WEIGHTED ORLICZ-RIESZ CAPACITY OF BALLS

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Abstract. Our aim in this paper is to estimate the weighted Orlicz-Riesz capacity of balls.

1. Introduction and statement of results

Several versions of capacities for Orlicz-Riesz spaces have appeared in research papers, for example those by Aissaoui and Benkirane [5], Kuznetsov [14], Mizuta [17], Adams and Hurri-Syrjänen [3, 4], and Joensuu [12]. The notion of capacity offers a standard way to characterize exceptional sets and is indispensable to an understanding of the local behavior of functions in Orlicz-Riesz spaces. Various capacity estimates also play an important role in the study of solutions to partial differential equations.

Recently the authors [8] gave an estimate of the Orlicz-Riesz capacity of balls as an extension of Adams and Hurri-Syrjänen [4] and Joensuu [12]. An estimate of the weighted Sobolev capacity of balls can be found e.g. in Heinonen, Kilpeläinen and Martio [9].

A positive measurable function \( w \) on \( \mathbb{R}^n \) is called an \( A_p \) weight (written as \( w \in A_p \)) if there exists a positive constant \( C_p \) such that

\[
\left( \frac{w(B)}{|B|} \right) \left( \frac{w^{1/(1-p)}(B)}{|B|} \right)^{p-1} \leq C_p \quad (< \infty)
\]

for all balls \( B \), where \( 1 < p < \infty \), \(|\cdot|\) denotes the \( n \)-dimensional Lebesgue measure and

\[
w(B) = \int_B w(y) \, dy.
\]

As an example, we have that the function \( w(x) = |x|^{\delta} \) is an \( A_p \) weight if and only if \( -n < \delta < n(p-1) \). It is well known that, for an \( A_p \) weight \( w \), the corresponding measure \( w \) is doubling, that is, \( w(2B) \leq cw(B) \) for all balls \( B = B(x, r) \); here the constant \( c \) depends only on \( n, p \) and \( C_p \) and \( 2B \) stands for the enlarged ball \( B(x, 2r) \). For these and other fundamental properties of \( A_p \) weights, see, for example, Heinonen, Kilpeläinen and Martio [9].

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For $0 < \alpha < n$ and a locally integrable function $f$ on $\mathbb{R}^n$, we define the Riesz potential $I_\alpha f$ of order $\alpha$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x - y|^{n-\alpha} f(y) \, dy.$$ 

In the present paper, we treat functions $f$ satisfying an Orlicz condition with an $A_p$ weight $\omega$:

$$\int_{\mathbb{R}^n} \varphi_p(|f(y)|)\omega(y) \, dy < \infty. \tag{1.1}$$

Here $\varphi_p(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ of the form

$$\varphi_p(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive quasi-increasing function on $(0, \infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$(\varphi) \quad c_1^{-1} \varphi(r) \leq \varphi(r^2) \leq c_1 \varphi(r) \quad \text{whenever } r > 0.$$ 

We say that $\varphi(r)$ is quasi-increasing if there exists $c > 0$ such that

$$\varphi(s) \leq c \varphi(t) \quad \text{whenever } 0 < s < t.$$ 

We set

$$\varphi_p(0) = 0,$$

because we will see from $(\varphi 4)$ below that

$$\lim_{r \to 0^+} \varphi_p(r) = 0;$$

see [19] p. 205.

For an open set $G \subset \mathbb{R}^n$, we denote by $L^{\varphi_p, \omega}(G)$ the family of all locally integrable functions $f$ on $G$ such that

$$\int_G \varphi_p(|f(y)|)\omega(y) \, dy < \infty,$$

and define

$$\|f\|_{\varphi_p, \omega, G} = \inf \left\{ \lambda > 0 : \int_G \varphi_p(|f(y)|/\lambda)\omega(y) \, dy \leq 1 \right\}.$$

This is a quasi-norm in $L^{\varphi_p, \omega}(G)$. For $E \subset G$, the relative $(\alpha, \varphi_p, \omega)$-capacity is defined by

$$B_{\alpha, \varphi_p, \omega}(E; G) = \inf \int_G \varphi_p(|f(y)|)\omega(y) \, dy,$$

where the infimum is taken over all functions $f$ such that $f = 0$ outside $G$ and $I_\alpha f(x) \geq 1$ for all $x \in E$.

(cf. Adams and Hedberg [2], Meyers [15], Ziemer [22] and the first author [16, 17]).

Our first aim in the present note is to give an estimate of the modular capacity $B_{\alpha, \varphi_p, \omega}$ of open balls $B(x, r)$ centered at $x$ of radius $r$, as an extension of Adams and Hurri-Syrjänen [4, Theorem 2.11], Joensuu [12], Heinonen, Kilpeläinen and Martio [8, Theorems 1 and 2.19] and the authors [8, Theorem A]. In fact, our first theorem is stated in the following.
Theorem A (cf. [10 Lemma 7.3]). Suppose \( p > 1 \) and \( \omega \in A_p \). For \( R > 0 \), there exists a constant \( A > 0 \) such that

\[
A^{-1} \left( \int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t)) \}^{1/(1-p)} \, dt/t \right)^{1-p} \leq B_{\alpha,\varphi,\omega}(B(x,r); B(0,R))
\]

whenever \( B(x, r) \subset B(0, R/4) \).

We write \( f \sim g \) if there exists a constant \( A \) so that \( A^{-1}g \leq f \leq Ag \).

Example 1.1. Let \( \omega(x) = |x|^\delta \) and \( \varphi(t) = (\log(e + t))^{\beta} \).

1. If \( \alpha p - n < \delta < n(p - 1) \), then

\[
\int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t)) \}^{1/(1-p)} \, dt/t \sim r^{-(\alpha p + n + \delta)/(1-p)}(\log(e + 1/r))^{\beta/(1-p)}
\]

for \( 0 < r < R/2 < 1 \). In this case,

\[
B_{\alpha,\varphi,\omega}(B(0,r); B(0,R)) \sim r^{-\alpha p + n + \delta}(\log(e + 1/r))^{\beta}.
\]

2. If \( \alpha p - n = \delta \) and \( \beta < p - 1 \), then

\[
\int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t)) \}^{1/(1-p)} \, dt/t \sim (\log(e + 1/r))^{\beta/(1-p) + 1}
\]

for \( 0 < r < R/2 < 1 \). In this case,

\[
B_{\alpha,\varphi,\omega}(B(0,r); B(0,R)) \sim (\log(e + 1/r))^{\beta + 1 - p}.
\]

3. If \( \alpha p - n = \delta \) and \( \beta = p - 1 \), then

\[
\int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t)) \}^{1/(1-p)} \, dt/t \sim \log(e + (\log(e + 1/r))
\]

for \( 0 < r < R/2 < 1 \). In this case,

\[
B_{\alpha,\varphi,\omega}(B(0,r); B(0,R)) \sim (\log(e + (\log(e + 1/r))))^{-(p-1)}.
\]

4. If \( \alpha p - n > \delta > -n \) or \( \alpha p - n = \delta \) and \( \beta > p - 1 \), then

\[
B_{\alpha,\varphi,\omega}([0]; B(0,1)) > 0.
\]

Next we are concerned with the norm capacity. For \( E \subset G \), we define

\[
C_{\alpha,\varphi,\omega}(E; G) = \inf \| f \|_{\varphi,\omega,G},
\]

where the infimum is taken over all functions \( f \) such that \( f = 0 \) outside \( G \) and

\[
I_\alpha f(x) \geq 1 \quad \text{for all } x \in E.
\]
Theorem B. Suppose \( p > 1 \) and \( \omega \in A_p \). For \( R > 0 \), there exists a constant \( A > 0 \) such that
\[
A^{-1} \left( \int_0^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t)) \}^{1/(1-p)} \frac{dt}{t} \right)^{(1-p)/p} \leq C_{\alpha, \varphi, \omega}(B(x,r); B(0,R))
\]
\[
\leq A \left( \int_0^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t)) \}^{1/(1-p)} \frac{dt}{t} \right)^{(1-p)/p}
\]
whenever \( B(x,r) \subset B(0,R/4) \).

In view of Theorems A and B, we have the following result, which extends the results by Adams and Hurri-Syrjänen [4], Joensuu [12] and the authors [8].

Corollary 1.2. Suppose \( p > 1 \) and \( \omega \in A_p \). For \( R > 0 \), there is a constant \( A > 0 \) such that
\[
A^{-1}B_{\alpha, \varphi, p, \omega}(B(x,r); B(0,R))^{1/p} \leq C_{\alpha, \varphi, p, \omega}(B(x,r); B(0,R)) \leq AB_{\alpha, \varphi, p, \omega}(B(x,r); B(0,R))^{1/p}
\]
whenever \( B(x,r) \subset B(0,R/4) \).

For further related results, we refer the reader to Adams [1], Adams and Hurri-Syrjänen [3], Edmunds and Evans [7], Kilpeläinen [13] and Mizuta and Shimomura [19, 20, 21].

Throughout this paper, let \( A \) denote various constants independent of the variables in question and let \( A(a,b,\cdots) \) be a constant that depends on \( a, b, \cdots \).

2. Proof of Theorem A

First we collect properties which follow from condition (\( \varphi 1 \)) (see [17], [19, Lemma 2.3], [18, Section 7]).

(\( \varphi 2 \)) \( \varphi \) satisfies the doubling condition; that is, there exists \( c_2 > 1 \) such that
\[
c_2^{-1} \varphi(r) \leq \varphi(2r) \leq c_2 \varphi(r)
\]
whenever \( r > 0 \).

(\( \varphi 3 \)) For each \( \gamma > 0 \), there exists \( c_3 = c_3(\gamma) \geq 1 \) such that
\[
c_3^{-1} \varphi(r) \leq \varphi(r^\gamma) \leq c_3 \varphi(r)
\]
whenever \( r > 0 \).

(\( \varphi 4 \)) For each \( \gamma > 0 \), there exists \( c_4 = c_4(\gamma) \geq 1 \) such that
\[
s^{-\gamma} \varphi(s) \leq c_4 t^\gamma \varphi(t)
\]
whenever \( 0 < s < t \).

(\( \varphi 5 \)) For each \( \gamma > 0 \), there exists \( c_5 = c_5(\gamma) \geq 1 \) such that
\[
t^{-\gamma} \varphi(t) \leq c_5 s^{-\gamma} \varphi(s)
\]
whenever \( 0 < s < t \).

(\( \varphi 6 \)) If \( \varphi \) and \( \psi \) are positive monotone functions on \([0, \infty)\) satisfying (\( \varphi 1 \)), then for each \( \gamma > 0 \), there exists a constant \( c_6 = c_6(\gamma) \geq 1 \) such that
\[
c_6^{-1} \varphi(r) \leq \varphi(r^{\gamma} \psi(r)) \leq c_6 \varphi(r)
\]
whenever \( r > 0 \).

Let us begin with an upper estimate for the modular \( B_{\alpha, \varphi, p, \omega} \)-capacity of balls.
Theorem 2.1. Suppose $p > 1$ and $\omega$ is a positive locally integrable function on $\mathbb{R}^n$. Then there exists a constant $A > 0$ such that

$$B_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \leq A \left( \int_{2r}^{2R} \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \frac{dt}{t} \right)^{1-p}$$

whenever $0 < r < R/2 < \infty$.

Proof. For $r > 0$, consider the function

$$f_r(y) = |y|^{-\alpha}$$

for $r < |y| < 2r$ and $f_r = 0$ elsewhere. If $x \in B(0, r)$ and $y \in B(0, 2r) \setminus B(0, r)$, then $|x - y| < 3r$, so that

$$I_\alpha f_r(x) \geq (3r)^{\alpha-n} \int_{B(0, 2r) \setminus B(0, r)} |y|^{-\alpha} dy = A_1$$

with a constant $A_1 = A_1(\alpha, n) > 0$.

Now let $0 < r < R/2$, and take $j_0$ such that $2^{j_0+1}r \leq R < 2^{j_0+2}r$. For $\{a_j\}$ such that $a_j \geq 0$ and $\sum_{j=0}^{j_0} a_j = 1$, set

$$f = \sum_{j=0}^{j_0} a_j f_{2^j r} / A_1.$$ 

Then

$$I_\alpha f(x) = A_1^{-1} \sum_{j=0}^{j_0} a_j I_\alpha f_{2^j r}(x) \geq 1$$

for $x \in B(0, r)$. Therefore we have by $(\varphi 2)$ and $(\varphi 3)$ that

$$B_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \leq \int_{B(0, R)} \varphi_p(f(y)) \omega(y) dy$$

$$= \sum_{j=0}^{j_0} \int_{B(0, R)} [a_j f_{2^j r}(y) / A_1]^p \varphi(a_j f_{2^j r}(y) / A_1) \omega(y) dy$$

$$\leq A_2 \sum_{j=0}^{j_0} a_j^p (2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1} r)).$$

Now, letting $K = \sum_{j=0}^{j_0} ((2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1} r)))^{1/(1-p)}$ and

$$a_j = \frac{((2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1} r)))^{1/(1-p)}}{K},$$

we find

$$B_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \leq A_2 K^{-p} \sum_{j=0}^{j_0} ((2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1} r)))^{1/(1-p)}$$

$$= A_2 K^{1-p}.$$
Here we have
\[ K \geq A_3 \sum_{j=0}^{2^{j+3}r} \int_{2r}^{2^{j+1}r} \{t^{-\alpha p}(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \frac{dt}{t} \]
\[ \geq A_3 \int_{2r}^{2^{j+3}r} \{t^{-\alpha p}(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \frac{dt}{t}, \]
so that
\[ K \geq A_3 \int_{2r}^{2R} \{t^{-\alpha p}(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \frac{dt}{t}, \]
which proves the result. \(\square\)

**Corollary 2.2.** If \(\omega\) is doubling, then there exists a constant \(A > 0\) such that
\[ B_{\alpha,\phi,p,\omega}(B(x,r); B(0,R)) \leq A \left( \int_r^R \{t^{-\alpha p}(t^{-1})\omega(B(x,t))\}^{1/(1-p)} \frac{dt}{t} \right)^{1-p} \]
whenever \(B(x,r) \subset B(0,R/4)\). In fact,
\[ B_{\alpha,\phi,p,\omega}(B(x,r); B(0,R)) \leq B_{\alpha,\phi,p,\omega}(B(x,r); B(x,R/2)) \leq A \left( \int_r^R \{t^{-\alpha p}(t^{-1})\omega(B(x,t))\}^{1/(1-p)} \frac{dt}{t} \right)^{1-p} \]
whenever \(B(x,r) \subset B(0,R/4)\).

Next we give a lower estimate for the modular \(B_{\alpha,\phi,p,\omega}\)-capacity of balls.

**Theorem 2.3.** Suppose \(p > 1\) and \(\omega \in A_p\). For \(R > 0\), there exists a constant \(A = A(R) > 0\) such that
\[ B_{\alpha,\phi,p,\omega}(B(0,r); B(0,R)) \geq A \left( \int_r^R \{t^{-\alpha p}(t^{-1})\omega(B(0,t))\}^{1/(1-p)} \frac{dt}{t} \right)^{1-p} \]
whenever \(0 < r < R/2 < \infty\).

**Proof.** For \(0 < r < R/2\), take a nonnegative measurable function \(f\) on \(B(0,R)\) such that
\[ I_\alpha f(x) \geq 1 \quad \text{for } x \in B(0,r). \]
Then we have by Fubini’s theorem
\[ \int_{B(0,r)} dx \leq \int_{B(0,r)} I_\alpha f(x) \; dx \]
\[ \leq \int_{B(0,R)} \left( \int_{B(0,r)} |x - y|^{\alpha - n} \; dx \right) f(y) \; dy \]
\[ \leq A_1 r^n \int_{B(0,R)} (r + |y|)^{\alpha - n} f(y) \; dy. \]
For $\varepsilon > 0$ and $0 < \delta < \alpha$, we see from Hölder’s inequality that
\[
\int_{B(0,R)} (r + |y|)^{\alpha-n} f(y) \, dy = 
\int_{\{y \in B(0,R) : f(y) > \varepsilon(r+|y|)^{-\delta}\}} (r + |y|)^{\alpha-n} f(y) \, dy
+ \int_{\{y \in B(0,R) : f(y) \leq (r+|y|)^{-\delta}\}} (r + |y|)^{\alpha-n} f(y) \, dy
\leq \left( \int_{B(0,R)} [(r + |y|)^{\alpha-n} \{\varphi((r + |y|)^{-\delta})\omega(y)\}]^{-1/p} dy \right)^{1/p'}
\times \left( \int_{B(0,R)} \varphi_p(f(y))\omega(y) \, dy \right)^{1/p} + \varepsilon \int_{B(0,R)} (r + |y|)^{\alpha-n-\delta} dy,
\]
where $1/p + 1/p' = 1$. Since $\omega \in A_p$, we find by (2.2) and (2.3) that
\[
\int_{B(0,R)} [(r + |y|)^{\alpha-n} \{\varphi((r + |y|)^{-\delta})\omega(y)\}]^{-1/p} dy
\leq A_2(\varepsilon) \int_0^R (r + t)^{(\alpha-n)p/(p-1)} \varphi((r + t)^{-1}) [\omega(y)]^{1/(1-p)} \frac{dt}{t}
\leq A_3(\varepsilon) \int_0^R (r + t)^{\alpha p/(p-1)} \varphi((r + t)^{-1}) [\omega(y)]^{1/(1-p)} \frac{dt}{t}
\leq A_4(\varepsilon) \int_r^{2R} [t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))]^{1/(1-p)} \frac{dt}{t}
\leq A_4(\varepsilon) \int_r^R [t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))]^{1/(1-p)} \frac{dt}{t}.
\]
Thus we derive
\[
\int_{B(0,R)} (r + |y|)^{\alpha-n} f(y) \, dy
\leq A_4(\varepsilon) \left( \int_r^R [t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))]^{1/(1-p)} \frac{dt}{t} \right)^{1/p'}
\times \left( \int_{B(0,R)} \varphi_p(f(y))\omega(y) \, dy \right)^{1/p} + A_5 \varepsilon,
\]
so that
\[
1 \leq A_6(\varepsilon) \left( \int_r^R [t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))]^{1/(1-p)} \frac{dt}{t} \right)^{1/p'}
\times \left( \int_{B(0,R)} \varphi_p(f(y))\omega(y) \, dy \right)^{1/p} + A_7 \varepsilon.
\]
If \( A_\varepsilon = 1/2 \), then we establish
\[
B_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \geq A \left( \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(t, 0))\}^{1/(1-p)} \, dt/t \right)^{1-p},
\]
as required.

**Corollary 2.4.** Suppose \( p > 1 \) and \( \omega \in A_p \). For \( R > 0 \), there exists a constant \( A = A(R) > 0 \) such that
\[
B_{\alpha, \varphi, \omega}(B(x, r); B(0, R)) \geq A \left( \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(x, t))\}^{1/(1-p)} \, dt/t \right)^{1-p}
\]
whenever \( B(x, r) \subset B(0, R/4) \).

Now Theorem A follows from Corollaries 2.2 and 2.4.

3. PROOF OF THEOREM B

Let us begin with an upper estimate for the norm \( C_{\alpha, \varphi, \omega} \)-capacity of balls.

**Theorem 3.1.** Suppose that \( p > 1 \) and that \( \omega(B(0, r)) \) satisfies the doubling condition. Then there exists a constant \( A > 0 \) such that
\[
C_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \leq A \left( \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(t, 0))\}^{1/(1-p)} \, dt/t \right)^{(1-p)/p}
\]
whenever \( 0 < r < R/2 < \infty \).

**Proof.** Set
\[
\varphi^*(r) = \int_r^R \{t^{-\alpha p}\varphi(t^{-1})\omega(B(t, 0))\}^{1/(1-p)} \, dt/t
\]
for \( r > 0 \). Consider the function
\[
f(y) = |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0, |y|))\}^{1/(1-p)} \varphi^*(r)^{-1/p}
\]
for \( r < |y| < R \) and \( f = 0 \) elsewhere. If \( x \in B(0, r) \) and \( y \in B(0, R) \setminus B(0, r) \), then \( |x - y| < 2|y| \), so that
\[
I_\alpha f(x) \geq 2^{\alpha - n} \varphi^*(r)^{-1/p} \int_{B(0, R) \setminus B(0, r)} |y|^{-n} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0, |y|))\}^{1/(1-p)} \, dy
\]
\[
= A_1 \varphi^*(r)^{(p-1)/p}
\]
with a constant \( A_1 = A_1(\alpha, n) > 0 \). It follows from the definition of capacity that
\[
C_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \leq A_1^{-1} \varphi^*(r)^{(1-p)/p} \| f \|_{\varphi^*, \omega, B(0, R)}.
\]
Thus it suffices to show that
\[
\| f \|_{\varphi^*, \omega, B(0, R)} \leq A_2.
\]
For this purpose, we first note that
\[
\int_{B(0,R)} \varphi_p(f(y))\omega(y)dy \\
= \int_{B(0,R) \setminus B(0,r)} f(y)^p \varphi(f(y))\omega(y)dy \\
= \varphi^*(r)^{-1} \int_{B(0,R) \setminus B(0,r)} |y|^{-\alpha p} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0,|y|))\}^{p/(1-p)} \\
\times \varphi \left( |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0,|y|))\}^{1/(1-p)} \varphi^*(r)^{-1/p} \right) \omega(y)dy.
\]

Here we see from the doubling condition of \(\omega(B(0,r))\) that
\[
|y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0,|y|))\}^{1/(1-p)} \varphi^*(r)^{-1/p} \\
\leq A_3 |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0,|y|))\}^{1/(1-p)} \\
\left( \int_r^{2R} \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} dt/t \right)^{-1/p} \\
\leq A_3 |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0,|y|))\}^{1/(1-p)} \\
\left( \int_{|y|}^{2|y|} \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} dt/t \right)^{-1/p} \\
\leq A_4 \varphi \left( |y|^{-1} \right)^{-1/p} \omega(B(0,|y|))^{-1/p}
\]
for \(y \in B(0,R) \setminus B(0,r)\). Further we can find constants \(A_0 > 0\) and \(\gamma > 0\) such that
\[
\omega(B(0,t)) \geq A_0 t^{\gamma}
\]
for all \(t > 0\). Hence, as in Theorem 2.1, we obtain by \(\varphi 6\):
\[
\int_{B(0,R)} \varphi_p(f(y))\omega(y)dy \\
\leq A_5 \varphi^*(r)^{-1} \int_{B(0,R) \setminus B(0,r)} |y|^{-\alpha p} \{ |y|^{-\alpha p} \varphi(|y|^{-1}) \omega(B(0,|y|))\}^{p/(1-p)} \varphi \left( |y|^{-1} \right)^{-1/p} \omega(y)dy \\
\leq A_6 \varphi^*(r)^{-1} \int_{R} \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} dt/t \\
= A_6,
\]
as required. \(\Box\)

**Corollary 3.2.** Suppose that \(p > 1\) and \(\omega\) is doubling. Then there exists a constant \(A > 0\) such that
\[
C_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) \leq A \left( \int_{R} \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))\}^{1/(1-p)} dt/t \right)^{(1-p)/p}
\]
whenever \(B(x,r) \subset B(0,R/4)\).

Next we give a lower estimate for the norm \(C_{\alpha,\varphi_p,\omega}\)-capacity of balls.
Theorem 3.3. Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that

$$
C_{\alpha, \phi, \omega}(B(0, r); B(0, R)) \geq A \left( \int_r^R \{ t^{-\alpha p} \phi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \, dt / t \right)^{(1-p)/p}
$$

whenever $0 < r < R/2 < \infty$.

Proof. For $0 < r < R/2$ take a nonnegative measurable function $f$ on $B(0, R)$ such that $I_\alpha f(x) \geq 1$ for $x \in B(0, r)$. Then we have by Fubini’s theorem

$$
\int_{B(0, r)} dx \leq \int_{B(0, r)} I_\alpha f(x) \, dx \leq \int_{B(0, R)} \left( \int_{B(0, r)} |x - y|^{\alpha - n} \, dx \right) f(y) \, dy \leq A_1 r^n \int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy,
$$

so that

$$
1 \leq A_1 \int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy.
$$

We show that

$$
\int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy \leq A_2 \left( \int_r^R \{ t^{-\alpha p} \phi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \, dt / t \right)^{(p-1)/p} \|f\|_{\phi, \omega, B(0, R)}.
$$

For this purpose, suppose that $\|f\|_{\phi, \omega, B(0, R)} \leq 1$. For $0 < \delta < \alpha$, we see from Hölder’s inequality that

$$
\int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy \leq \left( \int_{B(0, R)} [(r + |y|)^{\alpha - n} \{ \phi((r + |y|)^{-\delta}) \omega(y) \}^{-1/p}]^{p'} \, dy \right)^{1/p'} \times \left( \int_{B(0, R)} \phi_p(f(y)) \omega(y) \, dy \right)^{1/p} + \int_{B(0, R)} (r + |y|)^{\alpha - n - \delta} \, dy.
$$
Here note from \( \omega \in A_p \), (\( \varphi 2 \)) and (\( \varphi 3 \)) that
\[
\int_{B(0, R)} [(r + |y|)^{-n} \{ \varphi((r + |y|)^{-p}) \omega(y) \}]^{-1/p'} dy
\leq A_3 \int_0^R (r + t)^{\alpha - n + \alpha/\beta} \varphi((r + t)^{-1})^{1/\beta} \left( \int_{B(0, t)} \omega(y)^{1/\beta} dy \right) dt/t
\leq A_4 \int_0^R (r + t)^{\alpha - n + \alpha/\beta} \varphi((r + t)^{-1})^{1/\beta} \left( \int_{B(0, r + t)} \omega(y) dy \right)^{1/\beta} dt/t
\leq A_4 \int_0^{2R} \{ t^{-\alpha} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/\beta} dt/t
\leq A_5 \int_0^R \{ t^{-\alpha} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/\beta} dt/t,
\]
so that
\[
\int_{B(0, R)} (r + |y|)^{-n} f(y) dy
\leq A_5 \left( \int_0^R \{ t^{-\alpha} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/\beta} dt/t \right)^{1/p'} \\
\times \left( \int_{B(0, R)} \varphi(f(y)) \omega(y) dy \right)^{1/p} + A_6
\leq A_7 \left( \int_0^R \{ t^{-\alpha} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/\beta} dt/t \right)^{1/p'}.
\]
Hence we establish
\[
C_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \geq A \left( \int_0^R \{ t^{-\alpha} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/\beta} dt/t \right)^{(1/p)/\beta},
\]
as required. \( \square \)

**Corollary 3.4.** Suppose \( p > 1 \) and \( \omega \in A_p \). For \( R > 0 \), there exists a constant \( A = A(R) > 0 \) such that
\[
C_{\alpha, \varphi, \omega}(B(x, r); B(0, R)) \geq A \left( \int_0^R \{ t^{-\alpha} \varphi(t^{-1}) \omega(B(x, t)) \}^{1/\beta} dt/t \right)^{(1/p)/\beta},
\]
whenever \( B(x, r) \subset B(0, R/4) \).

As in the proof of Theorem A, Theorem B follows readily from Corollaries 3.2 and 3.3.

**References**


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