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TAUTOLOGICAL PAIRINGS ON MODULI SPACES OF CURVES

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ABSTRACT. We discuss analogs of Faber's conjecture for two nested sequences of partial compactifications of the moduli space of smooth pointed curves. We show that their tautological rings are one-dimensional in top degree but sometimes do not satisfy Poincaré duality.

The structure of the tautological ring of the moduli space of smooth curves of genus g is predicted by the *Faber conjecture*, which states that $R^*(\mathcal{M}_g)$ is Gorenstein with socle in dimension $g-2$ ([Fa]). We break this statement into two parts:

Socle: The tautological ring vanishes in high degree and is one-dimensional in top degree:

$$R^k(\mathcal{M}_g) = \begin{cases} 0, & \text{if } k > g-2, \\ \mathbb{Q}, & \text{if } k = g-2. \end{cases} \tag{1}$$

Poincaré duality: For $0 \leq k \leq g-2$, the bilinear pairing

$$R^k(\mathcal{M}_g) \times R^{g-2-k}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \tag{2}$$

is nondegenerate.

In [FP2], Faber and Pandharipande speculate that the tautological rings of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}^{\mathrm{ct}}_{g,n}$ and $\mathcal{M}^{\mathrm{rt}}_{g,n}$ satisfy analogous properties. (Here, $\mathcal{M}^{\mathrm{ct}}_{g,n}$ denotes the moduli space of curves of compact type and $\mathcal{M}^{\mathrm{rt}}_{g,n}$ the moduli space of curves with rational tails.) While the socle statements of these speculations have been proven, the Poincaré duality property remains open. We give evidence that the two properties are not necessarily immediately correlated.

We define two chains of partial compactifications (Definitions 1.2 and 1.3)

$$\mathcal{M}^{\mathrm{ct}}_{g,n} = \mathcal{M}^{\lambda_g}_{g,n} \subseteq \mathcal{M}^{\lambda_{g-1}}_{g,n} \subseteq \cdots \subseteq \mathcal{M}^{\lambda_1}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n}, \tag{3}$$

$$\mathcal{M}^{\mathrm{rt}}_{g,n} = \mathcal{M}^{\mathrm{ch}_{2g-1}}_{g,n} \subseteq \mathcal{M}^{\mathrm{ch}_{2g-3}}_{g,n} \subseteq \cdots \subseteq \mathcal{M}^{\mathrm{ch}_1}_{g,n} \subseteq \overline{\mathcal{M}}_{g,n} \tag{4}$$

and define their tautological rings by restriction. The main results of this paper address the analog of Faber's conjecture for these spaces. The socle statement extends but Poincaré duality fails for the first cases $\mathcal{M}^{\lambda_1}_{g,n} = \mathcal{M}^{\mathrm{ch}_1}_{g,n}$ if $g \geq 2$ and $(g,n) \neq 0$.

Proposition 1. *For $i = 1, 2, \ldots, g$,*

(1) $R^k(\mathcal{M}^{\lambda_i}_{g,n}) = 0$ *for* $k > 3g-3+n-i$ *and* $R^{3g-3+n-i}(\mathcal{M}^{\lambda_i}_{g,n}) \cong \mathbb{Q}$,

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(2) $R^k(\mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}}) = 0$ *for* $k > 3g-2+n-2i$ *and* $R^{3g-2+n-2i}(\mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}}) \cong \mathbb{Q}$.

Proposition 2. *For* $g \geq 2$ *and* $(g,n) \neq (2,0)$*, the pairing*

$$R^g(\mathcal{M}_{g,n}^{\lambda_1}) \times R^{2g-4+n}(\mathcal{M}_{g,n}^{\lambda_1}) \to R^{3g-4+n}(\mathcal{M}_{g,n}^{\lambda_1}) \tag{5}$$

is not perfect.

The tautological rings of the other intermediate spaces $\mathcal{M}_{g,n}^{\lambda_i}$ are addressed with the following asymptotic result.

Proposition 3. *For any fixed* g *and* $2 \leq i \leq g-1$ *and for every* $n \gg 0$*, either the tautological restriction sequence*

$$R^{i-1}(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\lambda_i}) \longrightarrow R^g(\overline{\mathcal{M}}_{g,n}) \longrightarrow R^g(\mathcal{M}_{g,n}^{\lambda_i}) \longrightarrow 0 \tag{6}$$

is not exact in the middle or the pairing

$$R^g(\mathcal{M}_{g,n}^{\lambda_i}) \times R^{2g-3-i+n}(\mathcal{M}_{g,n}^{\lambda_i}) \to R^{3g-3-i+n}(\mathcal{M}_{g,n}^{\lambda_i}) \tag{7}$$

is not perfect.

The Chow ring is known to be tautological in codimensions 0 and 1 ([M]). Therefore, the restriction sequence (6) is exact when $i = 1, 2$ and Proposition 3 immediately implies:

Corollary 4. *If* $g \geq 3$ *and* $i = 2$*, then the pairing* (7) *is not perfect for every* $n \gg 0$.

The restriction sequence (6) is known to be exact for $\mathcal{M}_g$ in degrees $\geq g-1$, and exactness is conjectured in all degrees for the $\mathcal{M}_{g,n}^{\mathrm{rt}}$ and $\mathcal{M}_{g,n}^{\mathrm{ct}}$ ([FP3]). Therefore we expect the following:

Conjecture. *For* $2 \leq i \leq g-1$*, and* $(g,n) \neq (2,0)$*, the pairing on the tautological rings of* $\mathcal{M}_{g,n}^{\lambda_i}$ *is never perfect.*

Remark 1. When $i = 1$, we have $\mathcal{M}_{g,n}^{\lambda_1} = \mathcal{M}_{g,n}^{\mathrm{ch}_1}$. However, the proof of Proposition 3 does not extend in a natural way to $\mathcal{M}_{g,n}^{\mathrm{ch}_i}$, for $i > 1$. In these cases, Poincaré duality remains an open question.

With Lemma 1.4, we show that for the extremal case $i = g$, $\mathcal{M}_{g,n}^{\lambda_g} = \mathcal{M}_{g,n}^{\mathrm{ct}}$ and $\mathcal{M}_{g,n}^{\mathrm{ch}_{2g-1}} = \mathcal{M}_{g,n}^{\mathrm{rt}}$. Our results and conjecture therefore address generalizations of spaces that appeared in [FP2]. The socle statements of the Faber conjectures for $\mathcal{M}_{g,n}^{\mathrm{rt}}$, $\mathcal{M}_{g,n}^{\mathrm{ct}}$, and $\overline{\mathcal{M}}_{g,n}$ were proved by Graber-Vakil ([GV2, §5.5–§5.7]) and Faber-Pandharipande ([FP3, §4.1]). Prior to these results, the socle statement for $\mathcal{M}_g$ was shown by Looijenga and Faber ([L; Fa, Theorem 2]). The Poincaré duality property is only known for $g = 0$ by Keel ([Ke]), and for $\mathcal{M}_g$ for $g \leq 23$ by Faber ([Fa]).

Remark 2. We also show that for $g \geq 2$, the dimension of the kernel of the map

$$R^g(\mathcal{M}_{g,n}^{\lambda_1}) \to \hom\left(R^{2g-4+n}(\mathcal{M}_{g,n}^{\lambda_1}), R^{3g-4+n}(\mathcal{M}_{g,n}^{\lambda_1})\right) \tag{8}$$

becomes arbitrarily large as either g or n grows (Corollary 5). It is easy to see that Poincaré duality fails in arbitrary degrees by taking nonzero elements of high degree in the ideal generated by the kernel of this map.

Remark 3. Note that $R^*(\mathcal{M}_{2,0}^{\lambda_1})$ is Gorenstein. It is straightforward to check that the intersection matrix for the two generators of $R^1(\mathcal{M}_{2,0}^{\lambda_1})$ is nondegenerate. Since $\pi_1^*(\mathcal{M}_{2,0}^{\lambda_1}) = \mathcal{M}_{2,1}^{\lambda_1}$, Proposition 2 shows that if $\mathcal{M}$ is a moduli space which satisfies Faber's conjecture, its universal family does not necessarily satisfy Faber's conjecture.

Remark 4. The classes λ_i and ch_{2i-1} vanish respectively on $\overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}^{\lambda_i}$ and $\overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}}$. This motivates our notation.

This paper is organized as follows: in the first section we recall some basic definitions and provide references to the existing literature. In each of the subsequent sections we prove the three propositions.

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1. Background

The *tautological ring* $R^*(\overline{\mathcal{M}}_{g,n})$ is a natural subring of the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$ elegantly defined in [FP3]: as g and n vary, the tautological rings form the smallest system of $\mathbb{Q}$-subalgebras of $A^*(\overline{\mathcal{M}}_{g,n})$ that are closed under pushforward and pullback by the natural forgetful morphisms

$$\pi_i\colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}, \tag{9}$$

and the gluing morphisms

$$\iota_{\mathrm{irr}}\colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}, \tag{10}$$

$$\iota_{g_1,n_1}\colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}. \tag{11}$$

The tautological ring contains boundary strata δ_Γ (fundamental classes of the closure of the locus of curves whose dual graph is Γ), cotangent line classes ψ_i, Mumford-Morita-Miller κ classes, and Chern classes λ_i of the Hodge bundle. Much is known about the intersection theory of such classes. An excellent, albeit unfinished and unpublished, reference is [Ko]. Other references include [Fu], [HKK$^+$], and [M].

The following formula will be used in the proof of Proposition 2.

Lemma 1.1. *For any value of n for which the integrals are defined:*

$$\int_{\overline{\mathcal{M}}_{0,n}} \kappa_{n-3} = 1, \tag{12}$$

$$\int_{\overline{\mathcal{M}}_{1,n}} \kappa_n = \frac{1}{24}, \tag{13}$$

$$\int_{\overline{\mathcal{M}}_{0,n}} \kappa_i \kappa_{n-i-3} = \binom{n-1}{i+1} - 1, \tag{14}$$

$$\int_{\overline{\mathcal{M}}_{1,n}} \kappa_1 \kappa_{n-1} = \binom{n+1}{2}\frac{1}{24}. \tag{15}$$

Proof. Equations (12) and (13) follow immediately from the pull-back formula for κ classes ([AC, Eq. (1.10)]),

$$\kappa_a = \pi_{n+1}^*(\kappa_a) + \psi_{n+1}^a. \tag{16}$$

From this it follows easily that

$$\int_{\overline{\mathcal{M}}_{0,n}} \kappa_i \kappa_{n-i-3} = \int_{\overline{\mathcal{M}}_{0,n}} \left(\pi_{n+1*}\psi_{n+1}^{i+1}\right) \kappa_{n-i-3} \tag{17}$$

$$= \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_{n+1}^{i+1} \left(\kappa_{n-i-3} - \psi_{n+1}^{n-i-3}\right) \tag{18}$$

$$= \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_{n+1}^{i+1} \left(\pi_{n+2*}\psi_{n+2}^{n-i-2}\right) - \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_{n+1}^{n-2} \tag{19}$$

$$= \int_{\overline{\mathcal{M}}_{0,n+2}} \left(\psi_{n+1} - D_{n+1,n+2}\right)^{i+1} \psi_{n+2}^{n-i-2} - \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_{n+1}^{n-2} \tag{20}$$

$$= \int_{\overline{\mathcal{M}}_{0,n+2}} \psi_{n+1}^{i+1} \psi_{n+2}^{n-i-2} - 0 - \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_{n+1}^{n-2} \tag{21}$$

$$= \binom{n-1}{i+1} - 1. \tag{22}$$

The proof for (15) is analogous. □

Definition 1.2. $\mathcal{M}_{g,n}^{\lambda_i} \subseteq \overline{\mathcal{M}}_{g,n}$ is the locus of curves whose dual graph has genus $\leq g-i$. Equivalently, $\mathcal{M}_{g,n}^{\lambda_i}$ is the locus of curves where the sum of the geometric genera of the components is at least i.

Definition 1.3. $\mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}} \subseteq \overline{\mathcal{M}}_{g,n}$ is the locus of curves with at least one component of genus at least i.

Lemma 1.4. *$\mathcal{M}_{g,n}^{\lambda_g} = \mathcal{M}_{g,n}^{ct}$ and $\mathcal{M}_{g,n}^{\mathrm{ch}_{2g-1}} = \mathcal{M}_{g,n}^{rt}$.*

Proof. The dual graph of any curve in $\mathcal{M}_{g,n}^{\lambda_g}$ is connected of genus 0 and thus is a tree. Any curve in $\mathcal{M}_{g,n}^{\mathrm{ch}_{2g-1}}$ has at least one (hence exactly one) component of genus g; the other components must necessarily form trees of rational curves. □

Lemma 1.5. *For $i = 1, 2, \ldots, g$,*

(1) *the class λ_i vanishes on $\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\lambda_i}$,*

(2) *the class ch_{2i-1} vanishes on $\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}}$.*

Proof. The boundary stratum δ_Γ is the image of the gluing morphism

$$\delta_\Gamma = \mathrm{im}\left(\iota_\Gamma \colon \prod \overline{\mathcal{M}}_{g_j, n_j + d_j} \to \overline{\mathcal{M}}_{g,n}\right). \tag{23}$$

The Hodge bundle splits when restricted to δ_Γ as in [FP1, equations (17) and (18)]:

$$\iota_\Gamma^*(\mathbb{E}) = \bigoplus \mathbb{E}_{g_j, n_j + d_j} \oplus \mathcal{O}^n. \tag{24}$$

To see (1), use the Whitney formula,

$$\iota_\Gamma^*\left(c(\mathbb{E})\right) = c(\iota_\Gamma^* \mathbb{E}) = \prod \left(1 + \lambda_{1,j} + \cdots + \lambda_{g_j,j}\right), \tag{25}$$

where $\lambda_{i,j}$ is the i-th Chern class of the Hodge bundle of the j-th factor $\overline{\mathcal{M}}_{g_j, n_j+d_j}$. If $\delta_\Gamma \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\lambda_i}$, then $\sum g_j < i$ and the term of degree i in equation (25) vanishes.

To see (2), use the additivity of the Chern character, on δ_Γ:

$$\iota_\Gamma^*\left(\mathrm{ch}_{2i-1}(\mathbb{E})\right) = \mathrm{ch}_{2i-1}(\iota_\Gamma^* \mathbb{E}) = \sum_j \mathrm{ch}_{2i-1,j}, \tag{26}$$

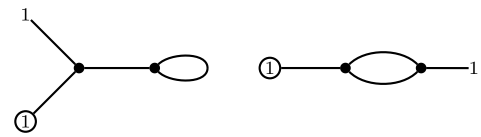


FIGURE 1. These two dual graphs differ by a Feynman move. Our notation has the genus of a component inside the corresponding vertex, unless the component is rational, in which case it is denoted by a black vertex.

where $\mathrm{ch}_{2i-1,j}$ is the $(2i-1)$-th Chern character of the Hodge bundle of the j-th factor $\overline{\mathcal{M}}_{g_j,n_j+d_j}$. Since $\delta_\Gamma \in \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}}$, all $g_j < i$ and thus $\mathrm{ch}_{2i-1} = 0$. □

We conclude this section by recalling theorem $\star$ by Graber-Vakil ([GV2]), which is a key ingredient in the proof of Proposition 1.

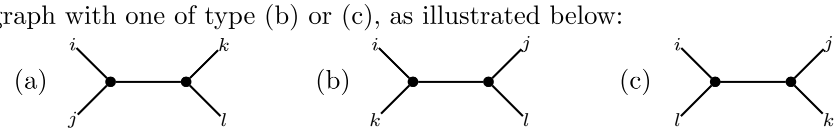
Theorem $\star$. *Any tautological class of degree k on $\mathcal{M}_{g,n}$ vanishes on the open set consisting of strata satisfying*

$$\#\{\text{genus } 0 \text{ components}\} < k - g + 1. \tag{27}$$

2. SOCLE

In this section we prove Proposition 1. The strategy of the proof is natural: Theorem $\star$ forces tautological classes of high degree to be supported on strata with many rational components. On the other hand, curves in $\mathcal{M}_{g,n}^{\lambda_i}$ and $\mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}}$ satisfy geometric conditions that limit the number of rational components. These constraints imply high degree vanishing and force tautological classes in the socle degree to be supported on exactly one boundary stratum up to rational equivalence.

Definition 2.1. A Feynman move replaces a portion of a graph of type (a) on a dual graph with one of type (b) or (c), as illustrated below:



The half-edges i, j, k, and l may be glued to other half-edges to form edges (see Figure 1).

Lemma 2.2. *If the dual graphs of two boundary strata differ by Feynman moves, then they are rationally equivalent.*

Proof. This is immediate by noting that $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}_1$, so its boundary points are rationally equivalent. This equivalence is preserved under gluing morphisms. □

Remark 5. It is a standard combinatorial fact that two trivalent graphs with the same number of vertices and edges differ by a finite number of Feynman moves ([W]).

The proof of Proposition 1 now follows from some careful bookkeeping.

Part 1. Theorem $\star$ implies that any class in $R^{3g-3+n-i+k}(\mathcal{M}_{g,n}^{\lambda_i})$ $(k \geq 0)$ must be supported on boundary strata containing at least $2g+n-2-i+k$ rational components. Let δ_Γ be any one such boundary stratum. By the definition of $\mathcal{M}_{g,n}^{\lambda_i}$, the sum of the genera of the vertices of graph Γ is at least i.

Stability implies that the incidence of any rational vertex is at least three, and there are at most n marked half-edges incident to these vertices. Thus there are at least

$$3(2g+n-2-i+k)-n = 6g+2n-6-3i+3k \tag{28}$$

half-edges which must be glued to other half-edges. Since $g_\Gamma \leq g-i$, at least

$$6g+2n-6-3i+3k-2(3g+n-2-2i+k) = i+k \tag{29}$$

half-edges must be glued to vertices of positive genus. The total genus of the curve represented by Γ is g, so the only possibility is $k=0$; i.e. all rational vertices are trivalent, and the remaining i half-edges are glued to i vertices of genus 1 and incidence 1. By Remark 5, any two such graphs differ by a finite number of Feynman moves and hence represent rationally equivalent tautological classes. It is immediate to check that such graphs actually live in degree $3g-3+n-i$. $\square$

Part 2. This is similar to the previous proof, except that now we require the graph Γ to have at least one vertex of genus $\geq i$. Theorem $\star$ forces any class in $R^{3g+n-2-2i+k}(\mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}})$ $(k \geq 0)$ to be supported on boundary strata with at least $2g+n-1-2i+k$ rational components, and stability implies that there are at least three times as many half-edges, only n of which are not glued to some other half-edge. The total genus of the graph must be g, where the only consistent possibility is for $k=0$; i.e., all rational vertices are trivalent and there is exactly one vertex of genus i and incidence 1. Again, all such strata represent rationally equivalent classes because any two graphs are equivalent up to a finite number of Feynman moves.

The degree of such strata is $3g+n-1-3i$, which means that one must have a class of degree $i-1$ decorating the genus i vertex. Since $R^{i-1}(\mathcal{M}_{i,1}^{\mathrm{rt}}) \cong \mathbb{Q}$ (this is the socle statement of Faber's conjecture for $\mathcal{M}_{i,1}^{\mathrm{rt}}$), there is only one such nonzero class up to scalar multiples. $\square$

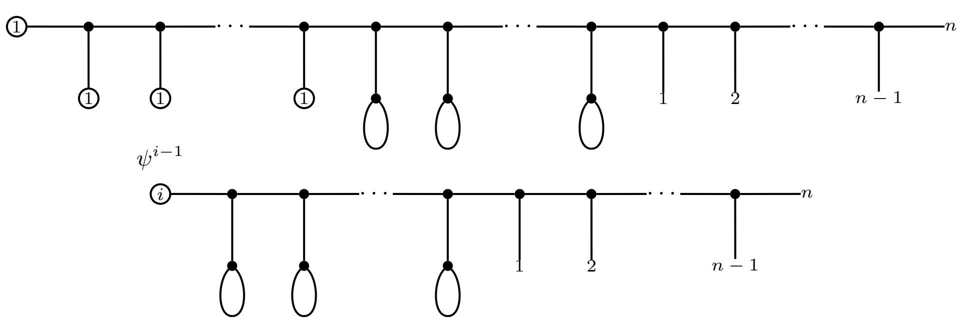


FIGURE 2. Generators in socle degree for $\mathcal{M}_{g,n}^{\lambda_i}$ and $\mathcal{M}_{g,n}^{\mathrm{ch}_i}$.

Remark 6. It is important to note that λ_i (resp. ch_{2i-1}) does not vanish on the unique generator of $R^{top}(\mathcal{M}_{g,n}^{\lambda_i})$ (resp. $R^{top}(\mathcal{M}_{g,n}^{\mathrm{ch}_{2i-1}})$), and hence it can be used as an evaluation class: multiplication by λ_i (resp. ch_{2i-i}) gives an isomorphism between the socle and $R^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$.

3. Failure of Poincaré duality for $\mathcal{M}^{\lambda_1}_{g,n}$

In this section we construct examples to illustrate the failure of Poincaré duality $\mathcal{M}^{\lambda_1}_{g,n} = \mathcal{M}^{\mathrm{ch}_1}_{g,n}$ for $g \geq 2$ and $(g,n) \neq (2,0)$. Choose any triple (a,b,c) of integers satisfying $a+b+c = g$, where a is nonnegative and b and c are positive. Choose any subset S of the n points. Let $\Gamma(a,b,c,S) \in R^g(\overline{\mathcal{M}}_{g,n})$ denote the graph with two vertices connected by b edges: one genus 1 vertex with a self-edges and carrying the points in S, and one genus 0 vertex with c self-edges and carrying the points in S^c (see Figure 3). Let $\delta(a,b,c,S)$ denote the associated boundary stratum.

When $(a,c,S) \neq (c-1,a+1,S^c)$, the strata $\delta(a,b,c,S)$ and $\delta(c-1,b,a+1,S^c)$ are not rationally equivalent, but their difference lies in the kernel of multiplication by λ_1.

Proof of Proposition 2. Let $\gamma_1 = \delta(a,b,c,S)$ and $\gamma_2 = \delta(c-1,b,a+1,S^c)$. Note that if $(a,c,S) = (c-1,a+1,S^c)$, then γ_1 and γ_2 denote the same graph. Therefore assume that this is not the case; we can always do this if $g \geq 2$ and $(g,n) \neq (2,0)$. The fact that $\gamma_1 - \gamma_2$ lies in the kernel of the map ϕ from (41) follows from the fact that λ_1 vanishes on $\overline{\mathcal{M}}_{0,k}$ and λ_1 is equivalent to $\frac{1}{24}\delta_{\mathrm{irr}}$ on $\overline{\mathcal{M}}_{1,k}$, for all k for which these spaces are defined. Thus $\lambda_1 \cdot \gamma_1$ and $\lambda_1 \cdot \gamma_2$ are both equal to $\frac{1}{24}\delta_A$. Here A is the graph with two genus 0 vertices connected by b edges: one has marked points indexed by S and $a+1$ self-loops, and the other has marked points indexed by S^c and c self-loops.

Suppose for a contraction that γ_1 and γ_2 were algebraically equivalent in $\mathcal{M}^{\lambda_1}_{g,n}$. The restriction sequence

$$R^0(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}^{\lambda_i}_{g,n}) \longrightarrow R^g(\overline{\mathcal{M}}_{g,n}) \longrightarrow R^g(\mathcal{M}^{\lambda_i}_{g,n}) \longrightarrow 0 \tag{30}$$

is exact since the first term has degree 0. Extending γ_1 and γ_2 to boundary strata in $\overline{\mathcal{M}}_{g,n}$,

$$\gamma_1 - \gamma_2 \in R^0(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}^{\lambda_1}_{g,n}). \tag{31}$$

However $R^0(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}^{\lambda_1}_{g,n})$ is generated by one element δ_B, where B is the graph with a unique vertex of genus 0, g self-loops and n half-edges.

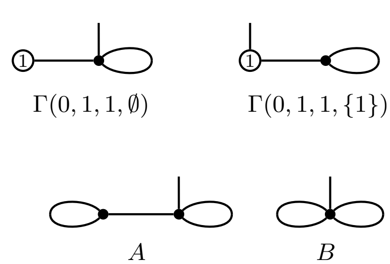


Figure 3. The graphs $\Gamma(0,1,1,\emptyset)$ and $\Gamma(0,1,1,\{1\})$ for $(g,n) = (2,1)$, and the corresponding graphs A and B.

Set

$$K_1 := 2a + b + |S|, \qquad L_1 := 2c + b + |S^c| - 3, \tag{32}$$

$$K_2 := 2c + b + |S^c| - 2, \qquad L_2 := 2a + b + |S| - 1. \tag{33}$$

For $i = 1, 2$ define the stratum

$$\gamma_i := \operatorname{im}\left(\iota_{\Gamma_i} \colon \overline{\mathcal{M}}_{1,K_i} \times \overline{\mathcal{M}}_{0,L_i+3} \to \overline{\mathcal{M}}_{g,n}\right) \tag{34}$$

and

$$\delta_B := \operatorname{im}\left(\iota_B \colon \overline{\mathcal{M}}_{0,2g+n} \to \overline{\mathcal{M}}_{g,n}\right). \tag{35}$$

Note that L_1 and L_2 cannot both be zero for $(g, n) \neq (2, 0)$. If they were, then necessarily $a = 0$, $b = 1$, and $c = 1$ (recall that a is nonnegative while b and c are positive), which implies $(g, n) = (2, 0)$.

If $L_1 = 0$, then $L_2 \neq 0$ and we have the following equations which follow from Lemma 1.1 and the fact that κ_a restricted to a boundary divisor is the sum of the pull-backs of κ_a on each factor of the gluing map:

$$\kappa_{K_1}\gamma_1 = \frac{1}{2^{g-1}(g-2)!}\frac{1}{24}, \qquad \kappa_1\kappa_{L_2}\gamma_1 = \frac{1}{2^{g-1}(g-2)!}\frac{1}{24}\binom{2g+n-2}{2}, \tag{36}$$

$$\kappa_{K_1}\gamma_2 = 0, \qquad \kappa_1\kappa_{L_2}\gamma_2 = \frac{1}{2^{g-1}(g-1)!}\frac{1}{24}, \tag{37}$$

$$\kappa_{K_1}\delta_B = \frac{1}{2^g g!}, \qquad \kappa_1\kappa_{L_2}\delta_B = \frac{1}{2^g g!}\left(\binom{2g+n-1}{2} - 1\right). \tag{38}$$

These are incompatible with equation (31). If $L_2 = 0$ and $L_1 \neq 0$, a similar argument holds.

Now consider the final case where both L_1 and L_2 are nonzero. The equations

$$\kappa_{K_1}\kappa_{L_1}\gamma_1 = \frac{1}{24}, \qquad \kappa_{K_1}\kappa_{L_1}\gamma_2 = 0, \qquad \kappa_{K_1}\kappa_{L_1}\delta_B = \binom{2g+n-1}{K_1+1} - 1, \tag{39}$$

$$\kappa_{2g-3+n}\gamma_1 = 0, \qquad \kappa_{2g-3+n}\gamma_2 = 0, \qquad \kappa_{2g-3+n}\delta_B = \frac{1}{2^g g!} \tag{40}$$

show the independence of the strata $\gamma_1 - \gamma_2$ and δ_B. Thus γ_1 and γ_2 cannot be albegraically equivalent in $\mathcal{M}_{g,n}^{\lambda_1}$. □

Corollary 5. *The dimension of the kernel of the map*

$$\phi\colon R^g(\mathcal{M}_{g,n}^{\lambda_1}) \to \hom\left(R^{2g-4+n}(\mathcal{M}_{g,n}^{\lambda_1}), R^{3g-4+n}(\mathcal{M}_{g,n}^{\lambda_1})\right) \tag{41}$$

goes to infinity as g or n go to infinity.

Proof. We exhibit a set of roughly $g + n/2$ linearly independent classes in $\ker\phi$. Let $\overline{n}$ denote the set $\{1, \ldots, n\}$. Set

$$\gamma_i = \delta(0, 1, g-1, \overline{2i}) - \delta(g-2, 1, 1, \overline{n}\backslash\overline{2i}) \tag{42}$$

for $i = 1, \ldots, \lfloor n/2 \rfloor$, and

$$\eta_j = \delta(j, 1, g-j-1, \overline{n}) - \delta(g-j-2, 1, j+1, \emptyset) \tag{43}$$

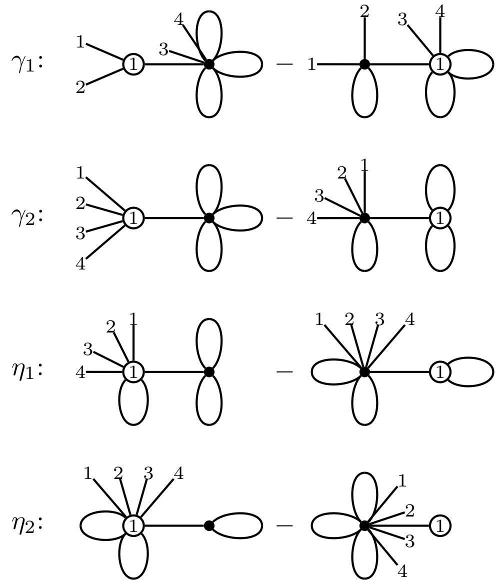


FIGURE 4. The classes γ_1, γ_2, η_1, and η_2 for $(g, n) = (4, 4)$

for $j = 1, \ldots, g-2$. Since $\lambda_1\gamma_i = \lambda_1\eta_j = 0$, the classes lie in the kernel of ϕ. The equations

$$\psi_1^{2i+1}\kappa_{2g+n-4-2i}\gamma_j = A_i\delta_{ij}, \tag{44}$$

$$\psi_1^{2i+1}\kappa_{2g+n-4-2i}\eta_j = 0, \tag{45}$$

$$\psi_1^{n+2i+1}\kappa_{2g-4-2i}\gamma_j = 0, \tag{46}$$

$$\psi_1^{n+2i+1}\kappa_{2g-4-2i}\eta_j = B_i\delta_{ij} \tag{47}$$

(here δ_{ij} is Kronecker's delta and not a boundary stratum, and A_i, B_i are nonzero real numbers) show that these classes are independent. Modulo $R^0(\overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}^{\lambda_1})$, which is one-dimensional, roughly $g + \lfloor n/2 \rfloor - 1$ of them must remain independent. $\square$

4. FAILURE OF POINCARÉ DUALITY FOR $\mathcal{M}_{g,n}^{\lambda_i}$

We first outline the strategy of the proof for Proposition 3. Fix g and i such that $2 \le i \le g-1$. A generalization of the construction in §3 produces a set $S_m \subseteq R^g(\overline{\mathcal{M}}_{g,(i+1)+m})$ in the annihilator of λ_i. The first problem in showing that at least one such class is nonzero in $R^g(\mathcal{M}_{g,(i+1)+m}^{\lambda_i})$ is that the kernel of the restriction sequence (6) is not known to be tautological. This is a difficult question that we cannot tackle at present. If the kernel is in fact tautological, the dimension of $R^{i-1}(\overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}^{\lambda_i})$ grows quickly as g or n increases. For a fixed g we bound the order of growth by i^n. By proving that S_m spans a linear subspace of dimension $(i+1)^m$, we conclude that eventually some classes in S_m cannot lie in the image of $R^{i-1}(\overline{\mathcal{M}}_{g,n}\backslash\mathcal{M}_{g,n}^{\lambda_i})$ and hence will be nonzero in $R^g(\mathcal{M}_{g,(i+1)+m}^{\lambda_i})$.

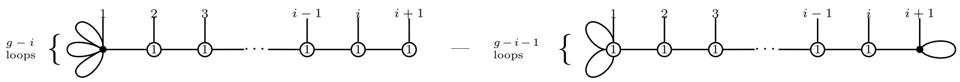


FIGURE 5. The class $\sigma \in R^g(\mathcal{M}^{\lambda_i}_{g,i+1})$. There are $g-i$ loops on each stratum. On the left hand side all loops are incident to the rational vertex; on the right hand side $g-i-1$ are incident to a genus one vertex and one loop connects the rational vertex to itself.

Proof of Proposition 3. Let $\sigma \in R^g(\overline{\mathcal{M}}_{g,i+1})$ be the difference of boundary classes illustrated in Figure 5. Intersecting either of the two strata with λ_i results in $1/24^i$ times the class of the graph where all genus one vertices are replaced with loops, and thus $\sigma \cdot \lambda_i = 0$. We set $S_0 = \{\sigma\}$.

For $m > 0$, and $\mathbf{a} = (a_1, \ldots, a_m)$ an m-tuple of numbers between 1 and $i+1$, let $\sigma_{\mathbf{a}} \in R^g(\overline{\mathcal{M}}_{g,(i+1)+m})$ be the class obtained by decorating both graphs of σ with the j-th mark on the a_j-th vertex for $j = 1, \ldots, m$. The set S_m of all possible such classes in $\mathcal{M}^{\lambda_i}_{g,(i+1)+m}$ has cardinality $(i+1)^m$.

We construct inductively a set T^m of classes of $R^{2g-3+(i+1)+m}(\overline{\mathcal{M}}_{g,(i+1)+m})$ which is dual to S_m. Our base case is $m = 0$, where the vector τ can be chosen to be a scalar multiple of an appropriate product of ψ classes (for example, τ can be defined by normalizing $\psi_1^{2(g-i-1)}\psi_{i+1}^2\prod_{j=2}^{i}\psi_j^3$).

We describe the first inductive step explicitly to illustrate the general step. In this case, $S_1 = \{\sigma_k\}_{k=1,\ldots,i+1}$, where σ_k is obtained by decorating σ with the $(i+2)$-th point on the k-th vertex. The dual set $T^1 = \{\tau^k\}_{k=1,\ldots,i+1}$ is constructed as follows: τ^k is the pull-back via the universal family of the class τ, intersected with the boundary divisor where the k-th and the $((i+1)+1)$-th point are on a genus 0 component.

In general consider the universal family

$$\pi\colon \overline{\mathcal{M}}_{g,(i+1)+m+1} \to \overline{\mathcal{M}}_{g,(i+1)+m}, \tag{48}$$

and note that

$$\pi^*\sigma_{\mathbf{a}} = \sum_{k=1}^{i+1} \sigma_{(\mathbf{a},k)}, \tag{49}$$

where $(\mathbf{a}, k)$ is the sequence with k appended to the end of $\mathbf{a}$. For $1 \leq j \leq (i+1)$ let $D_{j,(i+1)+m+1}$ denote the divisor image of the j-th section in $\overline{\mathcal{M}}_{g,(i+1)+m+1}$ (i.e. the boundary divisor where the j-th and the last mark are sitting on a genus 0 component). Then, for any m-tuple $\mathbf{a}$,

$$D_{j,(i+1)+m+1} \cdot \sigma_{(\mathbf{a},k)} = 0 \tag{50}$$

if $j \neq k$. By the projection formula and equations (49) and (50),

$$D_{j,(i+1)+m+1}\pi^*(\tau^{\mathbf{b}})\sigma_{(\mathbf{a},k)} \tag{51}$$

equals the class of a point if $j = k$ and $\mathbf{a} = \mathbf{b}$ and vanishes otherwise. Therefore the set

$$T^{m+1} := \{\pi^* \tau^{b_1,\ldots,b_m} D_{b_{m+1},(i+1)+m+1}\}_{\mathbf{b}} \tag{52}$$

gives a dual basis to S_{m+1}.

The growth of $\dim R^{i-1}(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\lambda_i})$ with respect to n is at most $O(i^n)$. To see this, note that the decorated dual graph of any class in $R^{i-1}(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\lambda_i})$ has at most i vertices, and e edges, where $g - i + 1 \leq e \leq g$. The total number of possibly unstable graphs without marked points satisfying these conditions is independent of n.

Graphs with exactly i vertices have g edges, and hence classes supported on these strata are pure boundary. For a given graph there are i^n possible ways of distributing the marked points on the vertices.

For graphs with strictly less than i vertices, there are at most $(i-1)^n$ ways to distribute the marks on the vertices. Each vertex can be decorated with a monomial in ψ and κ classes of degree $\leq i - 1$ ([GP, Proposition 11]). The number of ways to choose κ classes to decorate the vertices is independent of n. The number of monomials in ψ classes of degree $\leq i-1$ grows polynomially in n, yielding an order of $O((i-1)^n n^k) < O(i^n)$ for the number of classes supported on strata with less than i vertices.

Altogether, we have obtained that the dimension of $R^{i-1}(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\lambda_i})$ grows at most as $O(i^n)$. Thus $\dim R^g(\mathcal{M}_{g,n}^{\lambda_i}) > \dim R^{i-1}(\overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}^{\lambda_i})$ for large n, which implies that some classes in the annihilator of λ_i are nonzero in $R^g(\mathcal{M}_{g,n}^{\lambda_i})$. □

All nontrivial classes in the annihilators of λ_i that we construct are in codimension g or higher. This leaves us with a natural question:

Question. Is the map

$$\phi\colon R^j(\mathcal{M}_{g,n}^{\lambda_i}) \to \hom\left(R^{3g-3+n-i-j}(\mathcal{M}_{g,n}^{\lambda_i}), R^{3g-3+n-i}(\mathcal{M}_{g,n}^{\lambda_i})\right) \tag{53}$$

injective for $j \leq g - 1$?

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