Abstract. Let $\rho : G \hookrightarrow \text{GL}(n,F)$ be a faithful representation of a finite group $G$ over the field $F$ and let $V \cong F^n$ be an $F(G)$-module. It has been shown by L. Smith that if $n = 3$ and the order of $G$ is divisible by the positive characteristic $p$ of $F$, then $F[V]^G$ is Cohen-Macaulay. Under the condition $n = 3$ we prove the following conjecture through this remarkable result: If $F[V]^G$ is a Poincaré duality algebra, then $F[V]_{\text{Syl}_p(G)}$ is a complete intersection, where $\text{Syl}_p(G)$ is a Sylow $p$-subgroup of $G$.

1. Introduction

Let $G$ be a finite group, $F$ a field and $V$ an $n$-dimensional $F$-vector space. For a representation $\rho : G \hookrightarrow \text{GL}(n,F) \cong \text{GL}(V)$ the group $G$ acts on the algebra $F[V]$ of homogeneous polynomial functions on $V$. The subalgebra $F[V]^G := \{ f \in F[V] \mid \sigma f = f, \text{ for all } \sigma \in G \} \subseteq F[V]$ fixed under the $G$-action is called the ring of invariants of $G$, and the graded quotient algebra $F[V]^G := F[V]/h(G)$ is called the ring of coinvariants, where $h(G)$ is the ideal in $F[V]$ generated by all $G$-invariant homogeneous polynomials of strictly positive degree. The ring $F[V]^G$ has the Krull dimension $n = \dim F V$, and there is a homogeneous system of parameters $\{f_1, \ldots, f_n\}$ for $F[V]^G$ so that $F[V]^G$ is a finitely generated module over the polynomial ring $F[f_1, \ldots, f_n]$. The ring of coinvariants can be expressed as the tensor product

$$F[V]^G \cong F \otimes_{F[V]} F[V].$$

As a convenient reference for invariant theory, see [2] or [3]. Note that the graded polynomial ring $F[V]$ may be considered as a local ring with the unique maximal ideal $F[V] = \bigoplus_{i \geq 1} F[V]_i$.

Let $H$ be a graded connected commutative algebra over a field $F$. We say that $H$ is a Poincaré duality algebra of formal dimension $d$ if

1. $H_i = 0$ for all $i > d$,
2. $\dim H_d = 1$,
3. the bilinear form $H_i \otimes H_{d-i} \longrightarrow H_d, a \otimes b \mapsto a \cdot b$, is nonsingular; i.e. an element $a \in H_i$ is 0 if and only if $a \cdot b = 0$ for all $b \in H_{d-i}$,
where \( H_i \) is the homogeneous componet of \( H \) of degree \( i \). In the invariant theory a Poincaré duality quotient algebra \( \mathbb{F}[V]_G \) coincides with a zero-dimensional Gorenstein local ring.

The property of Poincaré duality plays an important role in observing whether a ring of invariants \( \mathbb{F}[V]^G \) is polynomial. It has been proved that in the nonmodular case the ring \( \mathbb{F}[V]^G \) is polynomial, while \( \mathbb{F}[V]_G \) is a Poincaré duality algebra (see [9] for the zero characteristic case and [10] for the case when the order \( | G | \) is relatively prime to the positive characteristic). In the modular case L. Smith ([10]) when proved:

**Theorem 1.1.** Let \( \rho : G \to GL(2, \mathbb{F}) \) be a representation of a finite group \( G \) over a field \( \mathbb{F} \) of positive characteristic. If \( \mathbb{F}[x, y]_G \) is a Poincaré duality algebra, then \( \mathbb{F}[x, y]_G \) is a complete intersection.

A ring \( R \) is said to be a complete intersection if there is a regular ring \( S \) and a regular sequence \( a_1, \ldots, a_n \) in \( S \) such that \( R \cong S/(a_1, \ldots, a_n) \). In the modular case we do not know whether a Poincaré duality algebra \( \mathbb{F}[x, y, z]_G \) is also a complete intersection. However, if \( P \) is a finite \( p \)-group, then the invariant ring \( \mathbb{F}[x, y, z]^P \) is not only a Cohen-Macaulay ring but also a unique factorization domain. Therefore, we may consider the situation of the ring of coinvariants \( \mathbb{F}[x, y, x]_{Syl_p(G)} \) of the Sylow \( p \)-subgroup of \( G \). The purpose of this paper is to prove that \( \mathbb{F}[x, y, z]_{Syl_p(G)} \) must be a complete intersection if \( \mathbb{F}[x, y, z]_G \) is a Poincaré duality algebra.

2. Main results

Throughout this paper we assume that \( \mathbb{F} \) is a field of characteristic \( p \neq 0 \), \( G \) is a finite group whose order is divisible by \( p \), and \( V \) is a 3-dimensional vector space over \( \mathbb{F} \). Let \( Syl_p(G) \) be a Sylow \( p \)-subgroup of \( G \). Then \( \mathbb{F}[V]^{Syl_p(G)} \) is integral over \( \mathbb{F}[V]^G \). If \( \{ f_1, f_2, f_3 \} \) is a homogeneous system of parameters for \( \mathbb{F}[V]^G \), so it is also for \( \mathbb{F}[V]^{Syl_p(G)} \), and \( \mathbb{F}[V]^{Syl_p(G)} \) is a finitely generated \( \mathbb{F}[f_1, f_2, f_3] \)-module.

Recall that a finitely generated graded commutative connected \( \mathbb{F} \)-algebra \( R \) is Cohen-Macaulay of the Krull dimension \( n \) if \( R \) is a free module over the graded polynomial subalgebra \( S = \mathbb{F}[f_1, \ldots, f_n] \), where \( \{ f_1, \ldots, f_n \} \) is a homogeneous system of parameters of \( R \). If \( R \) is a finitely generated graded module over \( S \) and \( R \) is Cohen-Macaulay, then \( R \) admits a canonical module \( w_R \cong \text{Hom}_S(R, S) \). A Cohen-Macaulay graded ring \( R \) with canonical graded module \( w_R \) is a Gorenstein ring if and only if \( w_R \) is isomorphic to \( R \), which is possible with a shift in degree. As a general reference for Gorenstein rings see [3].

It is well known that

**Theorem 2.1 ([9]).** If \( G \to GL(3, \mathbb{F}) \) is a representation of a finite group \( G \) over \( \mathbb{F} \), then \( \mathbb{F}[V]^G \) is Cohen-Macaulay.

Indeed, \( \mathbb{F}[V]^P \) is a unique factorization domain for a finite \( p \)-group \( P \) (see [8], Proposition 1.5.7). Therefore

**Theorem 2.2** (See, e.g., [4], Corollary 3.3.19). If \( P \to GL(3, \mathbb{F}) \) is a representation of a finite \( p \)-group \( P \) over \( \mathbb{F} \), then \( \mathbb{F}[V]^P \) is Gorenstein.

Suppose \( | G : Syl_p(G) | = m \) and let \( \sigma_1, \ldots, \sigma_m \) be coset representatives. The map \( \rho : \mathbb{F}[V]^{Syl_p(G)} \to \mathbb{F}[V]^G \) defined by \( \rho(f) = \frac{1}{m} \sum_{i=1}^m \sigma_i(f) \) splits. It follows
that \( \mathbb{F}[V]^{\mathrm{Syl}_p(G)} = \mathbb{F}[V]^G \oplus \ker \rho \) as an \( \mathbb{F}[V]^G \)-module, also as an \( \mathbb{F}[f_1, f_2, f_3] \)-module, and
\[
\mathrm{Hom}_{[f_1, f_2, f_3]}(\mathbb{F}[V]^{\mathrm{Syl}_p(G)}, \mathbb{F}[f_1, f_2, f_3]) \\
\cong \mathrm{Hom}_{[f_1, f_2, f_3]}(\mathbb{F}[V]^G, \mathbb{F}[f_1, f_2, f_3]) \oplus \mathrm{Hom}_{[f_1, f_2, f_3]}(\ker \rho, \mathbb{F}[f_1, f_2, f_3]).
\]
Since \( \mathbb{F}[V]^{\mathrm{Syl}_p(G)} \) is Gorenstein, so is \( \mathbb{F}[V]^G \).

**Lemma 2.3** (\([2]\)). If \((A, m, \mathbb{F})\) is a \(d\)-dimensional Gorenstein local ring and \(M\) is a finitely generated \(A\)-module with finite projective dimension, then
\[
\mathrm{Tor}_i^A(\mathbb{F}, M) \cong \mathrm{Ext}_{d-i}^d(\mathbb{F}, M), \quad 0 \leq i \leq d.
\]
Furthermore if \(P\) is a representation of a finite \(p\)-group \(P\) over \(\mathbb{F}\), then \(\mathbb{F}[V]_P \cong \mathrm{Ext}_{d}^d(\mathbb{F}, \mathbb{F}[V])\).

Suppose that \(H\) is a subgroup of a finite group \(G\). Because projective \(\mathbb{F}(G)\)-modules are projective \(\mathbb{F}(H)\)-modules, a projective resolution of \(\mathbb{F}\) as an \(\mathbb{F}(G)\)-module gives us a projective resolution of \(\mathbb{F}\) as an \(\mathbb{F}(H)\)-module. Thus there is a restriction homomorphism
\[
\mathrm{Res}_{i}^G : \mathrm{Ext}_{(G)}^i(\mathbb{F}, \mathbb{F}[V]) \rightarrow \mathrm{Ext}_{(H)}^i(\mathbb{F}, \mathbb{F}[V]).
\]
The \(\mathbb{F}(M)\)-module \(\mathrm{Hom}_{(M)}(\mathbb{F}, \mathbb{F}[V])\) can be regarded as the ring \(\mathbb{F}[V]^M\) of invariants, where \(M = G, H\). Therefore the transfer \(\mathrm{Tr}_i^G : \mathbb{F}[V]^H \rightarrow \mathbb{F}[V]^G\) given by \(f \mapsto \sum_{\sigma \in G/H} \sigma(f)\) for all \(f \in \mathbb{F}[V]^H\) induces a homomorphism, denoted as \(\mathrm{Tr}_i^G\) as well:
\[
\mathrm{Tr}_i^G : \mathrm{Ext}_{(H)}^i(\mathbb{F}, \mathbb{F}[V]) \rightarrow \mathrm{Ext}_{(G)}^i(\mathbb{F}, \mathbb{F}[V]).
\]
On the other hand, if \(\mathbb{F}[V]^M\) is regarded as an \(\mathbb{F}(M)\)-module, then it is easy to see that
\[
\mathrm{Hom}_{(M)}(\mathbb{F}, \mathbb{F}[V]) \cong \mathrm{Hom}_{(V)}^M(\mathbb{F} \otimes \mathbb{F}(M), \mathbb{F}[V]^M, \mathbb{F}[V]).
\]
We can extend this isomorphism to \(\mathrm{Ext}\). First note that if \(P\) is a projective \(\mathbb{F}(M)\)-module, where \(M = G, H\), then \(P \otimes_{(M)} \mathbb{F}[V]^M\) is a projective \(\mathbb{F}[V]^M\)-module. Thus, a projective \(\mathbb{F}(M)\)-resolution of \(\mathbb{F}\) corresponds to a projective \(\mathbb{F}[V]^M\)-resolution of \(\mathbb{F} \otimes \mathbb{F}[V]^M\); hence
\[
\mathrm{Ext}_{(M)}^i(\mathbb{F}, \mathbb{F}[V]) \cong \mathrm{Ext}_{(V)}^i(\mathbb{F}, \mathbb{F}[V]^M, \mathbb{F}[V]).
\]
By applying the inclusion \(i : \mathbb{F} \hookrightarrow \mathbb{F} \otimes \mathbb{F}[V]^M\), there exists an \(\mathbb{F}[V]^M\)-module homomorphism \(\mathrm{Ext}_{(V)}^i(\mathbb{F} \otimes \mathbb{F}[V]^M, \mathbb{F}[V]) \rightarrow \mathrm{Ext}_{(V)}^i(\mathbb{F}, \mathbb{F}[V])\) defined by \(\alpha \mapsto \alpha \circ i\). Thus, we obtain
\[
\mathrm{Tr}_i^G(\mathrm{Res}_{i}^H(\alpha)) \circ i = \mathrm{Tr}_i^G(\mathrm{Res}_{i}^H)(\alpha \circ i),
\]
for all \(\alpha \in \mathrm{Ext}_{(V)}^i(\mathbb{F} \otimes \mathbb{F}[V]^H, \mathbb{F}[V])\) and \(\alpha \circ i \in \mathrm{Ext}_{(V)}^i(\mathbb{F}, \mathbb{F}[V])\). Therefore, the restriction and transfer reduce from \(\mathrm{Ext}_{(V)}^i(\mathbb{F} \otimes \mathbb{F}[V]^M, \mathbb{F}[V])\) to \(\mathrm{Ext}_{(V)}^i(\mathbb{F}, \mathbb{F}[V])\), and then the following proposition is still valid for the \(\mathbb{F}[V]^G\)-module \(\mathrm{Ext}_{(V)}^i(\mathbb{F}, \mathbb{F}[V])\).

**Proposition 2.4** (See, e.g., \([3]\)). If \(G \hookrightarrow \mathrm{GL}(n, \mathbb{F})\) is a representation of a finite group \(G\) over \(\mathbb{F}\) and \(H\) is a subgroup of \(G\), then we have the composition
\[
\mathrm{Tr}_i^G \circ \mathrm{Res}_{i}^H(\beta) = |G : H| \cdot \beta,
\]
for any \(\beta \in \mathrm{Ext}_{(V)}^i(\mathbb{F}, \mathbb{F}[V])\).
Corollary 2.5. If $G \hookrightarrow \text{GL}(3, \mathbb{F})$ is a representation of a finite group $G$ over $\mathbb{F}$ and $F[V]_G$ is a Poincaré duality algebra, then

$$F[V]_G \cong F[V]_{\text{Syl}_p(G)} \oplus \bar{h}(\text{Syl}_p(G))/\bar{h}(G).$$

Proof. \begin{align*}
\text{Res}_{\text{Syl}_p(G)}^G : F[V]_G &\longrightarrow F[V]_{\text{Syl}_p(G)} \text{ is a split homomorphism for the Reynolds operator } \\
\pi^G_{\text{Syl}_p(G)} &\equiv \frac{1}{[G:\text{Syl}_p(G)]} \text{Tr}^G_{\text{Syl}_p(G)}.
\end{align*}

Applying the following lemma of H. Bass, we can prove the result (Theorem 2.7) that we want now.

Lemma 2.6 (H. Bass [1]). If $(A, m, \mathbb{F})$ is a d-dimensional Cohen-Macaulay local ring and $I$ is an $m$-primary ideal in $A$, then the following conditions are equivalent:

1. The quotient ring $A/I$ is Gorenstein.
2. $\text{Ext}_A^i(A/I, A) \equiv \begin{cases} A/I, & i = d, \\ 0, & \text{otherwise}. \end{cases}$

Theorem 2.7. Let $\rho : G \rightarrow \text{GL}(3, \mathbb{F})$ be a representation of a finite group $G$ over a field $\mathbb{F}$. If $F[V]_G$ is a Poincaré duality algebra, then so is $F[V]_{\text{Syl}_p(G)}$.

Proof. From the result of H. Bass (Lemma 2.6) and Corollary 2.5, it follows immediately that

$$F[V]_G \cong \text{Ext}^3_{\text{Syl}_p(G)}(F[V]_G, F[V]) \cong \text{Ext}^3_{\text{Syl}_p(G)}(F[V]_{\text{Syl}_p(G)}, F[V]) \cong \text{Ext}^3_{\text{Syl}_p(G)}(h(\text{Syl}_p(G))/h(G), F[V]).$$

Hence, $F[V]_{\text{Syl}_p(G)} \cong \text{Ext}^3_{\text{Syl}_p(G)}(F[V]_{\text{Syl}_p(G)}, F[V])$ is a Gorenstein ring.

Remark 2.8 ([1]). For a finite $p$-group $P$ the ideal $h(P)$ is a complete intersection ideal if the ring of coinvariants $F[x, y, z]_P$ is a Poincaré duality algebra.

Consequently we have simultaneously established

Corollary 2.9. Let $G \hookrightarrow \text{GL}(3, \mathbb{F})$ be a representation of a finite group $G$ over $\mathbb{F}$. If $F[V]_G$ is a Poincaré duality algebra, then $F[V]_{\text{Syl}_p(G)}$ is a complete intersection.

REFERENCES


1 An ideal is called a complete intersection ideal if it is generated by a regular sequence.

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