Picard group of $C$ is

$$\text{Pic } C = H^1(\mathcal{O}_{\mathbb{P}^1}(-g - 1)) \times \mathbb{Z} \cong k^g \times \mathbb{Z},$$

where the projection to $\mathbb{Z}$ is given by the degree of the restriction $L|_{\mathbb{P}^1}$.

**Proof.** See [BE] Props. 4.1, 4.2. \qed

Bayer and Eisenbud remarked that one must switch to torsion-free sheaves in order to obtain the analogue of line bundles of odd degree. For simplicity, here we only consider line bundles of even degree $2n$.

A line bundle $L$ can also be constructed by gluing. Suppose $L|_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(n)$. Using the above notation, we have

$$L|_{U_1} = k[s, \epsilon]e_1,$$

$$L|_{U_2} = k[t, \eta]e_2,$$

and

$$e_1 = (t + F\eta)^n(1 + G\eta)e_2$$
on $U_1 \cap U_2$ for some $G \in k[t, t^{-1}]$. Conversely, any such $G$ can be used to construct a line bundle.

If we change the coordinates by

$$e_1 = (1 + m(s)\epsilon)^{-1}e_1', e_2 = (1 + n(t)\eta)e_2'$$
on $U_1$ and $U_2$ with polynomials $m(s), n(t)$, then we get

$$e_1' = (t + F\eta)^n(1 + (G - m(t^{-1})t^{-g-1} + n(t))\eta)e_2'.$$

In order to classify $L$, it suffices to specify $G$ as an element of

$$k[t, t^{-1}]/(k[t] + t^{-g-1}k[t^{-1}]) = H^1(\mathcal{O}_{\mathbb{P}^1}(-g - 1)).$$

This also recovers the fact in Proposition 1.2 that $H^1(\mathcal{O}_{\mathbb{P}^1}(-g - 1))$ parameterizes line bundles of fixed degree. We will also write $G$ as

$$G = \sum_{j=1}^{g} G_j t^{-j}. \quad (2)$$

Let $L$ be a line bundle on $C$ of degree $2n$ given by the above gluing data. We would like to find out the space of global sections of $L$. Suppose $p = p(t)$ is a polynomial of degree $\leq n$. Then $pe_{2|\mathbb{P}^1}$ determines an element

$$\sigma \in H^0(L|_{\mathbb{P}^1}) = H^0(\mathcal{O}_{\mathbb{P}^1}(n)).$$

Define

$$\delta_L(p) = -(p'F + pG) \in k[t, t^{-1}]/(k[t] + t^{n-g-1}k[t^{-1}]), \quad (3)$$

where $p' = \frac{\partial p}{\partial t}$. 

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Theorem 1.3. The space of sections of $L$ restricted to $U_2 = \text{Spec } k[t, \eta]$ can be identified as the direct sum of the space of elements $q(t)\eta$ and the space of expressions $p(t) + p_1(t)\eta$, where $q$ is a polynomial of degree $\leq n - g - 1$, $p$ is a polynomial of degree $\leq n$ satisfying $\delta_L(p) = 0$ in $\mathbb{R}$, and $p_1 \in k[t]$ is the polynomial part of $p^*F + pG$, i.e. $p_1(t) \equiv p'(t)F(t) + p(t)G(t) \mod t^{-1}k[t^{-1}]$.

Proof. This is exactly [BR Thm. 4.3].

At first glance, the above way to identify $H^0(L)$ seems quite messy. Nevertheless, a further observation will imply an important conclusion immediately.

Corollary 1.4. Let $L$ be a line bundle of degree $2n$ on a ribbon $C$. If $n \geq g$, then $h^0(L) = 1 - g + 2n$.

Proof. Let a section of $L$ restricted to $U_2$ correspond to the data $(q(t)\eta, p(t) + p_1(t))$ in the above setup. When $n \geq g$, we have $k[t, t^{-1}]/(k[t] + t^{n-g-1}k[t^{-1}]) = 0$. Then by its definition, $\delta_L$ always takes the value of 0. So $\delta_L(p) = 0$ does not impose any condition on $p(t)$. The only constraint on $p(t)$ and $q(t)$ is the upper bound of their degree. Here $q(t)$ has degree $\leq n - g - 1$ and $p(t)$ has degree $\leq n$. In total, they yield a $(1 - g + 2n)$-dimensional space for sections of $L$.

Remark 1.5. Notice that if $n \geq g$, then the degree $d$ of $L$ satisfies $d = 2n > 2g - 2$. In case $C$ is a smooth curve of genus $g$, $h^0(L) = 1 - g + d$ holds for any line bundle $L$ on $C$ with degree $d > 2g - 2$. So the above corollary can be viewed as a similarity between ribbons and smooth curves.

2. The locus $W^r_{2n}$

For smooth curves, the theory of special linear series can be best characterized by the Brill-Noether theory. We refer readers to [ACGH Chap. V] for bibliographical notes on this topic. Let $C$ be a curve of genus $g$. We introduce the variety $W^r_d(C)$ as

$$W^r_d(C) = \{L \in \text{Pic}^d(C) \mid h^0(C, L) \geq r + 1\}.$$

We also define the Brill-Noether number $\rho$ as

$$\rho = g - (r + 1)(g - d + r).$$

The basic results of the Brill-Noether theory can be summarized as follows.

Theorem 2.1. Let $C$ be a smooth curve of genus $g$. Let $d, r$ be integers such that $d \geq 1, r \geq 0$.

(Existence) If $\rho \geq 0$, $W^r_d(C)$ is non-empty. Furthermore, every component of $W^r_d(C)$ has dimension at least equal to $\rho$ provided $r \geq d - g$.

(Connectedness) Assume that $\rho \geq 1$. Then $W^r_d(C)$ is connected.

(Dimension) Assume that $C$ is a general curve. If $\rho < 0$, then $W^r_d(C)$ is empty. If $\rho \geq 0$, then $W^r_d(C)$ is reduced and of dimension $\rho$.

We would like to investigate linear series for ribbons. The importance of such a study is three-fold. In the first place, $W^r_d(C)$ essentially carries a determinantal structure for a smooth curve $C$. In case $C$ is a ribbon, the determinantal characterization can even be made explicit. Secondly, the Brill-Noether theory for a special member in a family of curves usually reveals information for a general one. Ribbons do arise as the degeneration of smooth curves; cf. [F]. Finally, Lazarsfeld [L] proved that a general curve contained in certain K3 surfaces satisfies the above dimension
some results of Brill-Noether type for ribbons.

Let $C$ be a ribbon determined by $[F_1, \ldots, F_{g-2}]$, the coefficients of $F$ in (1), up to a scalar. Let $L$ be a line bundle of degree $2n$ on $C$ determined by $(G_1, \ldots, G_g)$, the coefficients of $G$ in (2). If $g \geq n$, there is no special linear system because of Corollary 1.4. Actually we only need to consider $2n \leq g - 1$, since the Riemann-Roch formula also holds for ribbons; cf. [BE, Sec. 5]. From now on, assume that $2n \leq g - 1$. Define a $(g - n) \times (n + 1)$ matrix $A_F(G)$ with entries $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}$, namely,

$$A_F(G) = \begin{pmatrix} G_1 & G_2 + F_1 & \cdots & G_{n+1} + nF_n \\ G_2 & G_3 + F_2 & \cdots & G_{n+2} + nF_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ G_{g-n} & G_{g-n+1} + F_{g-n} & \cdots & G_g \end{pmatrix}.$$ 

Now we can state our main result.

**Theorem 2.2.** In the above setting, the locus $W_{2n}^r(C)$ is isomorphic to the following affine algebraic set:

$$W_{2n}^r(C) = \{(G_1, \ldots, G_g) \in \mathbb{A}^g \mid \text{rank} A_F(G) \leq n - r\}.$$ 

**Proof.** By Theorem 1.3 the space $H^0(L)$ of global sections of $L$ can be identified as the direct sum of two spaces:

$$\langle q(t)\eta \rangle \oplus \langle p(t) + p_1(t)\eta \rangle.$$ 

Since $q(t)$ is a polynomial of degree $\leq n - g - 1$ and $n \leq g - 2$, the first summand is the null space. For the second, $p(t)$ is a polynomial of degree $\leq n$ satisfying $\delta_L(p) = 0$ as in [K. Then $p_1(t)$ is determined by $p(t)$, the polynomial part of $p^*F + pG$. Let $p(t) = \sum_{i=0}^n a_i t^i$. The condition $\delta_L(p) = 0$ means

$$p^*F + pG \in k[t] + t^{n-g-2}k[t^{-1}],$$

which is equivalent to the following:

$$A_F(G) \cdot \vec{a} = 0,$$

where $\vec{a}$ is the vector $(a_0, \ldots, a_n)^t$ determined by the coefficients of $p(t)$. Note that $g - n \geq n + 1$ by the assumption on $n$. Hence, $W_{2n}^r(C)$ can be identified as the desired determinantal locus. $\square$

The following Clifford theorem for ribbons is a direct consequence of Theorem 2.2.

**Theorem 2.3.** Let $C$ be a ribbon and let $L$ be a line bundle of degree $2n$ on $C$, $1 \leq n \leq g - 2$. Then $h^0(C, L) \leq n + 1$. The equality holds if and only if $C$ is a hyperelliptic ribbon and $L$ is the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ from $C_{\text{red}} \cong \mathbb{P}^1$ to $C$.

**Proof.** By the above determinantal description for $W_{2n}^r(C)$, we know that $h^0(C, L) \leq n + 1$. If the equality holds, then $r = n$. We have $G_i = 0$ and $F_j = 0$ for all $i, j$. Thus $C$ is hyperelliptic and $L$ is isomorphic to the pullback of $\mathcal{O}_{\mathbb{P}^1}(1)$ from $C_{\text{red}} \cong \mathbb{P}^1$. $\square$
When $C$ is a hyperelliptic ribbon, i.e. $F_i = 0$ for all $i$, $A_F(G)$ has entries $s_{ij} = G_{i+j-1}$:

$$
\begin{pmatrix}
G_1 & G_2 & \cdots & G_{n+1} \\
G_2 & G_3 & \cdots & G_{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
G_{g-n} & G_{g-n+1} & \cdots & G_g
\end{pmatrix}
$$

Such a matrix is called the catalecticant matrix. We cite the result [E, Prop. 4.3] as follows.

**Proposition 2.4.** The space of rank $\leq m$ catalecticant matrices is isomorphic to a cone over $S_m$, where $S_m$ is the union of $m$-secant $(m-1)$-planes to a rational normal curve of degree $g-1$.

This exactly describes $W_{2n}^r(C)$ for a hyperelliptic ribbon.

**Theorem 2.5.** Let $C$ be a hyperelliptic ribbon. Then $W_{2n}^r(C)$ is isomorphic to a cone over $S_{n-r}$ for $r < n$. In particular, $W_{2n}^r(C)$ has dimension equal to $2n-2r$.

**Proof.** $W_{2n}^r(C)$ can be identified as the space of rank $\leq n-r$ catalecticant matrices, which is isomorphic to a cone over $S_{n-r}$ by Proposition 2.4. Also, $S_{n-r}$ has dimension $2n-2r-1$, so $W_{2n}^r(C)$ has dimension $2n-2r$. \qed

We have seen that for a nonhyperelliptic ribbon, its structure can be determined by the data $[F_1, \ldots, F_{g-2}]$ in [1] as a point of $\mathbb{P}^{g-3}$. Note that the expected dimension of $W_{2n}^r(C)$ would still be $g-(r+1)(g-2n+r)$, which equals the Brill-Noether number $\rho$. We would like to study the actual dimension of $W_{2n}^r(C)$. First, let us focus on a natural compactification of $W_{2n}^r(C)$ as follows.

Define another $(g-n) \times (n+1)$ matrix $\overline{A}_F(G)$ with entries $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}G_0$:

$$
\overline{A}_F(G) = \begin{pmatrix}
G_1 & G_2 + F_1G_0 & \cdots & G_{n+1} + nF_nG_0 \\
G_2 & G_3 + F_2G_0 & \cdots & G_{n+2} + nF_{n+1}G_0 \\
\vdots & \vdots & \ddots & \vdots \\
G_{g-n} & G_{g-n+1} + F_{g-n}G_0 & \cdots & G_g
\end{pmatrix}
$$

Let

$$
\overline{W}_{2n}^r = \{ [G_0, G_1, \ldots, G_g] \in \mathbb{P}^g \mid \text{rank } \overline{A}_F(G) \leq n-r \}.
$$

There is an inclusion $W_{2n}^r(C) \subset \overline{W}_{2n}^r$ given by

$$
(G_1, \ldots, G_g) \rightarrow [1, G_1, \ldots, G_g].
$$

The complement of $W_{2n}^r(C)$ in $\overline{W}_{2n}^r$ is just the hyperplane section $\{G_0 = 0\} \cap \overline{W}_{2n}^r(C)$.

We also need to introduce generic determinantal varieties. Let $M$ be the space of $(g-n) \times (n+1)$ matrices. Denote by $M_l$ the locus of rank $\leq l$ matrices. Here $M_l$ is called the $l$-generic determinantal variety, $l \leq n+1$. Denote by $M$ and $M_l$ the projectivization of $M$ and $M_l$ respectively.

**Proposition 2.6.** $M_l$ is an irreducible subvariety of codimension $(g-n-l)(n+1-l)$ in $M$.

One can refer to [ACGH] Chap. II for a general discussion on determinantal varieties. Our next result is about the global geometry of $\overline{W}_{2n}^r(C)$.
Theorem 2.7. Let \( C \) be a ribbon of genus \( g \). For \( r < n \), \( W_{2n}^r(\mathcal{C}) \) is always non-empty and has dimension equal to \( 2n-2r-1 \) or \( 2n-2r \). Each irreducible component of \( W_{2n}^r(\mathcal{C}) \) has dimension at least equal to \( \rho \) provided \( \rho \geq 0 \). Furthermore, \( W_{2n}^r(\mathcal{C}) \) is connected provided \( \rho > 0 \).

Proof. \( W_{2n}^r(\mathcal{C}) \) is the intersection of \( M_{n-r} \) and a \( g \)-dimensional linear subspace of \( M \) determined by \( s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}G_0 \). Therefore, each irreducible component of \( W_{2n}^r(\mathcal{C}) \) has dimension \( \geq \dim M_{n-r} + g - \dim M = \rho \).

When \( G_0 = 0 \), the matrix \( \mathcal{A}_F(G) \) reduces to a catalecticant matrix with entries \( s_{ij} = G_{i+j-1} \). The space of rank \( \leq n-r \) catalecticant matrices has dimension \( 2n-2r \) by Proposition 2.3. It implies that the hyperplane section \( \{ G_0 = 0 \} \cap W_{2n}^r(\mathcal{C}) \) has dimension \( 2n-2r-1 \). If the top dimensional component of \( W_{2n}^r(\mathcal{C}) \) is contained in \( \{ G_0 = 0 \} \), then \( W_{2n}^r(\mathcal{C}) \) has dimension \( 2n-2r-1 \). Otherwise it has dimension \( 2n-2r \).

\( W_{2n}^r(\mathcal{C}) \) can also be regarded as the intersection of the \((n-r)\)-generic determinantal variety \( M_{n-r} \) and a \( g \)-dimensional linear subspace of \( M \) defined by \( s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}G_0 \) for a fixed lifting \((F_1, \ldots, F_{g-2})\). If \( \rho > 0 \), the sum of the dimensions of these two spaces is greater than the dimension of \( M \). The connectedness of their intersection \( W_{2n}^r(\mathcal{C}) \) follows as a consequence of \([L, Ex. 3.3.7]\).

Corollary 2.8. Assume that \( \rho \geq 0 \). If \( W_{2n}^r(\mathcal{C}) \) is non-empty for a ribbon \( C \), then \( W_{2n}^r(\mathcal{C}) \) has dimension at least equal to \( \rho \).

Proof. \( W_{2n}^r(\mathcal{C}) \) is the complement of the hyperplane section \( \{ G_0 = 0 \} \cap W_{2n}^r(\mathcal{C}) \) in \( W_{2n}(\mathcal{C}) \). By Theorem 2.7 we know that each component of \( W_{2n}^r(\mathcal{C}) \) has dimension \( \geq \rho \). So does \( W_{2n}^r(\mathcal{C}) \), assuming it is non-empty.

We can also let \((F_1, \ldots, F_{g-2})\) vary as a point of \( \mathbb{A}^{g-2} \) and define the global Brill-Noether locus \( W_{2n}^r \) as follows:

\[ W_{2n} = \{(G_1, \ldots, G_g; F_1, \ldots, F_{g-2}) \in \mathbb{A}^g \times \mathbb{A}^{g-2} \mid \text{rank } \mathcal{A}_F(G) \leq n-r \}. \]

\( W_{2n}^r \) is the intersection of the \((n-r)\)-generic determinantal variety \( M_{n-r} \) and a \((2g-2)\)-dimensional linear subspace \( S \) of \( M \), where \( S \) is determined by relations \( 2s_{ij} = s_{i-1,j+1} + s_{i+1,j-1} \). Note that the expected dimension of \( W_{2n}^r \) would be \( 2g-2 - (g-2n+r)(r+1) = g-2+\rho \), which implies the following conclusion right away.

Corollary 2.9. If \( W_{2n}^r \) has dimension equal to \( g-2+\rho \), then for \((F_1, \ldots, F_{g-2})\) corresponding to a general ribbon \( C \), \( W_{2n}^r(\mathcal{C}) \) has dimension at most equal to \( \rho \).

In order to calculate the actual dimension of \( W_{2n}^r \), we introduce the concept of \( l \)-generic spaces developed by Eisenbud \([E, Prop.-Def. 1.1]\).

Definition 2.10. Let \( L \) be a linear subspace of the space \( M \) of \((g-n) \times (n+1)\) matrices. Here \( L \) can be regarded as an associated \((g-n) \times (n+1)\) matrix of linear forms. We say that \( L \) is \( m \)-generic for some integer \( 1 \leq m \leq n+1 \) if after arbitrary invertible row and column operations, any \( m \) of the linear forms \( L_{ij} \) in \( L \) are linearly independent.

We also say that \( L \) meets \( M_l \) properly if their intersection has codimension equal to \((g-n-l)(n+1-l)\) in \( L \).
Theorem 2.11. Let $L \subset M$ be an $m$-generic space; then $L$ meets $M_{n+1-m}$ properly.

Proof. This is part of \[E\,\text{Thm. 2.1}].

Note that the space of catalecticant matrices is 1-generic. One can also prove the 2-genericity for the space $S$ of matrices of type $A_F(G)$.

Proposition 2.12. Consider $G_1, \ldots, G_g; F_1, \ldots, F_{g-2}$ as independent linear forms. The $(2g - 2)$-dimensional vector space $S$ of all matrices determined by $A_F(G)$ is 2-generic.

Proof. $A_F(G)$ is the matrix with entries $s_{ij} = G_{i+j-1} + (j-1)F_{i+j-2}$. Suppose there were two invertible matrices $A = (a_{ij})$ and $B = (b_{ij})$ corresponding to invertible row and column operations such that two entries of the new matrix $A \cdot A_F(G) \cdot B = (s'_{ij})$ became equal to each other. We can always assume that these two entries are $s'_{i1}$ and $s'_{22}$. The case that they are in the same row or column would be even easier.

Then the condition $s'_{i1} = s'_{22}$ is equivalent to

$$\sum_{i+j=k+1} (a_{i1}b_{j1} - a_{2j}b_{j2})(G_k + (j-1)F_{k-1}) = 0$$

for any $k$. Namely,

$$\sum_{i+j=k+1} (a_{i1}b_{j1} - a_{2j}b_{j2}) = 0 \text{ and } \sum_{i+j=k+1} (a_{i1}b_{j1} - a_{2j}b_{j2})(j-1) = 0$$

since $G_i$ and $F_j$ can vary independently.

Define four polynomials as follows:

$$A_k(x) = \sum_i a_{ki}x^i \text{ and } B_k(x) = \sum_j b_{kj}x^j \text{ for } k = 1, 2.$$  

We can deduce from the above two equalities that

$$A_1(x)B_1(x) = A_2(x)B_2(x) \text{ and } A_1(x)B'_1(x) = A_2(x)B'_2(x).$$

Since the matrices $A$ and $B$ are invertible, by these two relations we can get

$$B_1(x)B'_2(x) = B_2(x)B'_1(x),$$

which would imply that $(B_1(x)/B_2(x))' = 0$. Then $B_1(x)/B_2(x)$ would be a constant, which contradicts the assumption that the matrix $B$ is invertible. \qed

Corollary 2.13. For $r = 1$, $W_{2n}^1$ has dimension $4n - 4$ and $W_{2n}^1(C)$ has dimension at most equal to $\rho = 4n - g - 2$ for a general ribbon $C$, provided $\rho \geq 0$.

Proof. Since the space of matrices $A_F(G)$ is 2-generic, it intersects $M_{n-1}$ properly. So the intersection $W_{2n}^1$ has dimension equal to $g-2+\rho = 4n-4$ by Theorem 2.11. Then by Corollary 2.3, $W_{2n}^1(C)$ has dimension at most equal to $\rho = 4n - 2 - g$. \qed

Based on Corollaries 2.8 and 2.13 we obtain the following conclusion.

Corollary 2.14. For $r = 1$, if $W_{2n}^1(C)$ is non-empty for a general ribbon $C$, then $W_{2n}^1(C)$ has dimension equal to $\rho = 4n - 2 - g$ provided $\rho \geq 0$.

It would be interesting to pin down the following question in general.
**Question 2.15.** For $\rho \geq 0$, is the dimension of $W_{2n}^r$ equal to the expected dimension $g - 2 + \rho$? For a general ribbon $C$, is the locus $W_{2n}^r(C)$ non-empty and does it have dimension equal to $\rho$ provided $\rho \geq 0$?

By the determinantal descriptions for $W_{2n}^r$ and $W_{2n}^r(C)$, using Macaulay one can check that the above question does have a positive answer when the genus of $C$ is small.

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**References**


