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SPHERICAL POINTS IN RIEMANNIAN MANIFOLDS

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ABSTRACT. A point p in a Riemannian manifold M is weakly spherical if for each point $q \neq p$ there is either exactly one or at least three minimizing geodesic segments joining p to q. In this note, it is shown that round 2-dimensional spheres are the only Riemannian surfaces with a weakly spherical point realizing the injectivity radius.

Götz and Rybarski asked whether round spheres are the only convex surfaces with the property that every pair of points is joined either by a unique minimizing geodesic or by infinitely many minimizing geodesics [5, 4]. Zamfirescu answered this question affirmatively in [14] for C^3-smooth convex surfaces. In this note, round spheres are similarly characterized amongst all smooth Riemannian surfaces.

Definition 1. Let M denote a closed Riemannian manifold. A point $p \in M$ is defined to be *weakly spherical* if for each distinct point $q \in M$ the set of minimizing geodesics joining p to q has either one element or at least three elements. A point $p \in M$ is defined to be *strongly spherical* if for each distinct point $q \in M$ the set of minimizing geodesics joining p to q has either one element or infinitely many elements.

Every point in a constant curvature sphere is strongly spherical. More generally, every point in a product of simply connected compact rank one symmetric spaces is strongly spherical. This raises the following question.

Question. Assume that M is an irreducible closed Riemannian manifold with all points (strongly) spherical. Is M isometric to a simply connected compact rank one symmetric space?

When $\dim(M) = 2$ the answer is yes as proved below. Prior to stating this theorem, some notation is needed. Readers are referred to [3] for basic results about Riemannian manifolds.

Let M denote a closed Riemannian manifold and d its distance function. For $p \in M$, let T_pM denote the tangent space to p and $\mathrm{TCut}(p) \subset T_pM$ the *tangent cut locus* of p. Its image under the exponential map $\exp_p$ is the *cut locus* of p, denoted by $\mathrm{Cut}(p)$. The injectivity radius of M at p is defined by $\mathrm{inj}(p) = d(p, \mathrm{Cut}(p))$. The injectivity radius of M is defined by $\mathrm{inj}(M) = \inf_{p \in M}\{\mathrm{inj}(p)\}$ and is realized by at least one point of M.

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Theorem 2. *If M is a closed Riemannian surface with a weakly spherical point realizing the injectivity radius, then M is isometric to a constant curvature sphere.*

Proof. Let $p \in M$ be a weakly spherical point of M. A point $q \in \mathrm{Cut}(p)$ is a *cleave cut point* if there are precisely two minimizing geodesic segments joining p to q such that q is not conjugated to p along either of these segments.

The 1-dimensional Hausdorff measure of non-cleave cut points in $\mathrm{Cut}(p)$ is zero [8, Proposition 2.1]. The hypothesis that p is weakly spherical implies that no point in $\mathrm{Cut}(p)$ is a cleave cut point. Therefore, $\mathrm{Cut}(p)$ has zero 1-dimensional Hausdorff measure.

By [10], $\mathrm{Cut}(p)$ is a *local tree* (see [8, Theorem 2.3] for detailed definitions). A local tree with zero 1-dimensional Hausdorff measure is a single point. The conclusion of the theorem is now a consequence of the following lemma.

Lemma 3. *Assume that M is a closed Riemannian manifold and that $p \in M$ satisfies* $\mathrm{inj}(p) = \mathrm{inj}(M)$. *If* $\mathrm{Cut}(p)$ *is a single point, then M is isometric to a constant curvature sphere.*

Let $\mathrm{Cut}(p) = \{q\}$ and note that M is homeomorphic to a sphere. Let $\mathrm{diam}(M)$ denote the diameter of M. By the resolution of the Blaschke conjecture for spheres [1, 9, 12, 13], metrics on M with $\mathrm{inj}(M) = \mathrm{diam}(M)$ have constant curvature. It suffices to show that $\mathrm{diam}(M) \leq \mathrm{inj}(M)$ since $\mathrm{inj}(M) \leq \mathrm{diam}(M)$ always holds.

Choose points $x, y \in M$ such that $d(x, y) = \mathrm{diam}(M)$. The hypotheses imply that x and y lie in geodesics γ_x and γ_y of length $\mathrm{inj}(M)$ which join p to q.

If $\gamma_x = \gamma_y$, then $\mathrm{diam}(M) = d(x, y) \leq \mathrm{inj}(M)$, concluding the proof in this case. Otherwise, $\gamma_x \cup \gamma_y \subset M$ is an embedded circle in M of length $2\,\mathrm{inj}(M)$. It follows easily that $\mathrm{diam}(M) = d(x, y) \leq \mathrm{inj}(M)$, concluding the proof of the lemma. □

The final few results of this note begin to address the **Question** when M has dimension greater than two.

Proposition 4. *If M is a closed Riemannian manifold with all points weakly spherical, then M is simply connected.*

Proof. The proof is by contradiction. If M is not simply connected, then there is a closed curve $\gamma \subset M$ of minimal length amongst homotopically non-trivial closed curves in M. Necessarily, γ is a closed geodesic. Fix $p \in \gamma$ and let $p' \in \gamma$ denote its antipode. Then γ consists of two geodesic segments joining p to p', each of which is half the length of γ. Denote these geodesics by γ_0 and γ_1.

Claim. Both γ_0 and γ_1 are minimizing.

If not, then without loss of generality, γ_0 hits a cut point before arriving at p'. Let x be a point lying in γ_0 between this cut point and p' and let γ_2 be a minimizing geodesic joining p to x. Then the closed curve formed by traversing γ_1 from p to p', then γ_0 backwards from p' to x, and finally γ_2 backwards from x to p has length less than γ. By the choice of γ, this closed curve is homotopically trivial. It follows easily that the closed curve formed by traversing γ_0 from p to x and then γ_2 backwards from x to p is homotopic to γ. On the other hand, this curve has length less than γ. This contradiction concludes the proof of the claim.

The assumption that p is weakly spherical implies that there exists a third minimizing geodesic γ_3 joining p to p'. If γ_0 is homotopic to γ_3 fixing endpoints, then the closed curve consisting of γ_3 followed by γ_1 backwards is homotopic to γ and has equal length. However, this curve is not smooth at p' and can therefore be shortened in its free homotopy class, contradicting the choice of γ. Therefore, γ_0 and γ_3 are not homotopic fixing endpoints. It follows that the closed curve consisting of γ_0 followed by γ_3 backwards is homotopically non-trivial and has length equal to γ. On the other hand, this curve is not smooth at p' and can therefore be shortened in its free homotopy class. This contradiction completes the proof of the proposition. $\square$

Additional notation is needed for the final two results. For a point p in a closed Riemannian manfold M, let $\mathrm{TConj}(p) \subset T_pM$ denote the locus of all conjugate vectors. The locus of first conjugate vectors will be denoted by

$$\mathrm{FConj}(p) = \{v \in \mathrm{TConj}(p) \,|\, (0,1)v \cap \mathrm{TConj}(p) = \emptyset\}.$$

A conjugate vector $v \in \mathrm{TConj}(p)$ is defined to be *regular* if there is a neighborhood U of v in T_pM such that for each $u \in U$, the ray $[0,\infty)u$ intersects $\mathrm{TConj}(p) \cap U$ in a single point. When $\mathrm{TConj}(p)$ is non-empty, the locus of regular conjugate vectors forms a smooth codimension one submanifold of T_pM that is relatively open and dense in $\mathrm{TConj}(p)$ [11, Theorem 3.1]. Moreover, the multiplicity of conjugate vectors is constant in each connected component of the regular conjugate locus.

Lemma 5. *If M is a closed Riemannian manifold and $p \in M$ is strongly spherical, then $\mathrm{FConj}(p) = \mathrm{TCut}(p)$. Moreover, M is simply connected.*

Proof. The second conclusion of the lemma is implied by the first (cf. the remark following [6, Lemma 2.2, pg. 19]). To prove the first statement of the lemma, it suffices to prove that each point $q \in \mathrm{Cut}(p)$ is conjugated to p along some minimizing geodesic [7, Corollary 3.5].

Choose a vector $v \in \mathrm{TCut}(p)$ with $\exp_p(v) = q$. If $v \in \mathrm{FConj}(p)$, then q is conjugated to p along the minimizing geodesic $\eta(t) = \exp_p(tv)$. Otherwise, there is a vector $v' \in \mathrm{TCut}(p) \setminus \{v\}$ with $\exp_p(v) = \exp_p(v')$. Since p is strongly spherical, it follows that $X = \exp_p^{-1}(q) \cap \mathrm{TCut}(p)$ is a closed infinite set. Let $x \in X$ be an accumulation point of X. Since $\exp_p$ is not locally injective in a neighborhood of x, the inverse function theorem implies that $x \in \mathrm{TConj}(p)$. Since $\exp_p(x) = q$, the point q is conjugated to p along the minimizing geodesic $\gamma(t) = \exp_p(tx)$, concluding the proof. $\square$

Corollary 6. *Let M be a closed 3-manifold with a strongly spherical point $p \in M$ realizing the injectivity radius. If $\mathrm{FConj}(p)$ consists of regular conjugate vectors, then M is isometric to a constant curvature sphere.*

Proof. It suffices to prove that $\mathrm{Cut}(p)$ is a single point by Lemma 3. Lemma 5 implies that $\mathrm{FConj}(p) = \mathrm{TCut}(p)$, which by hypothesis consists of regular conjugate vectors. In particular, $S := \mathrm{FConj}(p) = \mathrm{TCut}(p)$ is a smooth 2-sphere consisting of conjugate vectors with a common multiplicity.

If the vectors in S all have multiplicity one, then the kernels of the derivative maps of $\exp_p$ at these vectors defines a non-vanishing line field on S by the proof of [6, Lemma 2.2, pg. 19]. This is a contradiction since S is a smooth 2-sphere. Therefore, all vectors in S have multiplicity two as conjugate vectors. The discussion immediately following [11, Theorem 3.2] implies that $\exp_p$ is locally constant on S. As S is connected, $\mathrm{Cut}(p) = \exp_p(S)$ is a single point, concluding the proof. $\square$

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