WEIGHTED ORLICZ-RIESZ CAPACITY OF BALLS

YOSHIHIRO MIZUTA, TAKAO OHNO, AND TETSU SHIMOMURA

(Communicated by Tatiana Toro)

Abstract. Our aim in this paper is to estimate the weighted Orlicz-Riesz capacity of balls.

1. Introduction and statement of results

Several versions of capacities for Orlicz-Riesz spaces have appeared in research papers, for example those by Aissaoui and Benkirane [5], Kuznetsov [14], Mizuta [17], Adams and Hurri-Syrjänen [3, 4], and Joensuu [12]. The notion of capacity offers a standard way to characterize exceptional sets and is indispensable to an understanding of the local behavior of functions in Orlicz-Riesz spaces. Various capacity estimates also play an important role in the study of solutions to partial differential equations.

Recently the authors [8] gave an estimate of the Orlicz-Riesz capacity of balls as an extension of Adams and Hurri-Syrjänen [4] and Joensuu [12]. An estimate of the weighted Sobolev capacity of balls can be found e.g. in Heinonen, Kilpeläinen and Martio [9].

A positive measurable function $w$ on $\mathbb{R}^n$ is called an $A_p$ weight (written as $w \in A_p$) if there exists a positive constant $C_p$ such that

$$\left(\frac{w(B)}{|B|}\right)^\frac{1}{p-1} \leq C_p \quad (\leq \infty)$$

for all balls $B$, where $1 < p < \infty$, $|\cdot|$ denotes the $n$-dimensional Lebesgue measure and

$$w(B) = \int_B w(y) \, dy.$$ 

As an example, we have that the function $w(x) = |x|^\delta$ is an $A_p$ weight if and only if $-n < \delta < n(p-1)$. It is well known that, for an $A_p$ weight $w$, the corresponding measure $w$ is doubling, that is, $w(2B) \leq cw(B)$ for all balls $B = B(x,r)$; here the constant $c$ depends only on $n, p$ and $C_p$ and $2B$ stands for the enlarged ball $B(x,2r)$. For these and other fundamental properties of $A_p$ weights, see, for example, Heinonen, Kilpeläinen and Martio [9].

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
For $0 < \alpha < n$ and a locally integrable function $f$ on $\mathbb{R}^n$, we define the Riesz potential $I_\alpha f$ of order $\alpha$ by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} |x-y|^{n-\alpha} f(y) \, dy.$$ 

In the present paper, we treat functions $f$ satisfying an Orlicz condition with an $A_p$ weight $\omega$:

$$(1.1) \quad \int_{\mathbb{R}^n} \varphi_p(|f(y)|) \omega(y) \, dy < \infty.$$ 

Here $\varphi_p(r)$ is a positive nondecreasing function on the interval $(0, \infty)$ of the form

$$\varphi_p(r) = r^p \varphi(r),$$

where $p > 1$ and $\varphi(r)$ is a positive quasi-increasing function on $(0, \infty)$ which is of logarithmic type; that is, there exists $c_1 > 0$ such that

$$\varphi_1 \leq c_1 \varphi(r^2) \leq c_2 \varphi(r) \quad \text{whenever } r > 0.$$ 

We say that $\varphi(r)$ is quasi-increasing if there exists $c > 0$ such that

$$\varphi(s) \leq c \varphi(t) \quad \text{whenever } 0 < s < t.$$ 

We set

$$\varphi_p(0) = 0,$$

because we will see from (\ref{phi4}) below that

$$\lim_{r \to 0^+} \varphi_p(r) = 0;$$


For an open set $G \subset \mathbb{R}^n$, we denote by $L^{\varphi_p,\omega}(G)$ the family of all locally integrable functions $f$ on $G$ such that

$$\int_G \varphi_p(|f(y)|) \omega(y) \, dy < \infty,$$

and define

$$\|f\|_{\varphi_p,\omega,G} = \inf \left\{ \lambda > 0 : \int_G \varphi_p(|f(y)|/\lambda) \omega(y) \, dy \leq 1 \right\}.$$ 

This is a quasi-norm in $L^{\varphi_p,\omega}(G)$. For $E \subset G$, the relative $(\alpha, \varphi_p, \omega)$-capacity is defined by

$$B_{\alpha, \varphi_p, \omega}(E; G) = \inf \int_G \varphi_p(|f(y)|) \omega(y) \, dy,$$

where the infimum is taken over all functions $f$ such that $f = 0$ outside $G$ and $I_\alpha f(x) \geq 1$ for all $x \in E$.

(cf. Adams and Hedberg [2], Meyers [15], Ziemer [22] and the first author [16, 17].)

Our first aim in the present note is to give an estimate of the modular capacity $B_{\alpha, \varphi_p, \omega}$ of open balls $B(x, r)$ centered at $x$ of radius $r$, as an extension of Adams and Hurri-Syrjänen [4, Theorem 2.11], Joensuu [12], Heinonen, Kilpeläinen and Martio [9, Theorems 2.18 and 2.19] and the authors [8, Theorem A]. In fact, our first theorem is stated in the following.
Theorem A (cf. [10 Lemma 7.3]). Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A > 0$ such that

$$A^{-1} \left( \int_R^R \{ t^{-\alpha_p} \varphi(t^{-1}) \omega(B(x,t)) \}^{1/(1-p)} \, dt/t \right)^{1-p} \leq B_{\alpha,\varphi,\omega}(B(x,r); B(0,R))$$

$$\leq A \left( \int_R^R \{ t^{-\alpha_p} \varphi(t^{-1}) \omega(B(x,t)) \}^{1/(1-p)} \, dt/t \right)^{1-p}$$

whenever $B(x,r) \subset B(0,R/4)$.

We write $f \sim g$ if there exists a constant $A$ so that $A^{-1}g \leq f \leq Ag$.

Example 1.1. Let $\omega(x) = |x|^\delta$ and $\varphi(t) = (\log(e + t))^{\beta}$.

(1) If $\alpha p - n < \delta < n(p - 1)$, then

$$\int_r^R \{ t^{-\alpha_p} \varphi(t^{-1}) \omega(B(0,t)) \}^{1/(1-p)} \, dt/t \sim r^{(-\alpha_p + n \beta)/(1-p)} (\log(e + 1/r))^{\beta/(1-p)}$$

for $0 < r < R/2 < 1$. In this case, $B_{\alpha,\varphi,\omega}(B(0,r); B(0,R)) \sim r^{-\alpha_p + n \beta} (\log(e + 1/r))^{\beta}$.

(2) If $\alpha p - n = \delta$ and $\beta < p - 1$, then

$$\int_r^R \{ t^{-\alpha_p} \varphi(t^{-1}) \omega(B(0,t)) \}^{1/(1-p)} \, dt/t \sim (\log(e + 1/r))^{\beta/(1-p) + 1}$$

for $0 < r < R/2 < 1$. In this case, $B_{\alpha,\varphi,\omega}(B(0,r); B(0,R)) \sim (\log(e + 1/r))^{\beta + 1 - p}$.

(3) If $\alpha p - n = \delta$ and $\beta = p - 1$, then

$$\int_r^R \{ t^{-\alpha_p} \varphi(t^{-1}) \omega(B(0,t)) \}^{1/(1-p)} \, dt/t \sim \log(e + (\log(e + 1/r)))$$

for $0 < r < R/2 < 1$. In this case, $B_{\alpha,\varphi,\omega}(B(0,r); B(0,R)) \sim (\log(e + (\log(e + 1/r))))^{-(p-1)}$.

(4) If $\alpha p - n > \delta > -n$ or $\alpha p - n = \delta$ and $\beta > p - 1$, then

$B_{\alpha,\varphi,\omega}(\{0\}; B(0,1)) > 0$.

Next we are concerned with the norm capacity. For $E \subset G$, we define

$$C_{\alpha,\varphi,\omega}(E; G) = \inf \| f \|_{\varphi,\omega; G},$$

where the infimum is taken over all functions $f$ such that $f = 0$ outside $G$ and

$$I_{\alpha} f(x) \geq 1 \quad \text{for all } x \in E.$$
Theorem B. Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A > 0$ such that
\[
A^{-1} \left( \int_r^R \left( \frac{t^{-\alpha p} \varphi(t^{-1}) \omega(B(x,t))}{t} \right)^{(1-p)/p} \frac{dt}{t} \right)^{(1-p)/p} \leq C_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R))
\]
whenever $B(x,r) \subset B(0,R/4)$.

In view of Theorems A and B, we have the following result, which extends the results by Adams and Hurri-Syrjänen [4], Joensuu [12] and the authors [8].

Corollary 1.2. Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there is a constant $A > 0$ such that
\[
A^{-1} B_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R))^{1/p} \leq C_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R)) \leq AB_{\alpha, \varphi_p, \omega}(B(x,r); B(0,R))^{1/p}
\]
whenever $B(x,r) \subset B(0,R/4)$.

For further related results, we refer the reader to Adams [1], Adams and Hurri-Syrjänen [3], Edmunds and Evans [7], Kilpeläinen [13] and Mizuta and Shimomura [19, 20, 21].

Throughout this paper, let $A$ denote various constants independent of the variables in question and let $A(a, b, \cdots)$ be a constant that depends on $a, b, \cdots$.

2. Proof of Theorem A

First we collect properties which follow from condition $(\varphi 1)$ (see [17], [19 Lemma 2.3], [18 Section 7]).

$(\varphi 2)$ $\varphi$ satisfies the doubling condition; that is, there exists $c_2 > 1$ such that
\[
c_2^{-1} \varphi(r) \leq \varphi(2r) \leq c_2 \varphi(r) \quad \text{whenever } r > 0.
\]

$(\varphi 3)$ For each $\gamma > 0$, there exists $c_3 = c_3(\gamma) \geq 1$ such that
\[
c_3^{-1} \varphi(r) \leq \varphi(r^\gamma) \leq c_3 \varphi(r) \quad \text{whenever } r > 0.
\]

$(\varphi 4)$ For each $\gamma > 0$, there exists $c_4 = c_4(\gamma) \geq 1$ such that
\[
s^{\gamma} \varphi(s) \leq c_4 t^{\gamma} \varphi(t) \quad \text{whenever } 0 < s < t.
\]

$(\varphi 5)$ For each $\gamma > 0$, there exists $c_5 = c_5(\gamma) \geq 1$ such that
\[
t^{-\gamma} \varphi(t) \leq c_5 s^{-\gamma} \varphi(s) \quad \text{whenever } 0 < s < t.
\]

$(\varphi 6)$ If $\varphi$ and $\psi$ are positive monotone functions on $[0, \infty)$ satisfying $(\varphi 1)$, then for each $\gamma > 0$, there exists a constant $c_6 = c_6(\gamma) \geq 1$ such that
\[
c_6^{-1} \varphi(r) \leq \varphi(r^\gamma \psi(r)) \leq c_6 \varphi(r) \quad \text{whenever } r > 0.
\]

Let us begin with an upper estimate for the modular $B_{\alpha, \varphi_p, \omega}$-capacity of balls.
Theorem 2.1. Suppose \( p > 1 \) and \( \omega \) is a positive locally integrable function on \( \mathbb{R}^n \). Then there exists a constant \( A > 0 \) such that
\[
B_{\alpha, \varphi_p, \omega}(B(0, r); B(0, R)) \leq A \left( \int_{2r}^{2R} \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \, dt/t \right)^{1-p}
\]
whenever \( 0 < r < R/2 < \infty \).

Proof. For \( r > 0 \), consider the function
\[
f_r(y) = |y|^{-\alpha}
\]
for \( r < |y| < 2r \) and \( f_r = 0 \) elsewhere. If \( x \in B(0, r) \) and \( y \in B(0, 2r) \setminus B(0, r) \), then \( |x - y| < 3r \), so that
\[
I_\alpha f_r(x) \geq (3r)^{\alpha-n} \int_{B(0, 2r) \setminus B(0, r)} |y|^{-\alpha} \, dy = A_1
\]
with a constant \( A_1 = A_1(\alpha, n) > 0 \).

Now let \( 0 < r < R/2 \), and take \( j_0 \) such that \( 2^{j_0+1}r \leq R < 2^{j_0+2}r \). For \( \{a_j\} \) such that \( a_j \geq 0 \) and \( \sum_{j=0}^{j_0} a_j = 1 \), set
\[
f = \sum_{j=0}^{j_0} a_j f_{2^jr}/A_1.
\]
Then
\[
I_\alpha f(x) = A_1^{-1} \sum_{j=0}^{j_0} a_j I_\alpha f_{2^jr}(x) \geq 1
\]
for \( x \in B(0, r) \). Therefore we have by (\( \varphi 2 \)) and (\( \varphi 3 \)) that
\[
B_{\alpha, \varphi_p, \omega}(B(0, r); B(0, R)) \leq \int_{B(0, R)} \varphi_p(f(y)) \omega(y) \, dy
\]
\[
= \sum_{j=0}^{j_0} \int_{B(0, R)} [a_j f_{2^jr}(y)/A_1]^{p} \varphi(a_j f_{2^jr}(y)/A_1) \omega(y) \, dy
\]
\[
\leq A_2 \sum_{j=0}^{j_0} a_j^p (2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1}r)).
\]

Now, letting \( K = \sum_{j=0}^{j_0} \{(2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1}r)) \}^{1/(1-p)} \) and
\[
a_j = \left\{(2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1}r)) \right\}^{1/(1-p)},
\]
we find
\[
B_{\alpha, \varphi_p, \omega}(B(0, r); B(0, R)) \leq A_2 K^{-p} \sum_{j=0}^{j_0} \{(2^j r)^{-\alpha p} \varphi((2^j r)^{-1}) \omega(B(0, 2^{j+1}r)) \}^{1/(1-p)}
\]
\[
= A_2 K^{-1-p}.
\]
Here we have
\[
K \geq A_3 \sum_{j=0}^{2^{1+3}r} \{t^{-\alpha p}(t^{-1})\omega(B(0, t))\}^{1/(1-p)} \frac{dt}{t}
\]
\[
\geq A_3 \int_{2r}^{2^{1+3}r} \{t^{-\alpha p}(t^{-1})\omega(B(0, t))\}^{1/(1-p)} \frac{dt}{t},
\]
so that
\[
K \geq A_3 \int_{2r}^{2R} \{t^{-\alpha p}(t^{-1})\omega(B(0, t))\}^{1/(1-p)} \frac{dt}{t},
\]
which proves the result. \(\Box\)

**Corollary 2.2.** If \(\omega\) is doubling, then there exists a constant \(A > 0\) such that
\[
B_{\alpha, \varphi_{\mu}, \omega}(B(x, r); B(0, R)) \leq A \left( \int_{B(0, r)} |t^{-\alpha p}(t^{-1})\omega(B(x, t))|^{1/(1-p)} \frac{dt}{t} \right)^{1-p}
\]
whenever \(B(x, r) \subset B(0, R/4)\). In fact,
\[
B_{\alpha, \varphi_{\mu}, \omega}(B(x, r); B(0, R)) \leq B_{\alpha, \varphi_{\mu}, \omega}(B(x, r); B(x, R/2)) \leq A \left( \int_{B(0, r)} |t^{-\alpha p}(t^{-1})\omega(B(x, t))|^{1/(1-p)} \frac{dt}{t} \right)^{1-p}
\]
whenever \(B(x, r) \subset B(0, R/4)\).

Next we give a lower estimate for the modular \(B_{\alpha, \varphi_{\mu}, \omega}\)-capacity of balls.

**Theorem 2.3.** Suppose \(p > 1\) and \(\omega \in A_p\). For \(R > 0\), there exists a constant \(A = A(R) > 0\) such that
\[
B_{\alpha, \varphi_{\mu}, \omega}(B(0, r); B(0, R)) \geq A \left( \int_{B(0, R)} |t^{-\alpha p}(t^{-1})\omega(B(0, t))|^{1/(1-p)} \frac{dt}{t} \right)^{1-p}
\]
whenever \(0 < r < R/2 < \infty\).

**Proof.** For \(0 < r < R/2\), take a nonnegative measurable function \(f\) on \(B(0, R)\) such that
\[
I_{\alpha} f(x) \geq 1 \quad \text{for } x \in B(0, r).
\]
Then we have by Fubini’s theorem
\[
\int_{B(0, r)} dx \leq \int_{B(0, r)} I_{\alpha} f(x) dx
\]
\[
\leq \int_{B(0, R)} \left( \int_{B(0, r)} |x-y|^{\alpha - n} dx \right) f(y) dy
\]
\[
\leq A_1 r^n \int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) dy.
\]
For $\varepsilon > 0$ and $0 < \delta < \alpha$, we see from Hölder’s inequality that

\[
\int_{B(0,R)} (r + |y|)^{\alpha-n} f(y) \, dy
= \int_{\{y \in B(0,R) : f(y) > \varepsilon (r + |y|)^{-\delta}\}} (r + |y|)^{\alpha-n} f(y) \, dy
+ \int_{\{y \in B(0,R) : f(y) \leq \varepsilon (r + |y|)^{-\delta}\}} (r + |y|)^{\alpha-n} f(y) \, dy
\leq \left( \int_{B(0,R)} [(r + |y|)^{\alpha-n} (\varphi(\varepsilon (r + |y|)^{-\delta}) \omega(y))^{-1/p} p']^{1/p'} \right)^{1/p} \times \left( \int_{B(0,R)} \varphi_p(f(y)) \omega(y) \, dy \right)^{1/p} + \varepsilon \int_{B(0,R)} (r + |y|)^{\alpha-n-\delta} \, dy,
\]

where $1/p + 1/p' = 1$. Since $\omega \in A_p$, we find by $(\varphi 2)$ and $(\varphi 3)$ that

\[
\int_{B(0,R)} [(r + |y|)^{\alpha-n} \{\varphi(\varepsilon (r + |y|)^{-\delta}) \omega(y))^{-1/p} p']^{1/p'} \, dy
\leq A_2(\varepsilon) \int_{0}^{R} (r + t)^{(\alpha-n)p/(p-1)} \varphi((r + t)^{-1})^{1/(1-p)} \left( \int_{B(0,t)} \omega(y)^{1/(1-p)} \, dy \right)^{1/(1-p)} \frac{dt}{t}
\leq A_3(\varepsilon) \int_{0}^{R} (r + t)^{ap/(p-1)} \varphi((r + t)^{-1})^{1/(1-p)} \left( \int_{B(0,r+t)} \omega(y) \, dy \right)^{1/(1-p)} \frac{dt}{t}
\leq A_3(\varepsilon) \int_{r}^{2R} \{t^{-ap} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt / t
\leq A_4(\varepsilon) \int_{r}^{R} \{t^{-ap} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt / t.
\]

Thus we derive

\[
\int_{B(0,R)} (r + |y|)^{\alpha-n} f(y) \, dy
\leq A_4(\varepsilon) \left( \int_{r}^{R} \{t^{-ap} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt / t \right)^{1/p'} \times \left( \int_{B(0,R)} \varphi_p(f(y)) \omega(y) \, dy \right)^{1/p} + A_5 \varepsilon,
\]

so that

\[
1 \leq A_6(\varepsilon) \left( \int_{r}^{R} \{t^{-ap} \varphi(t^{-1}) \omega(B(0,t))\}^{1/(1-p)} \, dt / t \right)^{1/p'} \times \left( \int_{B(0,R)} \varphi_p(f(y)) \omega(y) \, dy \right)^{1/p} + A_7 \varepsilon.
\]
If $A_\gamma = 1/2$, then we establish
\[
B_{\alpha,\varphi_p,\omega}(B(0, r); B(0, R)) \geq A \left( \int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \ dt/t \right)^{1-p},
\]
as required.

**Corollary 2.4.** Suppose $p > 1$ and $\omega \in A_p$. For $R > 0$, there exists a constant $A = A(R) > 0$ such that
\[
B_{\alpha,\varphi_p,\omega}(B(x, r); B(0, R)) \geq A \left( \int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(x, t)) \}^{1/(1-p)} \ dt/t \right)^{1-p}
\]
whenever $B(x, r) \subset B(0, R/4)$.

Now Theorem A follows from Corollaries 2.2 and 2.4.

3. **Proof of Theorem B**

Let us begin with an upper estimate for the norm $C_{\alpha,\varphi_p,\omega}$-capacity of balls.

**Theorem 3.1.** Suppose that $p > 1$ and that $\omega(B(0, r))$ satisfies the doubling condition. Then there exists a constant $A > 0$ such that
\[
C_{\alpha,\varphi_p,\omega}(B(0, r); B(0, R)) \leq A \left( \int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \ dt/t \right)^{(1-p)/p}
\]
whenever $0 < r < R/2 < \infty$.

**Proof.** Set
\[
\varphi^*(r) = \int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \ dt/t
\]
for $r > 0$. Consider the function
\[
f(y) = |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|) \omega(B(0, |y|)) \}^{1/(1-p)} \varphi^*(r)^{-1/p}
\]
for $r < |y| < R$ and $f = 0$ elsewhere. If $x \in B(0, r)$ and $y \in B(0, R) \setminus B(0, r)$, then $|x - y| < 2|y|$, so that
\[
I_{\alpha} f(x) \geq 2^{\alpha - n} \varphi^*(r)^{-1/p} \int_{B(0, R) \setminus B(0, r)} |y|^{-n} \{ |y|^{-\alpha p} \varphi(|y|) \omega(B(0, |y|)) \}^{1/(1-p)} \ dy
\]
\[
= A_1 \varphi^*(r)^{(p-1)/p}
\]
with a constant $A_1 = A_1(\alpha, n) > 0$. It follows from the definition of capacity that
\[
C_{\alpha,\varphi_p,\omega}(B(0, r); B(0, R)) \leq A_1^{-1} \varphi^*(r)^{(1-p)/p} \| f \|_{\varphi_p,\omega, B(0, R)}.
\]
Thus it suffices to show that
\[
\| f \|_{\varphi_p,\omega, B(0, R)} \leq A_2.
\]
Suppose that

Corollary 3.2.

as required.

For this purpose, we first note that
\[
\int_{B(0,R)} \varphi_p(f(y))\omega(y)dy \\
= \int_{B(0,R)\setminus B(0,r)} f(y)^p \varphi(f(y))\omega(y)dy \\
\leq \varphi^*(r)^{-1} \int_{B(0,r)\setminus B(0,r)} |y|^{-\alpha p} \{ |y|^{-\alpha p} \varphi(|y|^{-1})\omega(B(0,|y|))\}^{p/(1-p)} \\
\times \varphi\left( |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)} \right)^{-1/p} \omega(y)dy.
\]

Here we see from the doubling condition of \(\omega(B(0,r))\) that
\[
|y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)} \varphi^*(r)^{-1/p} \\
\leq A_3 |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)} \\
\times \left( \int_{r}^{2R} \{ t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t \right)^{-1/p} \\
\leq A_3 |y|^{-\alpha} \{ |y|^{-\alpha p} \varphi(|y|^{-1})\omega(B(0,|y|))\}^{1/(1-p)} \\
\times \left( \int_{|y|}^{2|y|} \{ t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t \right)^{-1/p} \\
\leq A_4 \varphi(|y|^{-1})^{-1/p} \omega(B(0,|y|))^{-1/p}
\]
for \(y \in B(0,R) \setminus B(0,r)\). Further we can find constants \(A_0 > 0\) and \(\gamma > 0\) such that
\[
\omega(B(0,t)) \geq A_0 t^\gamma
\]
for all \(t > 0\). Hence, as in Theorem 2.1, we obtain by (\(\varphi 6\)):
\[
\int_{B(0,R)} \varphi_p(f(y))\omega(y)dy \\
\leq A_5 \varphi^*(r)^{-1} \int_{B(0,r)\setminus B(0,r)} |y|^{-\alpha p} \{ |y|^{-\alpha p} \varphi(|y|^{-1})\omega(B(0,|y|))\}^{p/(1-p)} \varphi(|y|^{-1})\omega(y)dy \\
\leq A_6 \varphi^*(r)^{-1} \int_{r}^{R} \{ t^{-\alpha p} \varphi(t^{-1})\omega(B(0,t))\}^{1/(1-p)} dt/t \\
= A_6,
\]
as required.

\[\Box\]

Corollary 3.2. Suppose that \(p > 1\) and \(\omega\) is doubling. Then there exists a constant \(A > 0\) such that
\[
C_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) \leq A \left( \int_{r}^{R} \{ t^{-\alpha p} \varphi(t^{-1})\omega(B(x,t))\}^{1/(1-p)} dt/t \right)^{(1-p)/p}
\]
whenever \(B(x,r) \subset B(0,R/4)\).

Next we give a lower estimate for the norm \(C_{\alpha,\varphi_p,\omega}\)-capacity of balls.
Theorem 3.3. Suppose \( p > 1 \) and \( \omega \in A_p \). For \( R > 0 \), there exists a constant \( A = A(R) > 0 \) such that

\[
C_{\alpha, \varphi, \omega}(B(0, r); B(0, R)) \geq A \left( \int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \, dt/t \right)^{(1-p)/p}
\]

whenever \( 0 < r < R/2 < \infty \).

Proof. For \( 0 < r < R/2 \) take a nonnegative measurable function \( f \) on \( B(0, R) \) such that

\[
I_\alpha f(x) \geq 1 \quad \text{for } x \in B(0, r).
\]

Then we have by Fubini’s theorem

\[
\int_{B(0, r)} dx \leq \int_{B(0, r)} I_\alpha f(x) \, dx \\
\leq \int_{B(0, r)} \left( \int_{B(0, r)} |x - y|^{\alpha - n} \, dx \right) f(y) \, dy \\
\leq A_1 r^n \int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy,
\]

so that

\[
1 \leq A_1 \int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy.
\]

We show that

\[
\int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy \\
\leq A_2 \left( \int_r^R \{ t^{-\alpha p} \varphi(t^{-1}) \omega(B(0, t)) \}^{1/(1-p)} \, dt/t \right)^{(p-1)/p} \|f\|_{\varphi_p, \omega, B(0, R)}.
\]

For this purpose, suppose that \( \|f\|_{\varphi_p, \omega, B(0, R)} \leq 1 \). For \( 0 < \delta < \alpha \), we see from Hölder’s inequality that

\[
\int_{B(0, R)} (r + |y|)^{\alpha - n} f(y) \, dy \\
= \int_{\{y \in B(0, R): f(y) > (r + |y|)^{-\delta}\}} (r + |y|)^{\alpha - n} f(y) \, dy \\
+ \int_{\{y \in B(0, R): f(y) \leq (r + |y|)^{-\delta}\}} (r + |y|)^{\alpha - n} f(y) \, dy \\
\leq \left( \int_{B(0, R)} [(r + |y|)^{\alpha - n} \{ \varphi((r + |y|)^{-\delta}) \omega(y) \}^{-1/p}]^{p'} \, dy \right)^{1/p'} \\
\times \left( \int_{B(0, R)} \varphi_p(f(y)) \omega(y) \, dy \right)^{1/p} + \int_{B(0, R)} (r + |y|)^{\alpha - n - \delta} \, dy.
\]
Here note from \( \omega \in A_p \), (\( \varphi 2 \)) and (\( \varphi 3 \)) that
\[
\int_{B(0,R)} [(r + |y|)^{\alpha - n} \varphi((r + |y|)^{-\delta}) \omega(y)]^{-1/p} dy \\
\leq A_3 \int_0^R (r + t)^{(\alpha - n)p/(p-1)} \varphi((r + t)^{-1})^{1/(1-p)} \left( \int_{B(0,t)} \omega(y)^{1/(1-p)} dy \right) dt/t \\
\leq A_4 \int_0^R (r + t)^{\alpha p/(p-1)} \varphi((r + t)^{-1})^{1/(1-p)} \left( \int_{B(0,t)} \omega(y) dy \right)^{1/(1-p)} dt/t \\
\leq A_4 \int_r^{2R} (t^{-\alpha p} \varphi(t^{-1})) \omega(B(0,t)) \right)^{1/(1-p)} dt/t \\
\leq A_5 \int_r^R (t^{-\alpha p} \varphi(t^{-1})) \omega(B(0,t)) \right)^{1/(1-p)} dt/t,
\]
so that
\[
\int_{B(0,R)} (r + |y|)^{\alpha - n} f(y) dy \\
\leq A_5 \left( \int_r^R (t^{-\alpha p} \varphi(t^{-1})) \omega(B(0,t)) \right)^{1/(1-p)} dt/t \\
\times \left( \int_{B(0,R)} \varphi_p(f(y)) \omega(y) dy \right)^{1/p} + A_6 \\
\leq A_7 \left( \int_r^R (t^{-\alpha p} \varphi(t^{-1})) \omega(B(0,t)) \right)^{1/(1-p)} dt/t \right)^{1/p'}.
\]
Hence we establish
\[
C_{\alpha,\varphi_p,\omega}(B(0,r);B(0,R)) \geq A \left( \int_r^R (t^{-\alpha p} \varphi(t^{-1})) \omega(B(0,t)) \right)^{1/(1-p)} dt/t \right)^{(1/p')/p}
\]
as required. \( \square \)

Corollary 3.4. Suppose \( p > 1 \) and \( \omega \in A_p \). For \( R > 0 \), there exists a constant \( A = A(R) > 0 \) such that
\[
C_{\alpha,\varphi_p,\omega}(B(x,r);B(0,R)) \geq A \left( \int_r^R (t^{-\alpha p} \varphi(t^{-1})) \omega(B(x,t)) \right)^{1/(1-p)} dt/t \right)^{(1-p)/p}
\]
whenever \( B(x,r) \subset B(0,R/4) \).

As in the proof of Theorem A, Theorem B follows readily from Corollaries 3.2 and 3.3.

References


