RANK-ONE FLOWS OF TRANSFORMATIONS WITH INFINITE ERGODIC INDEX

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Abstract. A rank-one infinite measure preserving flow $T = (T_t)_{t \in \mathbb{R}}$ is constructed such that for each $t \neq 0$, the Cartesian powers of the transformation $T_t$ are all ergodic.

0. Introduction

In 1963 Kakutani and Parry discovered an interesting phenomenon in the theory of infinite measure preserving maps. They showed that for each $p > 0$ there exists a transformation whose $p$-th Cartesian power is ergodic but whose $(p+1)$-th Cartesian power is not \[ KP \]. Since then a number of other examples of transformations with exotic (from the point of view of the classical “probability preserving” ergodic theory) weak mixing properties were constructed. See surveys \[ Da2 \] and \[ DaS3 \] for a detailed discussion on that. In \[ Da1 \], \[ DaS1 \] these examples were extended to infinite measure preserving actions of discrete countable Abelian groups. Weak mixing properties of infinite measure preserving actions of continuous Abelian groups such as $\mathbb{R}$ and $\mathbb{R}^d$ were under consideration in \[ I–W \]. In particular, a rank-one flow (i.e. $\mathbb{R}$-action) whose Cartesian square is ergodic was constructed there. A rank-one infinite measure preserving flow $T = (T_t)_{t \in \mathbb{R}}$ with infinite ergodic index (i.e. the Cartesian powers of $T$ are all ergodic) appeared in a recent paper \[ DaSo \]. It can be deduced easily from \[ DaSo \] that there is a residual subset $D_T$ of $\mathbb{R}$ such that for each $t \in D_T$, the transformation $T_t$ has infinite ergodic index. However the following more subtle question by C. Silva remains open so far:

Is there a rank-one infinite measure preserving flow $T$ with $D_T = \mathbb{R} \setminus \{0\}$?

Our purpose in this paper is to answer his question in the affirmative.

Theorem 0.1. There is a rank-one infinite $\sigma$-finite measure preserving flow $T = (T_t)_{t \in \mathbb{R}}$ such that for each $t \neq 0$, the transformation $T_t$ has infinite ergodic index.

The main idea of the proof is different from those that were used in \[ I–W \] and \[ DaSo \]. It is based on a technique of forcing a dynamical property. Originating from \[ Ry1 \], such techniques were utilized in \[ Ry2 \], \[ DaR \], etc., to obtain mixing,
power weak mixing, etc., of some systems. In this paper the desired flow appears as a certain limit of a sequence of weakly mixing finite measure preserving flows. We construct this sequence in such a way as to retain the property of infinite ergodic index in the limit. The construction is implemented in the language of \((C,F)\)-actions (see [Da2]).

1. Preliminaries: Rank-one actions and \((C,F)\)-actions of \(\mathbb{R}^d\)

We first recall the definition of rank one. Let \(S = (S_g)_{g \in \mathbb{R}^d}\) be a measure preserving action of \(\mathbb{R}^d\) on a standard \(\sigma\)-finite measure space \((Y, \mathcal{E}, \nu)\).

**Definition 1.1.**

(i) A Rokhlin tower or column for \(S\) is a triple \((A, f, F)\), where \(A \in \mathcal{E}\), \(F\) is a cube in \(\mathbb{R}^d\) and \(f : A \to F\) is a measurable mapping such that for any Borel subset \(H \subset F\) and an element \(g \in \mathbb{R}^d\) with \(g + H \subset F\), one has \(f^{-1}(g + H) = S_g f^{-1}(H)\).

(ii) \(S\) is said to be of rank one (by cubes) if there exists a sequence of Rokhlin towers \((A_n, f_n, F_n)\) such that the volume of \(F_n\) goes to infinity and for any subset \(C \in \mathcal{E}\) of finite measure, there is a sequence of Borel subsets \(H_n \subset F_n\) such that

\[
\lim_{n \to \infty} \nu(C \triangle f_n^{-1}(H_n)) = 0.
\]

The \((C,F)\)-construction of measure preserving actions for discrete countable groups was introduced in [dJ] and [Da1]. It was extended to the case of locally compact second countable Abelian groups in [DaN2]. (See also [Da2].) Here we outline it briefly for \(\mathbb{R}^d\), \(d \in \mathbb{N}\).

Given two subsets \(E, F \subset \mathbb{R}^d\), by \(E + F\) we mean their algebraic sum, i.e. \(E + F = \{ e + f \mid e \in E, f \in F \}\). The algebraic difference \(E - F\) is defined in a similar way. If \(F\) is a singleton, say \(F = \{f\}\), then we will write \(E + f\) for \(E + F\). Two subsets \(E\) and \(F\) of \(\mathbb{R}^d\) are called independent if \((E - E) \cap (F - F) = \{0\}\); i.e. if \(e + f = e' + f'\) for some \(e, e' \in E, f, f' \in F\), then \(e = e'\) and \(f = f'\).

Fix \(p \in \mathbb{N}\) and consider two sequences \((F_n)_{n=0}^{\infty}\) and \((C_n)_{n=1}^{\infty}\) of subsets in \(\mathbb{R}^d\) such that \(F_n\) is a cube \([0, h_n) \times \cdots \times [0, h_n)\) \((d\) times\) for an \(h_n \in \mathbb{R}\), \(C_n \subset \mathbb{R}^d\) is a finite set, \(#C_n > 1\),

\[
\begin{align*}
F_n & \text{ and } C_{n+1} \text{ are independent, and} \\
F_n + C_{n+1} & \subset F_{n+1}.
\end{align*}
\]

This means that \(F_n + C_{n+1}\) consists of \(#C_{n+1}\) mutually disjoint ‘copies’ \(F_n + c\) of \(F_n\), \(c \in C_{n+1}\), and all these copies are contained in \(F_{n+1}\). We equip \(F_n\) with the measure \((#C_1 \cdots #C_n)^{-1}(\lambda_{\mathbb{R}^d} \restriction F_n)\), where \(\lambda_{\mathbb{R}^d}\) denotes Lebesgue measure on \(\mathbb{R}^d\). Endow \(C_n\) with the equidistributed probability measure. Let \(X_n := F_n \times \prod_{k>n} C_k\) stand for the product of measure spaces. Define an embedding \(X_n \to X_{n+1}\) by setting

\[
(f_n, c_{n+1}, c_{n+2}, \ldots) \mapsto (f_n + c_{n+1}, c_{n+2}, \ldots).
\]

It is easy to see that this embedding is measure preserving. Then \(X_0 \subset X_1 \subset \cdots\). Let \(X := \bigcup_{n=0}^{\infty} X_n\) denote the inductive limit of the sequence of measure spaces \(X_n\) and let \(\mu\) denote the corresponding measure on \(X\). Then \(\mu\) is \(\sigma\)-finite. It is infinite if and only if

\[
\lim_{n \to \infty} \frac{h_n^d}{#C_1 \cdots #C_n} = \infty.
\]
Given $g \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we set
\[
L_g^{(n)} := (F_n \cap (F_n - g)) \times \prod_{k>n} C_k \quad \text{and} \quad R_g^{(n)} := (F_n \cap (F_n + g)) \times \prod_{k>n} C_k.
\]

Clearly, $L_g^{(n)} \subseteq L_g^{(n+1)}$ and $R_g^{(n)} \subseteq R_g^{(n+1)}$. Define a map $T_g^{(n)} : L_g^{(n)} \to R_g^{(n)}$ by setting
\[
T_g^{(n)}(f_n, c_{n+1}, \ldots) := (f_n + g, c_{n+1}, \ldots).
\]
Put
\[
L_g := \bigcup_{n=1}^{\infty} L_g^{(n)} \subset X \quad \text{and} \quad R_g := \bigcup_{n=1}^{\infty} R_g^{(n)} \subset X.
\]

Then a Borel one-to-one map $T_g : L_g \to R_g$ is well defined by $T_g \mid L_g^{(n)} = T_g^{(n)}$. Since $h_n \to \infty$, it follows that $\mu(X \setminus L_g) = \mu(X \setminus R_g) = 0$ for each $g \in \mathbb{R}^d$. It is easy to verify that $T := (T_g)_{g \in \mathbb{R}^d}$ is a Borel $\mu$-preserving action of $\mathbb{R}^d$.

**Definition 1.2.** $T$ is called the $(C, F)$-action of $\mathbb{R}^d$ associated with $(C_{n+1}, F_n)_{n \geq 0}$.

Each $(C, F)$-action is of rank one.

Given a Borel subset $A \subset F_n$, we set $[A]_n := \{x = (x_i)_{i=1}^{\infty} \in X_n \mid x_n \in A\}$ and call it an $n$-cylinder in $X$. Clearly,
\[
[A]_n = \bigcup_{c \in C_{n+1}} [A + c]_{n+1}.
\]

Notice also that
\[
(1-4) \quad T_g[A]_n = [A + g]_n \quad \text{for all } g \in \mathbb{R}^d \text{ and } A \subset F_n \cap (F_n - g), \ n \in \mathbb{N}.
\]

The sequence of all $n$-cylinders approximates the entire Borel $\sigma$-algebra on $X$ when $n \to \infty$.

We state without proof the following standard lemma (see, e.g., Lemma 2.4 from [Da1]).

**Lemma 1.3.** Let $\mathcal{P}_n$ be a finite partition of $F_n$ into parallelepipeds such that for each atom $\Delta$ of $\mathcal{P}_n$ and an element $c \in C_{n+1}$, the parallelepiped $\Delta + c$ is $\mathcal{P}_{n+1}$-measurable and the maximal diameter of the atoms in $\mathcal{P}_n$ goes to zero. Let $S$ be a measure preserving transformation of $X$. Then the following hold:

(i) The sequence of collections of $n$-cylinders $\{[A]_n \mid A \subset F_n \text{ is } \mathcal{P}_n\text{-measurable}\}$ approximates the entire $\sigma$-algebra $\mathfrak{B}$ as $n \to \infty$.

(ii) If for each pair of atoms $\Delta_1, \Delta_2 \in \mathcal{P}_n$, there are a subset $A \subset [\Delta_1]_n$ and a $\mu$-preserving one-to-one map $\gamma : A \to [\Delta_2]_n$ such that $\mu(A) > 0.5\mu([\Delta_1]_n)$ and $\gamma x \in \{S^i x \mid i \in \mathbb{Z}\}$ for all $x \in A$, then $S$ is ergodic.

We will also use the following property of the $(C, F)$-actions. If $T$ is associated with $(C_{n+1}, F_n)_{n \geq 0}$, then for each $p > 1$, the product action
\[
(T_{t_1} \times \cdots \times T_{t_p})_{(t_1, \ldots, t_p) \in (\mathbb{R}^d)^p}
\]
is the $(C, F)$-action of $(\mathbb{R}^d)^p$ associated with $(C_{n+1}^p, F_n^p)_{n \geq 0}$. The upper index $p$ means the $p$-th Cartesian power.
2. TWO AUXILIARY FACTS

Given a \( \sigma \)-finite measure space \((X, \mu)\), we denote by \( \text{Aut}(X, \mu) \) the group of all \( \mu \)-preserving (invertible) transformations of \( X \). It is a Polish group when endowed with the weak topology \( \mathbb{A}_d \). Recall that the weak topology is the weakest topology in which the maps

\[
\text{Aut}(X, \mu) \ni T \mapsto \mu(TA \cap B) \in \mathbb{R}
\]

are continuous for all subsets \( A, B \subset X \) of finite measure.

Given \( S \in \text{Aut}(X, \mu) \) and two subsets \( A, B \subset X \) with \( \mu(A) = \mu(B) < \infty \), we define subsets \( A_0, A_1, \ldots \) of \( A \) as follows:

\[
A_0 := A \cap B;
A_i := \left( A \setminus \bigcup_{j=0}^{i-1} A_j \right) \cap S^{-i} \left( B \setminus \bigcup_{j=0}^{i-1} S^j A_j \right), \quad i > 0.
\]

We now let \( \mathcal{N}_{S,A,B} := \min \{ \{ i \geq 0 \mid \mu(A_0 \sqcup \cdots \sqcup A_i) > 0.5\mu(A) \} \} \). If \( S \) is ergodic, then \( A = \bigcup_{i \geq 0} A_i \) and hence \( \mathcal{N}_{S,A,B} \) is well defined. Denote by \( \mathcal{E} \) the subset of all ergodic transformations in \( \text{Aut}(X, \mu) \). It is well known that \( \mathcal{E} \) is a dense \( G_\delta \) in \( \text{Aut}(X, \mu) \). Since for each \( i \geq 0 \) the map

\[
\text{Aut}(X, \mu) \ni S \mapsto \mu(A_0 \sqcup \cdots \sqcup A_i) \in \mathbb{R}
\]

is continuous, we obtain the following lemma.

**Lemma 2.1.** The map \( \mathcal{E} \ni S \mapsto \mathcal{N}_{S,A,B} \in \mathbb{R} \) is upper semicontinuous for all subsets \( A, B \subset X \) with \( \mu(A) = \mu(B) < \infty \).

In the case of \((C, F)\)-actions we can say more about the “structure” of the sets \( A_i \), \( i = 0, \ldots, \mathcal{N}_{S,A,B} \). For \( q = (q_1, \ldots, q_d) \in \mathbb{R}^d \), we let \( \|q\| := \max_{1 \leq i \leq d} |q_i| \).

**Lemma 2.2.** Let \((X, \mu, (T_t)_{t \in \mathbb{R}^d})\) be a \((C, F)\)-action of \( \mathbb{R}^d \) associated with a sequence \((C_{n+1}, F_n)_{n \geq 0}\) such that

\[
a + F_n + C_{n+1} \subset F_{n+1}
\]

for each \( a = (a_1, \ldots, a_p) \) with \( a_i \geq 0 \), \( i = 1, \ldots, p \), and \( \|a\| \leq 1 \). Fix two \( n \)-cylinders \( A \) and \( B \) of equal measure and a transformation \( S = T_q \) for some \( q \in \mathbb{R}^d \). Then the subsets \( A_0, A_1, \ldots, A_{\mathcal{N}(S,A,B)} \) defined by \( (2-1) \) are \((n + Q \cdot \mathcal{N}(S,A,B))\)-cylinders, where \( Q \) is any integer greater than \( \|q\| \).

**Proof.** We let \( N := Q \cdot \mathcal{N}(S,A,B) \). If \( A = [\tilde{A}]_n \) for some \( \tilde{A} \subset F_n \), then \( A = [\tilde{A}]_{n+N} \), where \( \tilde{A} := \tilde{A} + C_{n+1} + \cdots + C_{n+N} \subset F_{n+N} \). From \((1-2)\) and \((2-2)\), we deduce that the sets \( \tilde{A}+q, \ldots, \tilde{A}+N(S,A,B)q \) are all contained in \( F_{n+N} \). It remains to use \((2-1)\) and \((1-4)\). \( \square \)

We also note that \( S^i A_i \subset B \) and \( S^i A_i \cap S^j A_j = \emptyset \) for all \( i, j = 0, \ldots, \mathcal{N}(S,A,B) \).

3. PROOF OF THE MAIN RESULT

**Theorem 3.1.** There exists a \((C, F)\)-flow \( T = (T_t)_{t \in \mathbb{R}} \) such that each transformation \( T_t, t \neq 0 \), has infinite ergodic index.
Proof. We will construct this flow via an inductive procedure. Fix a sequence of integers \((p_n)_{n \geq 1}\) in which every integer greater than 1 occurs infinitely many times. Suppose that after \(n - 1\) steps of the construction we have already defined

\[
F_0, C_1, F_1, \ldots, C_{m_{n-1}}, F_{m_{n-1}}.
\]

Suppose also that for each \(0 \leq i \leq m_{n-1}\), a finite partition \(P_i\) of \(F_i\) into intervals is chosen in such a way that

- the interval \(\Delta + c\) is \(P_{i+1}\)-measurable for each atom \(\Delta\) of \(P_i\), \(0 \leq i < m_{n-1}\), and each \(c \in C_{i+1}\)
- the length of any atom of \(P_i\) is no more than \(i^{-1}\), \(1 \leq i \leq m_{n-1}\).

Step \(n\). Consider a rank-one weakly mixing finite measure preserving \((C, F)\)-flow \(T^{(n)} = (T^{(n)}_t)_{t \in \mathbb{R}}\) associated with a sequence \((C_{k+1,n}, F_{k,n})_{k \geq 0}\) such that \(F_0,n := F_{m_{n-1}}\). Examples of weakly mixing rank-one finite measure preserving flows are well known—see, e.g., [dJP]. In [DaS2] one can find explicit \((C, F)\)-construction of mixing finite measure preserving flows. Let \((X^{(n)}, \mu_n)\) be the space of this action. Since \(T^{(n)}\) is weakly mixing, it follows that for each \(t > 0\), the transformation

\[
S_t := T^{(n)}_t \times \cdots \times T^{(n)}_t (p_n \text{ times})
\]

of the product space \((X^{(n)}, \mu_n)^{p_n}\) is ergodic. We note that this space is the space of the \((C, F)\)-action of \(\mathbb{R}^{p_n}\) associated with the sequence \((C_{k+1,n}^{p_n}, F_{k,n}^{p_n})_{k \geq 0}\) (see our remark at the end of §1). Given \(k \geq 0\), let \(P_{k,n}\) be a finite partition of \(F_{k,n}\) into intervals such that

- \(P_{0,n} = P_{m_{n-1}}\),
- the interval \(\Delta + c\) is \(P_{k+1,n}\)-measurable for each atom \(\Delta\) of \(P_{k,n}\) and \(c \in C_{k+1,n}\) and
- the length of any atom of \(P_{k,n}\) is no more than \((m_{n-1} + k)^{-1}\).

We now let \(P_{k,n}^{p_n} := P_{k,n} \times \cdots \times P_{k,n} (p_n \text{ times})\). Then \(P_{k,n}^{p_n}\) is a finite partition of \(F_{k,n}^{p_n}\) into parallelepipeds and

- the parallelepiped \(\Delta + c\) is \(P_{k+1,n}^{p_n}\)-measurable for each atom \(\Delta\) of \(P_{k,n}^{p_n}\) and \(c \in C_{k+1,n}^{p_n}\) and
- the diameter of an atom of \(P_{k,n}^{p_n}\) is no more than \((m_{n-1} + k)^{-p_n}\).

Denote by \(D_n\) the maximum of \(N(S_t, [\Delta]_0, [\Delta']_0)\) when \(\Delta\) and \(\Delta'\) run independently over the atoms of \(P_{0,n}^{p_n}\) and \(t\) runs over the segment \([n^{-1}, n) \subset \mathbb{R}\). It exists by Lemma 2.1. It now follows from Lemma 2.2 that for any pair of parallelepipeds \(\Delta, \Delta' \in P_{0,n}^{p_n}\) and a real \(t \in [n^{-1}, n]\), there exist \(nD_n\)-cylinders \(A_1, \ldots, A_{D_n} \subset [\Delta]_0\) such that

\[
\mu_n^{p_n} \left( \bigcup_{i=1}^{D_n} A_i \right) > \frac{1}{2} \mu_n^{p_n} ([\Delta]_0),
\]

(3-3)

\[
S_t^i A_i \subset [\Delta']_0 \text{ for each } 1 \leq i \leq D_n \text{ and } \\
S_t^i A_i \cap S_t^j A_j = \emptyset \text{ if } 1 \leq i \neq j \leq D_n.
\]

We now “continue” the sequence \((3-1)\) by setting

\[
C_{m_{n-1}+1} := C_{1,n}, \quad F_{m_{n-1}+1} := F_{1,n}, \ldots, \quad C_{m_{n-1}+nD_n} := C_{nD_n,n}.
\]

Next, to define \(F_{m_{n-1}+nD_n}\) we “double” the set \(F_{nD_n,n}\), i.e.

\[
F_{m_{n-1}+nD_n} := [0, 2a) \text{ if } F_{nD_n,n} = [0, a) \text{ for some } a > 0.
\]
It remains to put \( m_n := m_{n-1} + nD_n \). The \( n \)-th step is now completed.

Continuing this procedure infinitely many times, we obtain the entire sequence \((C_{i+1}, F_i)_{i \geq 0}^\times\). Denote by \( T = (T_i)_{i \leq \infty} \) the associated \((C, F)\)-flow. Let \((X, \mu)\) be the space of this flow. It follows from (3.4) that \( \lambda_H(F_i) > 2\lambda_H(F_{i-1}) \#C_i \) for infinitely many \( i \). Hence \( \mu(X) = \infty \). Moreover, a finite partition \( \mathcal{P}_i \) of \( F_i \) into intervals is fixed such that the conditions of Lemma 1.3 are satisfied. Next, there are one-to-one correspondences (natural identifications) between

- the collection of 0-cylinders in \( X^{(n)} \) and the collection of \( m_{n-1} \)-cylinders in \( X \)
- the collection of \( nD_n \)-cylinders in \( X^{(n)} \) and the collection of \( m_n \)-cylinders in \( X \).

Moreover, the “dynamics” of \( T^{(n)} \) on the \( nD_n \)-cylinders is the same as the dynamics of \( T \) on the \( m_n \)-cylinders. This means the following: if \( A, B \subseteq F_{nD_n}^{(n)} \) and \( [B]_{nD_n} = T_w^{(n)}[A]_{nD_n} \) for some \( w \in \mathbb{R} \), then \( [B]_{m_n} = T_w[A]_{m_n} \). Therefore we deduce from (3.5) that for any pair of parallelepipeds \( \Delta, \Delta' \in \mathcal{P}_{m_{n-1}}^{(n)} \) and a real \( t \in [n^{-1}, n] \), there exist \( m_n \)-cylinders \( A_1, \ldots, A_{D_n} \subset [\Delta]_{m_{n-1}} \) such that

\[
\mu^{p_n}\left( \bigcup_{i=1}^{D_n} A_i \right) > \frac{1}{2} \mu^{p_n}(\Delta)_{m_{n-1}}.
\]

\[ V_t^i A_i \subset [\Delta]_{m_{n-1}} \text{ for each } 1 \leq i \leq D_n \]

\[ V_t^i A_i \cap V_t^j A_j = \emptyset \text{ if } 1 \leq i \neq j \leq D_n, \]

where \( V_t := T_t \times \cdots \times T_t \) \( (p_n \text{ times}) \). Fix \( p > 0 \). Passing to a subsequence where \( p_n = p \) we now deduce from Lemma 1.3(ii) that \( T_t \times \cdots \times T_t \) \( (p \text{ times}) \) is ergodic for each \( t > 0 \). Hence \( T_t \) has infinite ergodic index. \( \square \)

4. CONCLUDING REMARKS

4.1. When constructing \( T \), we use only finitely many initial terms of the sequence \((C_{k+1}, F_k)_{k \geq 0}^\times\) for each \( n > 0 \). However to determine “where to stop” (i.e. to determine \( D_n \)) we use the weak mixing properties of the auxiliary flow \( T^{(n)} \) which depends on the entire infinite sequence \((C_{k+1}, F_k)_{k \geq 0}^\times\). No upper bound on \( D_n \) is found. This means that the construction of \( T \) is not effective. In this connection we raise a question:

\textit{Is it possible to find an effective construction for the flow from Theorem 0.1?}

We note that the construction in \([\text{DaSo}]\) is effective.

4.2. It is possible to strengthen Theorem 0.1 by replacing the infinite ergodic index with a stronger property of power weak mixing. Recall that a measure preserving transformation \( S \) is called power weak mixing if for each finite sequence \( n_1, \ldots, n_k \) of nonzero integers the transformation \( S^{n_1} \times \cdots \times S^{n_k} \) is ergodic. Only a slight modification of our argument is needed to show the following theorem. (The idea is to define \( S_t \) in (3.2) as the product of different nonzero powers \( T_t^{(n)} \). The product is ergodic since \( T^{(n)} \) is finite measure preserving and weakly mixing.)

**Theorem 4.1.** There exists a rank-one infinite \( \sigma \)-finite measure preserving flow \( T = (T_t)_{t \in \mathbb{R}} \) such that the transformation \( T_t \) is power weakly mixing for each \( t \neq 0 \).

Also, it is easy to extend Theorems 0.1 and 4.1 to actions of \( \mathbb{R}^d \).
Theorem 4.2. For each $d > 1$, there exists a rank-one infinite $\sigma$-finite measure preserving action $T = (T_g)_{g \in \mathbb{R}^d}$ of $\mathbb{R}^d$ such that the transformation $T_g$ is power weakly mixing for each $g \neq 0$.

We leave the proofs of Theorems 4.1 and 4.2 to the reader.

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