ON SIMULTANEOUS UNIFORM APPROXIMATION TO A $p$-ADIC NUMBER AND ITS SQUARE

YANN BUGEAUD

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ABSTRACT. Let $p$ be a prime number. We show that a result of Teulié is nearly best possible by constructing a $p$-adic number $\xi$ such that $\xi$ and $\xi^2$ are uniformly simultaneously very well approximable by rational numbers with the same denominator. The same conclusion was previously reached by Zelo in his PhD thesis, but our approach using $p$-adic continued fractions is more direct and simpler.

1. Introduction

Throughout this paper we set $\lambda = (\sqrt{5} - 1)/2$. In 1969, Davenport and Schmidt established the following statement.

**Theorem DS.** Let $\xi$ be a real number that is neither rational nor quadratic. Then, there exists a positive real number $c$ such that the system of inequalities

$$
|x_0 \xi - x_1| \leq cX^{-\lambda}, \quad |x_0 \xi^2 - x_2| \leq cX^{-\lambda}, \quad |x_0| \leq X
$$

has no non-zero integer solution $(x_0, x_1, x_2)$ for arbitrarily large real numbers $X$.

It was rather unexpected when, in 2003, Roy proved that Theorem DS cannot be improved.

**Theorem R.** There exist a real number $\xi$ which is neither rational nor quadratic and a positive real number $c$ such that the system of inequalities

$$(1.1) \quad |x_0 \xi - x_1| \leq cX^{-\lambda}, \quad |x_0 \xi^2 - x_2| \leq cX^{-\lambda}, \quad |x_0| \leq X$$

has a non-zero integer solution $(x_0, x_1, x_2)$ for every real number $X > 1$.

Theorem R is quite surprising, since the volume of the convex bodies defined by (1.1) tends rapidly to zero as $X$ grows to infinity. Any real number $\xi$ satisfying a Diophantine condition as in Theorem R was termed by Roy an extremal number. He proved that the set of extremal (real) numbers is countable and gave some explicit examples of extremal (real) numbers.

Throughout the present paper, $p$ always denotes a prime number. The absolute value $|\cdot|_p$ is normalised in such a way that $|p|_p = p^{-1}$. In 2002, Teulié established the $p$-adic analogue of Theorem DS.
Theorem T. Let \( \xi \) be a \( p \)-adic number that is neither rational nor quadratic. Then, there exists a positive real number \( c \) such that the system of inequalities
\[
|x_0\xi - x_1|_p \leq cX^{-1-\lambda}, \quad |x_0\xi^2 - x_2|_p \leq cX^{-1-\lambda}, \quad \max\{|x_0|,|x_1|,|x_2|\} \leq X
\]
has no non-zero integer solution \((x_0, x_1, x_2)\) for arbitrarily large real numbers \( X \).

In analogy with the real case, we define an extremal \( p \)-adic number to be a \( p \)-adic number \( \xi \) with the property that there is a positive constant \( c \) such that, for every real number \( X > 1 \), the system (1.2) has a non-zero integer solution \((x_0, x_1, x_2)\).

Very recently, in his PhD thesis, Zelo [9] adapted the method initiated by Roy [5] to show that Teulié’s result is nearly best possible. The next result follows from his Corollary 2.5.9.

Theorem Z. Let \( \varepsilon \) be a positive real number. There exist a \( p \)-adic number \( \xi \) which is neither rational nor quadratic and a positive real number \( c \) such that the system of inequalities
\[
|x_0\xi - x_1|_p \leq cX^{-1-\lambda+\varepsilon}, \quad |x_0\xi^2 - x_2|_p \leq cX^{-1-\lambda+\varepsilon}, \quad \max\{|x_0|,|x_1|,|x_2|\} \leq X
\]
has a non-zero integer solution \((x_0, x_1, x_2)\) for every real number \( X > 1 \).

The purpose of the present note is to give an alternative, simpler proof of Zelo’s result. Our approach is inspired by Roy’s construction [5] of an extremal number using continued fractions and properties of the infinite Fibonacci word.

2. Result

Let \( a \) and \( b \) be two symbols. Set \( f_1 = b, f_2 = a \) and let \( f_n = f_{n-1}f_{n-2} \) be the concatenation of the words \( f_{n-1} \) and \( f_{n-2} \), for \( n \geq 3 \). Then,
\[
f_\infty = \lim_{n \to +\infty} f_n = abaababaabaab\ldots
\]
is the Fibonacci word on the alphabet \([a, b]\). Roy [5] proved that the real number \( \xi = [0; 1, 2, 1, 2, 1, 2, 1, \ldots] \)
whose sequence of partial quotients is given by the Fibonacci word on \([1, 2]\), is an extremal real number.

In this note we show that a similar construction works in the \( p \)-adic setting. Before stating our main result, it is convenient to define an exponent of approximation.

Definition. Let \( n \geq 1 \) be an integer and let \( \xi \) be a \( p \)-adic number. We denote by \( \hat{\lambda}_n(\xi) \) the supremum of the real numbers \( \hat{\lambda} \) such that, for every sufficiently large real number \( X \), the system of inequalities
\[
\max_{1 \leq m \leq n} |x_0\xi^m - x_m|_p \leq X^{-1-\hat{\lambda}}, \quad 0 < \max\{|x_0|,|x_1|,\ldots,|x_n|\} \leq X
\]
has a solution in integers \( x_0, \ldots, x_n \).

It follows from the Dirichlet Schubfachprinzip that \( \hat{\lambda}_n(\xi) \geq 1/n \) for every positive integer \( n \) and every irrational number \( \xi \). Teulié [8] derived upper bounds for \( \hat{\lambda}_n(\xi) \) when \( \xi \) is not algebraic of degree at most \( n \). His Theorem T implies that \( \hat{\lambda}_2(\xi) \leq \lambda \) for every \( p \)-adic number \( \xi \) which is neither rational nor quadratic, while Theorem Z asserts that
\[
(2.1) \quad \sup\{\hat{\lambda}_2(\xi) : \xi \in \mathbb{Q}_p, \xi \text{ is neither rational nor quadratic}\} = \lambda.
\]
As in the real case, it remains unknown whether there are transcendental $p$-adic numbers $\xi$ and integers $n \geq 3$ such that $\lambda_n(\xi) > 1/n$.

Our Theorem gives a constructive proof of (2.1).

**Theorem.** Let $\nu$ be a positive integer and let $(\nu_n)_{n \geq 1}$ be the Fibonacci word on $\{\nu, \nu + 1\}$ starting with $\nu$. Let $\xi_\nu$ denote the $p$-adic number

$$
\xi_\nu := 1 + \lim_{n \to +\infty} \frac{p^{v_1}}{1 + \frac{p^{v_2}}{1 + \frac{p^{v_3}}{\cdots + p^{v_n}}}}.
$$

Then we have $\hat{\lambda}_2(\xi_\nu) \geq (1 - 7/\nu)\lambda$ and

$$
\sup\{\hat{\lambda}_2(\xi_\nu) : \nu \geq 1\} = \lambda.
$$

**Remark 1.** It does not seem that Zelo’s approach allows him to replace $X$ in Theorem Z by a function of $X$ which increases less rapidly, like e.g. $X^{1/\log \log X}$. The same applies for the constructive method described in the present note. In particular, it remains an interesting open problem to decide whether there exist extremal $p$-adic numbers and even whether there exist $p$-adic numbers $\xi$ with $\hat{\lambda}_2(\xi) = \lambda$.

**Remark 2.** It follows from the $p$-adic version of the Schmidt Subspace Theorem that any $p$-adic number $\xi$ satisfying $\hat{\lambda}_2(\xi) > 1/2$ is either rational, quadratic, or transcendental.

**Remark 3.** Zelo’s approach is more complicated than ours, but it gives more information. Indeed, it yields a characterization of extremal $p$-adic numbers (if such numbers exist) as well as a characterization of $p$-adic numbers $\xi$ with $\hat{\lambda}_2(\xi)$ sufficiently close to $\lambda$. One may hope that, combined with ideas from [6], it could be used to prove the existence of $p$-adic numbers that are very badly approximable by cubic integers.

3. Proof

Before proceeding with the construction of $p$-adic numbers enjoying special approximation properties, we make several general remarks which were inspired by [3].

- **Definition of $p$-adic continued fractions.**

Set $p_{-1} = 1$, $q_{-1} = 0$, $p_0 = 1$, $q_0 = 1$.

Let $\nu = (\nu_n)_{n \geq 1}$ be a sequence of positive integers and set

$$
p_n = p^{\nu_n}p_{n-2} + p_{n-1}, \quad q_n = p^{\nu_n}q_{n-2} + q_{n-1} \quad (n \geq 1).
$$

Observe that

$$
\left| \frac{p_1}{q_1} - \frac{p_0}{q_0} \right|_p = p^{-v_1}
$$

and that, for $n \geq 2$, we have

$$
\left| \frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} \right|_p = \left| \frac{(p^{\nu_n}p_{n-2} + p_{n-1})q_{n-1} - (p^{\nu_n}q_{n-2} + q_{n-1})p_{n-1}}{q_nq_{n-1}} \right|_p
$$

$$
= p^{-\nu_n} \left| \frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}} \right|_p,
$$
since $p$ does not divide $q_n q_{n-1} q_{n-2}$.

Consequently, for $n \geq 0$ and $k \geq 1$, we have

$$\left| \frac{p_{n+k}}{q_{n+k}} - \frac{p_n}{q_n} \right|_p = \left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right|_p = p^{-v_{n+1} - v_n - \ldots - v_1}. \quad (3.1)$$

This shows that the sequence $(p_n/q_n)_{n \geq 1}$ converges $p$-adically. Let $\xi_v$ denote its limit. It follows from (3.1) that

$$\left| \frac{\xi_v - p_n}{q_n} \right|_p = p^{-v_{n+1} - v_n - \ldots - v_1}, \quad n \geq 1, \quad (3.2)$$

and we can write

$$\xi_v := 1 + \lim_{n \to +\infty} \frac{p^{v_1}}{1 + \frac{p^{v_2}}{1 + \frac{p^{v_3}}{\ldots + p^{v_n}}}}.$$  

**Palindromes.**

Let $n$ be a positive integer. We have

$$\frac{p_n}{q_n} = 1 + \frac{p^{v_1}}{1 + \frac{p^{v_2}}{1 + \frac{p^{v_3}}{\ldots + p^{v_n}}}}.$$  

Furthermore, the classical mirror formula (see [2], page 12) asserts that

$$\frac{p_n}{p_{n-1}} = 1 + \frac{p^{v_n}}{1 + \frac{p^{v_{n-1}}}{\ldots + p^{v_1}}},$$

Consequently, if the word $v_1 \ldots v_n$ is a palindrome, that is, if $v_j = v_{n+1-j}$ for $j = 1, \ldots, n$, then

$$\frac{p_n}{q_n} = \frac{p_n}{p_{n-1}},$$

hence,

$$q_n = p_{n-1}.$$  

This implies that

$$\left| \xi_v^2 - \frac{p_{n-1}}{q_{n-1}} \frac{p_n}{q_n} \right|_p = \left| \left( \xi_v - \frac{p_{n-1}}{q_{n-1}} \right) \cdot \left( \xi_v + \frac{p_n}{q_n} \right) + \xi_v \left( \frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} \right) \right|_p \leq p^{-v_{n-1} - v_{n-2} - \ldots - v_1},$$

by (3.1) and (3.2). We then derive from (3.2) that

$$\max \{ |q_{n-1} \xi_v - p_{n-1}|_p, |q_{n-1} \xi_v^2 - p_n|_p \} \leq p^{-v_{n-1} - v_{n-2} - \ldots - v_1},$$

showing that $\xi_v$ and its square are simultaneously well approximable by rational numbers of denominator $q_{n-1}$. 

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• Completion of the proof.

In the sequel, \( v \) denotes a positive integer and we assume that the sequence \( v = (v_n)_{n \geq 1} \) takes its values in the set \( \{v, v + 1\} \). We assume that \( v \geq 8 \) since the theorem obviously holds for \( v \leq 7 \). From the inequalities
\[
p^v q_{n-2} \leq q_n \leq q_{n-1} + p^{v+1} q_{n-2}, \quad n \geq 1,
\]
we deduce that there exist positive constants \( c_1 \) and \( c_2 \) such that
\[
(3.4) \quad c_1 p^{nv/2} \leq q_n \leq c_2 p^{(v+2)/2}, \quad n \geq 1.
\]
Furthermore, we observe that
\[
(3.5) \quad nv \leq v_1 + \ldots + v_n \leq n(v + 1), \quad n \geq 1.
\]
Take for \( (v_n)_{n \geq 1} \) the Fibonacci word on \( \{v, v + 1\} \) starting with \( v \). For simplicity, let us write \( \xi_v \) instead of \( \xi_v \). Let \( (F_k)_{k \geq 0} \) be the Fibonacci sequence defined by \( F_0 = 0, F_1 = 1 \) and \( F_{k+2} = F_{k+1} + F_k \) for \( k \geq 0 \). For \( k \geq 4 \), set \( n_k = F_k - 3 \). It is well known (see e.g. [1]) that, for \( k \geq 4 \), the prefix of length \( n_k + 1 \) of the word
\[
v_1 v_2 v_3 \ldots = v(v + 1)v(v + 1)v(v + 1)v \ldots
\]
is a palindrome.

In view of the preceding discussion, for \( k \geq 4 \), we have
\[
(3.6) \quad \max\{|q_{n_k} \xi_v - p_{n_k} v, |q_{n_k} v^2 - p_{n_k+1} v\} \leq p^{-v_{n_k+1} - v_{n_k} - \ldots - v_1} \leq c_3 q_{n_k}^{-2 + 4/v},
\]
by (3.3), (3.4) and (3.5). Here and below, \( c_3, \ldots, c_7 \) denote positive real numbers independent of \( k \).

Let \( Q \) be a large positive integer. Let \( k \geq 4 \) be the integer defined by the inequalities
\[
q_{n_k} \leq Q < q_{n_k+1}.
\]
Since \( n_k/n_{k+1} \) tends to \( \lambda \) as \( k \) tends to infinity, we may assume that \( Q \) is sufficiently large in order to guarantee that
\[
\lambda n_{k+1} \leq \frac{v + 3}{v + 2} n_k.
\]
Let \( u \) be the largest non-negative integer such that \( q_{n_k} p^u \leq Q \), and set
\[
q_{n_k}' = p^n q_{n_k}, \quad p_{n_k}' = p^n p_{n_k}, \quad p_{n_k+1}' = p^n p_{n_k+1}.
\]
We then have
\[
Q^\lambda \leq q_{n_k}'^\lambda \leq c_2 p^{\lambda n_{k+1}(v+2)/2} \leq c_2 p^{n_k(v+3)/2} \leq c_4 q_{n_k}^{-1+3/v},
\]
and it follows from (3.6) that
\[
\max\{|q_{n_k}' \xi_v - p_{n_k}' v, |q_{n_k}' v^2 - p_{n_k+1}' v\} \leq c_3 p^{-u} q_{n_k}^{2+4/v} \leq c_5 Q^{-1} q_{n_k}^{-1+4/v} \leq c_6 Q^{-1} Q^{-(1-7/v)\lambda}.
\]
Since \( 0 < p_{n_k}', p_{n_k+1}', q_{n_k}' \leq c_7 Q \), this shows that
\[
\hat{\lambda}_2(\xi_v) \geq (1 - 7/v)\lambda,
\]
and the proof of the theorem is complete. \( \square \)
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REFERENCES


DÉPARTEMENT DE MATHEMATIQUES, UNIVERSITÉ DE STRASBOURG, 7, RUE RENÉ DESCARTES, 67084 STRASBOURG, FRANCE

E-mail address: bugeaud@math.unistra.fr

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