A NOTE ON COMPLETE RESOLUTIONS

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Abstract. It is shown that the Eckmann-Shapiro Lemma holds for complete cohomology if and only if complete cohomology can be calculated using complete resolutions. It is also shown that for an \( LH \) group \( G \) the kernels in a complete resolution of a \( ZG \)-module coincide with Benson’s class of cofibrant modules.

1. Introduction

Let \( G \) be a group and \( ZG \) its integral group ring. A \( ZG \)-module \( M \) is said to admit a complete resolution \(( F, P, n )\) of coincidence index \( n \) if there is an acyclic complex \( F = \{(F_i, \vartheta_i) | i \in \mathbb{Z} \} \) of projective modules and a projective resolution \( P = \{(P_i, d_i) | i \in \mathbb{Z}, i \geq 0 \} \) of \( M \) such that \( F \) and \( P \) coincide in dimensions greater than \( n \); that is,

\[
\begin{align*}
F : & \cdots \to F_{n+1} \to F_n \xrightarrow{\partial_n} F_{n-1} \to \cdots \to F_0 \to F_{-1} \to F_{-2} \to \cdots \\
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P : & \cdots \to P_{n+1} \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to M \to 0
\end{align*}
\]

A \( ZG \)-module \( M \) is said to admit a complete resolution in the strong sense if there is a complete resolution \(( F, P, n )\) with \( \text{Hom}_{ZG}(F, Q) \) acyclic for every \( ZG \)-projective module \( Q \).

It was shown by Cornick and Kropholler in \cite{7} that if \( M \) admits a complete resolution \(( F, P, n )\) in the strong sense, then

\[
\widehat{\text{Ext}}_{ZG}(M, B) \simeq H^*(\text{Hom}_{ZG}(F, B))
\]

where \( \widehat{\text{Ext}}^*_{ZG}(M, \_ \) is the \( P \)-completion of \( \text{Ext}^*_{ZG}(M, \_ \) , defined by Mislin for any group \( G \) \cite{13} as

\[
\widehat{\text{Ext}}^k_{ZG}(M, B) = \lim_{r \to k} S_{-r}^m \text{Ext}^r_{ZG}(M, B)
\]

where \( S_{-m}^T \) is the \( m \)-th left satellite of a functor \( T \).

Alternative but equivalent definitions of the complete \( \widehat{\text{Ext}} \)-groups were given by Benson and Carlson \cite{5} and Vogel \cite{9}.
Complete cohomology $\hat{H}^*(G, \_)$ is defined as $\hat{\text{Ext}}_{Z^G}^*(Z, \_)$, where $Z$ is the trivial $ZG$-module, generalizing the Tate cohomology defined for finite groups and the Farrell-Tate cohomology defined for groups that have a finite-index subgroup of finite cohomological dimension.

A group $G$ is said to admit a complete resolution if the trivial $ZG$-module $Z$ admits a complete resolution.

It turns out that $G$ admits a complete resolution in the strong sense if and only if the generalized cohomological dimension $\text{cd}_G$ is finite [3], where

$$\text{cd}_G = \sup \{ n \in \mathbb{N} \mid \exists M \text{-free, } \exists F \text{-free : } \text{Ext}_{ZG}^n(M, F) \neq 0 \}$$

was defined by Ikenaga in his study of generalized Farrell-Tate cohomology in [10].

Note that $\text{cd}_G = \text{Gcd}_2 G$ [3], the Gorenstein projective dimension of $G$, which is defined via resolutions of the trivial $ZG$-module $Z$ by Gorenstein projective modules and is related to the $G$-dimension defined by Auslander in [1] (see also [2] and [6]).

A $ZG$-module $M$ is said to be Gorenstein projective if it admits a complete resolution in the strong sense of coincidence index 0, i.e. if $M$ is a kernel in a complete resolution in the strong sense.

Complete resolutions do not always exist; for example, if $G$ has a free abelian subgroup of infinite rank, then $G$ does not admit a complete resolution [14].

If the complete cohomology can be calculated using complete resolutions, then one has certain advantages such as the Eckmann-Shapiro Lemma and certain spectral sequences.

Here we show that the validity of the Eckmann-Shapiro Lemma for complete cohomology actually implies that complete resolutions exist and that complete cohomology can be calculated using complete resolutions:

**Theorem A.** The following are equivalent for a group $G$.

1. The Eckmann-Shapiro Lemma holds for complete cohomology.
2. $G$ has a complete resolution and every complete resolution of $G$ is a complete resolution in the strong sense.
3. Complete cohomology can be calculated using complete resolutions.

We also show the following:

**Theorem B.** If a $ZG$-module $M$ is a kernel in a complete resolution and $A$ is a $ZG$-module that is $ZF$-projective for every finite subgroup $F$ of $G$, then $M \otimes A$ (with diagonal $G$-action) is projective as a $ZH$-module for every $LH\mathbb{F}$-subgroup $H$ of $G$.

Theorem B in particular implies the two results below:

**Corollary C.** If $G$ is an $LH\mathbb{F}$-group, then the kernels of a complete resolution of a $ZG$-module coincide with the cofibrant modules.

**Corollary D.** If $G$ is an $LH\mathbb{F}$-group, then every complete resolution of a $ZG$-module $M$ is a complete resolution in the strong sense.

Cofibrant modules were introduced by Benson in [4] for his study of $ZG$-modules that admit a projective resolution by finitely generated projective modules, when $G$ is an $LH\mathbb{F}$-group. They are defined as follows: for any group $G$, a $ZG$-module $M$ is said to be cofibrant if $M \otimes B(G, \mathbb{Z})$ is a projective $ZG$-module, where $B(G, \mathbb{Z})$ is the set of bounded functions from $G$ to $\mathbb{Z}$. 

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The class $\mathcal{H}_F$ was defined by Kropholler in [11] as the smallest class of groups which contains the class of finite groups and is such that whenever a group $G$ admits a finite dimensional contractible $G$-CW-complex with stabilizers in $\mathcal{H}_F$, we have that $G$ is in $\mathcal{H}_F$. $\mathcal{LH}_F$ is the class of groups $G$ such that every finitely generated subgroup of $G$ is in $\mathcal{H}_F$. The class $\mathcal{LH}_F$ includes, for example, all soluble-by-finite groups and all groups with a faithful representation as endomorphisms of a Noetherian module over a commutative ring, and is extension-closed, closed under ascending unions and closed under amalgamated free products and HNN extensions.

In [3] it was proved that if $G$ is an $\mathcal{LH}_F$-group and $M$ is a $\mathbb{Z}G$-module that admits a projective resolution by finitely generated projective modules, then $M$ is a Gorenstein projective $\mathbb{Z}G$-module if and only if it is a cofibrant module.

Corollary C shows that for $\mathcal{LH}_F$-groups the Gorenstein projective modules coincide with the cofibrant modules. We believe the following to be true:

**Conjecture A.** For any group $G$ the Gorenstein projective modules coincide with the cofibrant modules.

**Conjecture B** (see also Conj. B in [15]). A $\mathbb{Z}G$-module $M$ admits a complete resolution if and only if it admits a complete resolution in the strong sense.

Conjecture B in particular implies that Gorenstein projectivity is a subgroup-closed property. Here we show that this is so if the subgroup is in $\mathcal{LH}_F$ (Corollary 2.2).

2. **Proof of the results**

**Proof of Theorem A.** (1) $\Rightarrow$ (2): Assume that the Eckmann-Shapiro Lemma holds for the complete cohomology of $G$, i.e.

$$\hat{\text{Ext}}^*_{\mathbb{Z}G}(A, \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B)) \simeq \hat{\text{Ext}}^*_{\mathbb{Z}H}(A|_{\mathbb{Z}H}, B)$$

for every $\mathbb{Z}G$-module $A$, every $\mathbb{Z}H$-module $B$ and every subgroup $H \leq G$. Applying the Eckmann-Shapiro Lemma with the trivial subgroup $H = 1$, we have that $\hat{\text{Ext}}^0_{\mathbb{Z}G}(A, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, B)) \simeq \hat{\text{Ext}}^0_{\mathbb{Z}}(A|_\mathbb{Z}, B)$ for every $\mathbb{Z}G$-module $A$ and every $\mathbb{Z}$-module $B$. We have that $\hat{\text{Ext}}^0_{\mathbb{Z}}(A|_\mathbb{Z}, B) = 0$, since $A$ and $B$ have finite projective dimension over $\mathbb{Z}$ [11, 4.2].

It follows that $\hat{\text{Ext}}^0_{\mathbb{Z}G}(A, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)) = 0$ for any $\mathbb{Z}G$-module $A$. For any $\mathbb{Z}G$-module $A$ there is a canonical injection $A \hookrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, A)$. So, if $I$ is injective, then it is a direct summand of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}G, I)$. It follows that $\hat{\text{Ext}}^0_{\mathbb{Z}G}(I, I) = 0$, and thus $I$ has finite projective dimension [11, 4.2]. It is easy to see that the supremum of the projective dimensions of the injective $\mathbb{Z}G$-modules, spil $\mathbb{Z}G$, is finite; hence (2) follows from [8, §4].

(2) $\Rightarrow$ (3) by Theorem 1.2 of [7].

(3) $\Rightarrow$ (1): It follows from the implication (4) $\Rightarrow$ (2) of Theorem 2.2 in [15] and [7] that every $\mathbb{Z}G$-module $M$ admits a complete resolution $(\mathcal{F}, \mathcal{P}, n)$ in the strong sense and $\hat{\text{Ext}}^n_{\mathbb{Z}G}(M, B) \simeq H^n(\text{Hom}_{\mathbb{Z}G}(\mathcal{F}, B))$. Clearly the computation of the complete cohomology via complete resolutions implies the validity of the Eckmann-Shapiro Lemma.

**Lemma 2.1.** Let $G$ be a group.
(a) If $M_i, i \in I$, is a family of $\mathbb{Z}G$-modules that admit complete resolutions (in the strong sense) of coincidence index 0, then the direct sum $\bigoplus_{i \in I} M_i$ admits a complete resolution (in the strong sense) of coincidence index 0.

(b) If $M$ admits a complete resolution of coincidence index $n$ and $A$ is a $\mathbb{Z}$-free $\mathbb{Z}G$-module, then $M \otimes A$ (with diagonal $G$-action) admits a complete resolution of coincidence index $n$.

(c) Let $\{(F_i, \vartheta_i), i \in \mathbb{Z}\}$ be an acyclic complex of projective $\mathbb{Z}G$-modules with kernels $M_i, i \in \mathbb{Z}$. If there is a bound on the projective dimensions of the $M_i$’s, then $M_i$ is projective for every $i \in \mathbb{Z}$.

Proof. The proofs of (a) and (b) are straightforward.

(c) Let $pd_{\mathbb{Z}G}M_r = k > 0$ for some $r \in \mathbb{Z}$. It follows from the short exact sequence $0 \to M_r \to F_r \to M_{r-1} \to 0$ that $pd_{\mathbb{Z}G}M_{r-1} = k + 1$ and thus inductively $pd_{\mathbb{Z}G}M_{r-s} = k + s$ for all $s \in \mathbb{N}$, which is a contradiction since there is a bound on the projective dimensions of the $M_i$’s. □

Proof of Theorem B. We first prove the theorem for $\mathbb{H}_{\beta, \gamma}$-subgroups, using transfinite induction on the ordinal number $\alpha$ such that the subgroup $H$ belongs to $\mathbb{H}_{\alpha, \gamma}$.

For $\alpha = 0$, $M \otimes A$ is projective over any finite subgroup of $G$, because $M$ is $\mathbb{Z}$-free and $A$ is projective over the finite subgroups of $G$. Assume that the result is true for all $\mathbb{H}_{\beta, \gamma}$-subgroups of $G$ for all $\beta < \alpha$, and let $H$ be a $\mathbb{H}_{\alpha, \gamma}$-subgroup of $G$.

There is an exact sequence of $ZH$-modules

$$0 \to C_r \to \cdots \to C_0 \to \mathbb{Z} \to 0$$

where each $C_i$ is a direct sum of modules of the form $\mathbb{Z}[H/F]$ with $F \leq H$ and $F \in \mathbb{H}_{\beta, \gamma}$ for $\beta < \alpha$. By the inductive hypothesis, $M \otimes A$ is projective over each $\mathbb{Z}F$, so each $\mathbb{Z}[H/F] \otimes M \otimes A$ is projective over $ZH$ and thus each $C_i \otimes M \otimes A$ is projective over $ZH$. Tensoring the above sequence with $M \otimes A$ gives us an exact sequence

$$0 \to C_r \otimes M \otimes A \to \cdots \to C_0 \otimes M \otimes A \to M \otimes A \to 0.$$ 

It follows that $pd_{ZH}M \otimes A < \infty$ for any $\mathbb{Z}G$-module $M$ that has a complete resolution of coincidence index 0. If there is not a bound on these projective dimensions, there is a family of $\mathbb{Z}G$-modules $M_n, n \in \mathbb{N}$, that have complete resolutions of coincidence index 0 and are such that $pd_{ZH}M_n \otimes A \geq n$ for all $n \in \mathbb{N}$. Then the direct sum $M = \bigoplus_{n \in \mathbb{N}} M_n$ has a complete resolution of coincidence index 0 by Lemma 2.1(a), and $pd_{ZH}M \otimes A \geq pd_{ZH}M_n \otimes A \geq n$ for all $n \in \mathbb{N}$ so that $pd_{ZH}M \otimes A$ is infinite, which is a contradiction. It follows that there is a bound on the projective dimension $pd_{ZH}M \otimes A$ where $M$ is a $\mathbb{Z}G$-module that has a complete resolution of coincidence index 0. From Lemma 2.1 parts (b) and (c), we obtain that $M \otimes A$ is projective as a $ZH$-module.

Next we prove the result for $\mathbb{LH}_{\beta, \gamma}$-subgroups, using induction on the cardinality of the subgroup. If $H$ is a countable subgroup, then it belongs to $\mathbb{H}_{\gamma}$, so the result has been proved. Assume that $H$ is uncountable and that the result is true for any subgroup with cardinality strictly smaller than $|H|$, and let $M$ be a $\mathbb{Z}G$-module that has a complete resolution of coincidence index 0. Since $H$ is uncountable it can be expressed as the union of an ascending chain of subgroups $H_\alpha, \alpha < \gamma$, for some ordinal number $\gamma$ such that each $H_\alpha$ has cardinality strictly smaller than $|H|$. By the inductive hypothesis $M \otimes A$ is projective over each $ZH_\alpha$, so by Lemma 5.6 in [3] we have $pd_{ZH}M \otimes A \leq 1$. Thus we have proved that $M \otimes A$ has finite projective dimension over $ZH$, for any $\mathbb{Z}G$-module $M$ that has a complete resolution
of coincidence index 0. It follows from the above argument that $M \otimes A$ is projective over $H$, for any $\mathbb{Z}G$-module $M$ that has a complete resolution of coincidence index 0.

Proof of Corollary C. Let $P_\ast$ be a complete resolution with kernels $M_i, i \in \mathbb{Z}$. It is known [12] that $B(G, \mathbb{Z})$ is a projective $ZF$-module for any finite subgroup $F$ of $G$. Hence it follows from Theorem B that each $M_i$ is cofibrant. The converse was shown in [7].

Proof of Corollary D. Let $P_\ast$ be a complete resolution with kernels $M_i, i \in \mathbb{Z}$. It follows from Corollary C that each $M_i$ is a cofibrant module. There is a $\mathbb{Z}$-split $\mathbb{Z}G$-monomorphism $\mathbb{Z} \rightarrow B(G, \mathbb{Z})$; hence if $Q$ is a projective $\mathbb{Z}G$-module, then $Q$ is a $\mathbb{Z}G$-direct summand of $\text{Hom}_{\mathbb{Z}}(B(G, \mathbb{Z}), Q)$ (with diagonal action), and so the complex $\text{Hom}_{\mathbb{Z}}(P_\ast, Q)$ is a direct summand of the complex

$$\text{Hom}_{\mathbb{Z}}(P_\ast, \text{Hom}_{\mathbb{Z}}(B(G, \mathbb{Z}), Q)) \simeq \text{Hom}_{\mathbb{Z}}(P_\ast \otimes B(G, \mathbb{Z}), Q).$$

This complex is exact because $M_i \otimes B(G, \mathbb{Z})$ are projective $\mathbb{Z}G$-modules, so $P_\ast \otimes B(G, \mathbb{Z})$ splits.

It is not known whether Gorenstein projectivity is a subgroup-closed property. Here we show that this is so if the subgroup is an $\textbf{LH}_F$-group.

Corollary 2.2. If $G$ is a group and $M$ is a Gorenstein projective module, then $M|_H$ is Gorenstein projective for any $\textbf{LH}_F$-subgroup $H$ of $G$.

Proof. Since $M$ is Gorenstein projective it admits a complete resolution in the strong sense of coincidence index 0 over $\mathbb{Z}G$, which is a complete resolution of coincidence index 0 over $\mathbb{Z}H$ for any subgroup $H$ of $G$. If the subgroup is in $\textbf{LH}_F$, then the resolution is a complete resolution in the strong sense by Corollary D. Hence $M|_H$ is a Gorenstein projective module.

References

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