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RINGS OF COINVARIANTS AND p-SUBGROUPS

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Abstract. Let $\varrho : G \hookrightarrow GL(n, \mathbb{F})$ be a faithful representation of a finite group G over the field $\mathbb{F}$ and let $V \cong \mathbb{F}^n$ be an $\mathbb{F}(G)$-module. It has been shown by L. Smith that if $n = 3$ and the order of G is divisible by the positive characteristic p of $\mathbb{F}$, then $\mathbb{F}[V]^G$ is Cohen-Macaulay. Under the condition $n = 3$ we prove the following conjecture through this remarkable result: If $\mathbb{F}[V]_G$ is a Poincaré duality algebra, then $\mathbb{F}[V]_{\mathrm{Syl}_p(G)}$ is a complete intersection, where $\mathrm{Syl}_p(G)$ is a Sylow p-subgroup of G.

1. Introduction

Let G be a finite group, $\mathbb{F}$ a field and V an n-dimensional $\mathbb{F}$-vector space. For a representation $\varrho : G \hookrightarrow GL(n, \mathbb{F}) \cong GL(V)$ the group G acts on the algebra $\mathbb{F}[V]$ of homogeneous polynomial functions on V. The subalgebra $\mathbb{F}[V]^G := \{f \in \mathbb{F}[V] \mid \sigma f = f, \text{ for all } \sigma \in G\} \subseteq \mathbb{F}[V]$ fixed under the G-action is called the ring of invariants of G, and the graded quotient algebra $\mathbb{F}[V]_G := \mathbb{F}[V]/h(G)$ is called the ring of coinvariants, where $h(G)$ is the ideal in $\mathbb{F}[V]$ generated by all G-invariant homogeneous polynomials of strictly positive degree. The ring $\mathbb{F}[V]^G$ has the Krull dimension $n = \dim_{\mathbb{F}} V$, and there is a homogeneous system of parameters $\{f_1, \cdots, f_n\}$ for $\mathbb{F}[V]^G$ so that $\mathbb{F}[V]^G$ is a finitely generated module over the polynomial ring $\mathbb{F}[f_1, \cdots, f_n]$. The ring of coinvariants can be expressed as the tensor product

$$\mathbb{F}[V]_G \cong \mathbb{F} \otimes_{\mathbb{F}[V]^G} \mathbb{F}[V].$$

As a convenient reference for invariant theory, see [2] or [8]. Note that the graded polynomial ring $\mathbb{F}[V]$ may be considered as a local ring with the unique maximal ideal $\overline{\mathbb{F}[V]} = \bigoplus_{i \geq 1} \mathbb{F}[V]_i$.

Let H be a graded connected commutative algebra over a field $\mathbb{F}$. We say that H is a Poincaré duality algebra of formal dimension d if

(1) $H_i = 0$ for all $i > d$,
(2) $\dim_{\mathbb{F}}(H_d) = 1$,
(3) the bilinear form $H_i \otimes_{\mathbb{F}} H_{d-i} \longrightarrow H_d$, $a \otimes b \mapsto a \cdot b$, is nonsingular; i.e. an element $a \in H_i$ is 0 if and only if $a \cdot b = 0$ for all $b \in H_{d-i}$,

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where H_i is the homogeneous componet of H of degree i. In the invariant theory a Poincaré duality quotient algebra $\mathbb{F}[V]_G$ coincides with a zero-dimensional Gorenstein local ring.

The property of Poincaré duality plays an important role in observing whether a ring of invariants $\mathbb{F}[V]^G$ is polynomial. It has been proved that in the nonmodular case the ring $\mathbb{F}[V]^G$ is polynomial, while $\mathbb{F}[V]_G$ is a Poincaré duality algebra (see [6] for the zero characteristic case and [7] for the case when the order $\mid G \mid$ is relatively prime to the positive characteristic). In the modular case L. Smith ([10]) when proved:

Theorem 1.1. *Let $\varrho : G \hookrightarrow GL(2, \mathbb{F})$ be a representation of a finite group G over a field $\mathbb{F}$ of positive characteristic. If $\mathbb{F}[x, y]_G$ is a Poincaré duality algebra, then $\mathbb{F}[x, y]_G$ is a complete intersection.*

A ring R is said to be a complete intersection if there is a regular ring S and a regular sequence $a_1, \cdots, a_n$ in S such that $R \cong S/(a_1, \cdots, a_n)$. In the modular case we do not know whether a Poincaré duality algebra $\mathbb{F}[x, y, z]_G$ is also a complete intersection. However, if P is a finite p-group, then the invariant ring $\mathbb{F}[x, y, z]^P$ is not only a Cohen-Macaulay ring but also a unique factorization domain. Therefore, we may consider the situation of the ring of coinvariants $\mathbb{F}[x, y, x]_{\mathrm{Syl}_p(G)}$ of the Sylow p-subgroup of G. The purpose of this paper is to prove that $\mathbb{F}[x, y, z]_{\mathrm{Syl}_p(G)}$ must be a complete intersection if $\mathbb{F}[x, y, z]_G$ is a Poincaré duality algebra.

2. Main results

Throughout this paper we assume that $\mathbb{F}$ is a field of characteristic $p \neq 0$, G is a finite group whose order is divisible by p, and V is a 3-dimensional vector space over $\mathbb{F}$. Let $\mathrm{Syl}_p(G)$ be a Sylow p-subgroup of G. Then $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is integral over $\mathbb{F}[V]^G$. If $\{f_1, f_2, f_3\}$ is a homogeneous system of parameters for $\mathbb{F}[V]^G$, so it is also for $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$, and $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is a finitely generated $\mathbb{F}[f_1, f_2, f_3]$-module.

Recall that a finitely generated graded commutative connected $\mathbb{F}$-algebra R is Cohen-Macaulay of the Krull dimension n if R is a free module over the graded polynomial subalgebra $S = \mathbb{F}[f_1, \cdots, f_n]$, where $\{f_1, \cdots, f_n\}$ is a homogeneous system of parameters of R. If R is a finitely generated graded module over S and R is Cohen-Macaulay, then R admits a canonical module $w_R \cong \mathrm{Hom}_S(R, S)$. A Cohen-Macaulay graded ring R with canonical graded module w_R is a Gorenstein ring if and only if w_R is isomorphic to R, which is possible with a shift in degree. As a general reference for Gorenstein rings see [5].

It is well known that

Theorem 2.1 ([9])**.** *If $G \hookrightarrow GL(3, \mathbb{F})$ is a representation of a finite group G over $\mathbb{F}$, then $\mathbb{F}[V]^G$ is Cohen-Macaulay.*

Indeed, $\mathbb{F}[V]^P$ is a unique factorization domain for a finite p-group P (see [8], Proposition 1.5.7). Therefore

Theorem 2.2 (See, e.g., [4], Corollary 3.3.19)**.** *If $P \hookrightarrow GL(3, \mathbb{F})$ is a representation of a finite p-group P over $\mathbb{F}$, then $\mathbb{F}[V]^P$ is Gorenstein.*

Suppose $\mid G : \mathrm{Syl}_p(G) \mid = m$ and let $\sigma_1, \cdots, \sigma_m$ be coset representatives. The map $\rho : \mathbb{F}[V]^{\mathrm{Syl}_p(G)} \longrightarrow \mathbb{F}[V]^G$ defined by $\rho(f) = \frac{1}{m} \sum_{i=1}^{m} \sigma_i(f)$ splits. It follows

that $\mathbb{F}[V]^{\mathrm{Syl}_p(G)} = \mathbb{F}[V]^G \oplus \ker\rho$ as an $\mathbb{F}[V]^G$-module, also as an $\mathbb{F}[f_1, f_2, f_3]$-module, and

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{F}[f_1,f_2,f_3]}(\mathbb{F}[V]^{\mathrm{Syl}_p(G)}, \mathbb{F}[f_1, f_2, f_3])\\ &\qquad \cong \mathrm{Hom}_{\mathbb{F}[f_1,f_2,f_3]}(\mathbb{F}[V]^G, \mathbb{F}[f_1, f_2, f_3]) \oplus \mathrm{Hom}_{\mathbb{F}[f_1,f_2,f_3]}(\ker\rho, \mathbb{F}[f_1, f_2, f_3]).\end{aligned}$$

Since $\mathbb{F}[V]^{\mathrm{Syl}_p(G)}$ is Gorenstein, so is $\mathbb{F}[V]^G$.

Lemma 2.3 ([2]). *If $(A, \mathfrak{m}, \mathbb{F})$ is a d-dimensional Gorenstein local ring and M is a finitely generated A-module with finite projective dimension, then*

$$\mathrm{Tor}_i^A(\mathbb{F}, M) \cong \mathrm{Ext}_A^{d-i}(\mathbb{F}, M),\ 0 \le i \le d.$$

Furthermore if $P \hookrightarrow GL(3, \mathbb{F})$ is a representation of a finite p-group P over $\mathbb{F}$, then $\mathbb{F}[V]_P \cong \mathrm{Ext}^3_{\mathbb{F}[V]^P}(\mathbb{F}, \mathbb{F}[V])$.

Suppose that H is a subgroup of a finite group G. Because projective $\mathbb{F}(G)$-modules are still projective $\mathbb{F}(H)$-modules, a projective resolution of $\mathbb{F}$ as an $\mathbb{F}(G)$-module gives us a projective resolution of $\mathbb{F}$ as an $\mathbb{F}(H)$-module. Thus there is a restriction homomorphism

$$\mathrm{Res}_G^H : \mathrm{Ext}^i_{\mathbb{F}(G)}(\mathbb{F}, \mathbb{F}[V]) \longrightarrow \mathrm{Ext}^i_{\mathbb{F}(H)}(\mathbb{F}, \mathbb{F}[V]).$$

The $\mathbb{F}(M)$-module $\mathrm{Hom}_{\mathbb{F}(M)}(\mathbb{F}, \mathbb{F}[V])$ can be regarded as the ring $\mathbb{F}[V]^M$ of invariants, where $M = G,\ H$. Therefore the transfer $\mathrm{Tr}_H^G : \mathbb{F}[V]^H \longrightarrow \mathbb{F}[V]^G$ given by $f \mapsto \sum_{\sigma \in G/H} \sigma(f)$ for all $f \in \mathbb{F}[V]^H$ induces a homomorphism, denoted as Tr_H^G as well:

$$\mathrm{Tr}_H^G : \mathrm{Ext}^i_{\mathbb{F}(H)}(\mathbb{F}, \mathbb{F}[V]) \longrightarrow \mathrm{Ext}^i_{\mathbb{F}(G)}(\mathbb{F}, \mathbb{F}[V]).$$

On the other hand, if $\mathbb{F}[V]^M$ is regarded as an $\mathbb{F}(M)$-module, then it is easy to see that

$$\mathrm{Hom}_{\mathbb{F}(M)}(\mathbb{F}, \mathbb{F}[V]) \cong \mathrm{Hom}_{\mathbb{F}[V]^M}(\mathbb{F} \otimes_{\mathbb{F}(M)} \mathbb{F}[V]^M, \mathbb{F}[V]).$$

We can extend this isomorphism to Ext. First note that if P is a projective $\mathbb{F}(M)$-module, where $M = G,\ H$, then $P \otimes_{\mathbb{F}(M)} \mathbb{F}[V]^M$ is a projective $\mathbb{F}[V]^M$-module. Thus, a projective $\mathbb{F}(M)$-resolution of $\mathbb{F}$ corresponds to a projective $\mathbb{F}[V]^M$-resolution of $\mathbb{F} \otimes \mathbb{F}[V]^M$; hence

$$\mathrm{Ext}^i_{\mathbb{F}(M)}(\mathbb{F}, \mathbb{F}[V]) \cong \mathrm{Ext}^i_{\mathbb{F}[V]^M}(\mathbb{F} \otimes_{\mathbb{F}(M)} \mathbb{F}[V]^M, \mathbb{F}[V]).$$

By applying the inclusion $i : \mathbb{F} \hookrightarrow \mathbb{F} \otimes \mathbb{F}[V]^M$, there exists an $\mathbb{F}[V]^M$-module homomorphism $\mathrm{Ext}^i_{\mathbb{F}[V]^M}(\mathbb{F} \otimes \mathbb{F}[V]^M, \mathbb{F}[V]) \to \mathrm{Ext}^i_{\mathbb{F}[V]^M}(\mathbb{F}, \mathbb{F}[V])$ defined by $\alpha \mapsto \alpha \circ i$. Thus, we obtain

$$\mathrm{Tr}_H^G(\mathrm{Res}_G^H(\alpha)) \circ i = \mathrm{Tr}_H^G(\mathrm{Res}_G^H)(\alpha \circ i),$$

for all $\alpha \in \mathrm{Ext}^i_{\mathbb{F}[V]^H}(\mathbb{F} \otimes \mathbb{F}[V]^H, \mathbb{F}[V])$ and $\alpha \circ i \in \mathrm{Ext}^i_{\mathbb{F}[V]^H}(\mathbb{F}, \mathbb{F}[V])$. Therefore, the restriction and transfer reduce from $\mathrm{Ext}^i_{\mathbb{F}[V]^M}(\mathbb{F} \otimes \mathbb{F}[V]^M, \mathbb{F}[V])$ to $\mathrm{Ext}^i_{\mathbb{F}[V]^M}(\mathbb{F}, \mathbb{F}[V])$, and then the following proposition is still valid for the $\mathbb{F}[V]^G$-module $\mathrm{Ext}_{\mathbb{F}[V]^G}(\mathbb{F}, \mathbb{F}[V])$.

Proposition 2.4 (See, e.g., [3]). *If $G \hookrightarrow GL(n, \mathbb{F})$ is a representation of a finite group G over $\mathbb{F}$ and H is a subgroup of G, then we have the composition*

$$\mathrm{Tr}_H^G \circ \mathrm{Res}_G^H(\beta) = \mid G : H \mid \cdot \beta,$$

for any $\beta \in \mathrm{Ext}^i_{\mathbb{F}[V]G}(\mathbb{F}, \mathbb{F}[V])$.

Corollary 2.5. *If $G \hookrightarrow GL(3,\mathbb{F})$ is a representation of a finite group G over $\mathbb{F}$ and $\mathbb{F}[V]_G$ is a Poincaré duality algebra, then*

$$\mathbb{F}[V]_G \cong \mathbb{F}[V]_{\mathrm{Syl}_p(G)} \oplus h(\mathrm{Syl}_p(G))/h(G).$$

Proof. $\mathrm{Res}_G^{\mathrm{Syl}_p(G)} : \mathbb{F}[V]_G \longrightarrow \mathbb{F}[V]_{\mathrm{Syl}_p(G)}$ is a split homomorphism for the Reynolds operator $\pi^G_{\mathrm{Syl}_p(G)} = \frac{1}{|G:\mathrm{Syl}_p(G)|}\,\mathrm{Tr}^G_{\mathrm{Syl}_p(G)}$. □

Applying the following lemma of H. Bass, we can prove the result (Theorem 2.7) that we want now.

Lemma 2.6 (H. Bass [1])**.** *If $(A,\mathfrak{m},\mathbb{F})$ is a d-dimensional Cohen-Macaulay local ring and I is an $\mathfrak{m}$-primary ideal in A, then the following conditions are equivalent:*

(1) *The quotient ring A/I is Gorenstein.*
(2) $\mathrm{Ext}^i_A(A/I,A) \cong \begin{cases} A/I, & i=d, \\ 0, & \text{otherwise}. \end{cases}$

Theorem 2.7. *Let $\varrho : G \hookrightarrow GL(3,\mathbb{F})$ be a representation of a finite group G over a field $\mathbb{F}$. If $\mathbb{F}[V]_G$ is a Poincaré duality algebra, then so is $\mathbb{F}[V]_{\mathrm{Syl}_p(G)}$.*

Proof. From the result of H. Bass (Lemma 2.6) and Corollary 2.5, it follows immediately that

$$\begin{aligned} \mathbb{F}[V]_G &\cong \mathrm{Ext}^3_{\mathbb{F}[V]}(\mathbb{F}[V]_G,\mathbb{F}[V]) \\ &\cong \mathrm{Ext}^3_{\mathbb{F}[V]}(\mathbb{F}[V]_{\mathrm{Syl}_p(G)},\mathbb{F}[V]) \oplus \mathrm{Ext}^3_{\mathbb{F}[V]}(h(\mathrm{Syl}_p(G))/h(G),\mathbb{F}[V]). \end{aligned}$$

Hence, $\mathbb{F}[V]_{\mathrm{Syl}_p(G)} \cong \mathrm{Ext}^3_{\mathbb{F}[V]}(\mathbb{F}[V]_{\mathrm{Syl}_p(G)},\mathbb{F}[V])$ is a Gorenstein ring. □

Remark 2.8 ([10]). For a finite p-group P the ideal $h(P)$ is a complete intersection ideal[1] if the ring of coinvariants $\mathbb{F}[x,y,z]_P$ is a Poincaré duality algebra.

Consequently we have simultaneously established

Corollary 2.9. *Let $G \hookrightarrow GL(3,\mathbb{F})$ be a representation of a finite group G over $\mathbb{F}$. If $\mathbb{F}[V]_G$ is a Poincaré duality algebra, then $\mathbb{F}[V]_{\mathrm{Syl}_p(G)}$ is a complete intersection.*

References

1. H. Bass, *On the ubiquity of Gorenstein rings*, Math. Zeitschr. 82(1963). MR0153708 (27:3669)
2. D.J. Benson, *Polynomial Invariants of Finite Groups*, Cambridge University Press, London, 1993. MR1249931 (94j:13003)
3. D.J. Benson, *Representaions and Cohomology I*, Cambridge University Press, 1998. MR1644252 (99f:20001a)
4. W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1996. MR1251956 (95h:13020)
5. D. Eisenbud, *Commutative Algebra*, Springer, 1996. MR1322960 (97a:13001)
6. R. Kane, *Poincaré Duality and the Ring of Coinvariants*, Canad. Math. Bull. 37(1)(1994), 82-88. MR1261561 (96e:51016)
7. T.-C. Lin, *Poincaré Duality Algebra and Rings of Coinvariants*, Proc. of the Amer. Math. Soc. 134(2006), 1599-1604. MR2204269 (2006k:13013)
8. L. Smith, *Polynomial Invariants of Finite Groups*, A K Peters, Wellesley, Massachusetts, 1995. MR1328644 (96f:13008)

[1] An ideal is called a **complete intersection ideal** if it is generated by a regular sequence.

9. L. Smith, *Some Rings of Invariants That Are Cohen-Macaulay*, Canad. Math. Bull. 39(2)(1996), 238-240. MR1390361 (97a:13010)
10. L. Smith, *On a Theorem of R. Steinberg on Rings of Coinvariants*, Proc. of the Amer. Math. Soc. 131(4)(2002), 1043-1048. MR1948093 (2003k:13008)

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