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# CONTINUITY OF SPECTRAL AVERAGING

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Abstract. We consider averages $\kappa$ of spectral measures of rank one perturbations with respect to a $\sigma$-finite measure $\nu$. It is examined how various degrees of continuity of $\nu$ with respect to $\alpha$-dimensional Hausdorff measures $(0 \leq \alpha \leq 1)$ are inherited by $\kappa$. This extends Kotani's trick where $\nu$ is simply the Lebesgue measure.

## 1. Introduction

Let $A$ be a bounded self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Fix a normalized vector $\phi \in \mathcal{H}$. Consider the family of rank one perturbations

$$A_\lambda := A + \lambda\langle\phi, .\rangle\phi \ , \tag{1.1}$$

indexed by the real parameter $\lambda$. Despite its simple form, the family in (1.1) proves to be a very useful tool in the study of discrete random Schrödinger operators. There, rank one perturbations correspond to fluctuations of the potential at a lattice site. Reference [1] summarizes several of these applications, among them the Simon-Wolf criterion for spectral localization, the theory of Aizenman-Molchanov for the Anderson model, and Wegner's estimate.

Crucial to most of these applications is a result known as spectral averaging or Kotani's trick. It allows one to relate the spectral behavior for fixed values of $\lambda$ to the spectral properties inherent to the entire family $\{A_\lambda\}$, i.e. upon a variation of $\lambda$.

Denote by $\mathrm{d}\mu(x)$ and $\mathrm{d}\mu_\lambda(x)$ the spectral measure with respect to $\phi$ for the operator $A$ and $A_\lambda$, respectively. Kotani's trick is the following result:

**Theorem 1.1** (Kotani's trick)**.** *Let $B$ be a Borel set on the real line. Then*

$$|B| = \int \mu_\lambda(B)\mathrm{d}\lambda \ . \tag{1.2}$$

*Here, $|.|$ denotes the Lebesgue measure.*

Different proofs and applications of this result were given in [2, 3, 4, 5, 6, 7, 8, 9]. We note that for some purposes, among them the Simon-Wolff criterion, a weaker formulation is sufficient. This weaker result states that the Borel measure on the right hand side of (1.2) is absolutely continuous with respect to Lebesgue. In fact,

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in the original proof of the Simon-Wolff criterion (see [9], Theorem 5) the authors show that the measure

$$\kappa(.) = \int \mu_\lambda(.) \frac{1}{1+\lambda^2} \mathrm{d}\lambda \tag{1.3}$$

is mutually equivalent to the Lebesgue measure.

Equation (1.3) suggests the following generalization: For $\nu$ a $\sigma$-finite Borel measure on $\mathbb{R}$, define a measure $\kappa$ by

$$\kappa(.) = \int \mu_\lambda(.) \mathrm{d}\nu(\lambda) \ . \tag{1.4}$$

Such averages were first considered in [10] for a finite measure $\nu$. There, relation (3.4) was discovered for a finite measure $\nu$ and was used to estimate the Hausdorff dimension of set $\{\lambda : A_\lambda \text{ has some continuous spectrum}\}$ (see Theorem 5.2 therein).

In view of the measure defined in (1.4), Kotani's trick ($\mathrm{d}\nu(\lambda) = \mathrm{d}\lambda$) and the result for $\mathrm{d}\nu(\lambda) = \frac{1}{1+\lambda^2}\mathrm{d}\lambda$ in (1.3) become statements about continuity properties of the measure $\nu$ being inherited by $\kappa$.

In this article we pursue this continuity-based approach to spectral averaging. We will show how various degrees of continuity of $\nu$ with respect to $\alpha$-dimensional Hausdorff measures ($\alpha \leq 1$) are inherited by $\kappa$. For a definition of $\alpha$ continuity see Definition 4.1. Our main result is the following theorem:

**Theorem 1.2.** *If $\nu$ is absolutely continuous with respect to Lebesgue, so is $\kappa$. Additionally, if $\nu$ is $\alpha c$, $0 < \alpha < 1$, then $\kappa$ is $\delta c$ for all $\delta < \alpha$.*

Kotani's trick then arises as a special case, where the density of $\mathrm{d}\kappa(x)$ can be calculated explicitly.

The paper is organized as follows: Section 2 summarizes some results of the theory of Borel transforms and rank one perturbations as needed for further development. After showing that mere continuity of $\nu$ is inherited by $\kappa$ (Theorem 3.1), we examine the situation for $\nu$ being uniformly-$\alpha$-Hölder continuous (see Definition 3.3). In particular, we shall show that uniform 1-Hölder continuity of $\nu$ is inherited by $\kappa$. Kotani's trick follows as a special case if $\mathrm{d}\nu(x) = \mathrm{d}x$. Finally, in Sections 4 and 5 we employ the Rogers-Taylor decomposition of measures with respect to Hausdorff measures to prove Theorem 1.2.

## 2. Borel transforms and rank one perturbations

The key quantity to understanding the spectral properties of the family (1.1) is the Borel transform of the spectral measure $\mathrm{d}\mu$ associated with the *unperturbed* operator $A$ and the vector $\phi$. In general, if $\eta$ is a Borel measure with $\int \frac{1}{1+|y|}\mathrm{d}\eta(y) < \infty$, for $z \in \mathbb{H}^+$ we define

$$F_\eta(z) := \int \frac{\mathrm{d}\eta(y)}{y-z} \ , \tag{2.1}$$

the Borel transform of the measure $\eta$. Letting $z = x + i\epsilon, \epsilon > 0$, we may split $F_\eta(x+i\epsilon)$ into its real and imaginary parts, i.e. $F_\eta(x+i\epsilon) =: Q_\eta(x+i\epsilon)+iP_\eta(x+i\epsilon)$, where

$$\begin{aligned} Q_\eta(x+i\epsilon) &= \int \frac{y-x}{(y-x)^2+\epsilon^2}\mathrm{d}\eta(y) \ , \\ P_\eta(x+i\epsilon) &= \int \frac{\epsilon}{(y-x)^2+\epsilon^2}\mathrm{d}\eta(y) \ . \end{aligned}$$

We shall refer to $P_\eta$ and $Q_\eta$ as Poisson and conjugate Poisson transforms of the measure $\eta$, respectively. Whereas $Q_\eta(x+i\epsilon)$ depends on the "symmetry" of $\eta$ around $x$, $P_\eta(x+i\epsilon)$ carries information about the growth of the measure $\eta$ at $x$. A detailed analysis about the asymptotic behavior of $P_\eta$ and $Q_\eta$ is given in [11].

The relation between the local growth of a measure and its Poisson transform follows from the following simple estimate: Given $\alpha \in [0,1]$, then for $x \in \mathbb{R}$ and $\epsilon > 0$

$$\epsilon^{1-\alpha} P_\eta(x+i\epsilon) \geq \epsilon^{1-\alpha} \int_{(x-\epsilon,x+\epsilon)} \frac{\epsilon}{(x-y)^2+\epsilon^2} \mathrm{d}\eta(y) \geq \tfrac{1}{2\epsilon^\alpha} M_\eta(x;\epsilon) \ , \tag{2.2}$$

where $M_\eta(x;\epsilon) := \eta(x-\epsilon, x+\epsilon)$ denotes the growth function of $\eta$ at $x$.

*Remark* 2.1. As will be seen below (see Theorem 2.2), it is useful to consider the Poisson transform of a measure even if its Borel transform does not exist. A necessary and sufficient condition for $P_\eta(x+i\epsilon) < \infty$, $\epsilon > 0$, is $\int \frac{1}{1+x^2} \mathrm{d}\eta(x) < \infty$.

For $\alpha \geq 0$ we define

$$\overline{D}_\eta^\alpha(x) \quad := \quad \limsup_{\epsilon \to 0^+} \frac{\eta(x-\epsilon, x+\epsilon)}{\epsilon^\alpha} \ , \tag{2.3}$$

the upper-$\alpha$-derivative of a measure $\eta$ at a point $x \in \mathbb{R}$.

The above estimate (2.2) leads to the following result proven e.g. in [10]:

**Proposition 2.1.** *Let $\alpha \in [0,1]$ and $x \in \mathbb{R}$ be fixed. Then $\overline{D}_\eta^\alpha(x)$ and $\limsup_{\epsilon\to 0^+} \epsilon^{1-\alpha} P_\eta(x+i\epsilon)$ are either* both *infinite, zero, or in* $(0,+\infty)$.

Proposition 2.1 will be used to analyze continuity with respect to Hausdorff measures of the measure $\kappa$ defined in (1.4).

The following theorem is the key to spectral analysis of rank one perturbations. It provides a characterization of the components of $\eta$ in a Lebesgue decomposition. Proof can be found e.g. in [1, 12].

**Theorem 2.2.** *Let $\eta$ be a Borel measure on the real line such that $\int \frac{1}{1+y^2} \mathrm{d}\eta(y) < \infty$. The following statements characterize the components in the Lebesgue decomposition of $\eta = \eta_{\mathrm{sing}} + \eta_{\mathrm{ac}}$:*

(i) $\mathrm{d}\eta_{\mathrm{ac}}(x) = \frac{1}{\pi} P_\eta(x+i0)\mathrm{d}x$.

(ii) $\eta_{\mathrm{sing}}$ *is supported on* $\{x : P_\eta(x+i0) = +\infty\}$.

Theorem 2.2 implies a characterization of the spectral properties of the family $\{A_\lambda\}$. Out of this we shall only need the following statement related to the singular (pp+sc)-spectrum of $\{A_\lambda\}$ (see [1], Theorem 12.2).

**Proposition 2.2.** (i) $\mu_{\lambda,\mathrm{sing}}$ *is supported on the set* $\{x : F_\mu(x+i0) = -\frac{1}{\lambda}\}$.

(ii) *The family of measures* $\{\mathrm{d}\mu_{\lambda,\mathrm{sing}}\}$ *are mutually singular. In particular, a point $x \in \mathbb{R}$ can be an atom for at most one value of $\lambda$.*

## 3. Spectral averaging

For a fixed $\sigma$-finite Borel measure $\nu$, consider the measure $\kappa$ introduced in (1.4). $\kappa$ is well defined since for any polynomial $p(x)$, $\langle \phi, p(A_\lambda)\phi \rangle$ is a polynomial in $\lambda$. Stone-Weierstraß and functional calculus then imply that $\lambda \mapsto \mu_\lambda(B)$ is Borel measurable for any Borel set $B \subseteq \mathbb{R}$.

We start our analysis of the continuity of $\kappa$ in relation to the continuity of $\nu$ with the following simple observation:

**Theorem 3.1.** *If $\nu$ is continuous, so is $\kappa$.*

*Proof.* Apply part (ii) of Proposition 2.2 to $\kappa(\{x\}) = \int \mu_\lambda(\{x\}) \mathrm{d}\nu(\lambda)$, $x \in \mathbb{R}$. $\square$

The following simple relation between the Poisson transforms of $\kappa$ and $\nu$ is crucial to further analyze the continuity properties of $\kappa$.

**Proposition 3.1.** *Assume* $\int \frac{1}{1+y^2} \mathrm{d}\nu(y) < \infty$. *Then*

$$P_\kappa(z) = P_\nu\left(-\frac{1}{F_\mu(z)}\right) \tag{3.1}$$

*for* $z \in \mathbb{H}^+$.

*Proof.* Using the definition of $\kappa$ in (1.4), the monotone convergence theorem implies

$$\int f(x) \mathrm{d}\kappa(x) = \int \left\{ \int f(x) \mathrm{d}\mu_\lambda(x) \right\} \mathrm{d}\nu(\lambda)$$

for any measurable $0 \leq f$.

In particular, for $z \in \mathbb{H}^+$,

$$\begin{aligned} P_\kappa(z) &= \int P_{\mu_\lambda}(z) \mathrm{d}\nu(\lambda) \\ &= \int \frac{P_\mu(z)}{|1 + \lambda F_\mu(z)|^2} \mathrm{d}\nu(\lambda) = P_\nu\left(-\frac{1}{F_\mu(z)}\right) . \end{aligned} \tag{3.2}$$

Here, the second equality follows from the Aronszajn-Krein formula [1],

$$F_{\mu_\lambda}(z) = \frac{F_\mu(z)}{1 + \lambda F_\mu(z)} , \tag{3.3}$$

which relates the Borel transforms of the spectral measures $\mu_\lambda$ and $\mu$. $\square$

*Remark* 3.2. If $\nu$ is a finite measure an analogous result between the respective Borel transforms was first obtained in [10]:

$$F_\kappa(z) = F_\nu\left(-\frac{1}{F_\mu(z)}\right). \tag{3.4}$$

Note that for non-finite $\nu$ the Borel transform will in general not exist (e.g. take $\nu$ to be the Lebesgue measure). In fact for $\sigma$-finite $\nu$, often the Poisson transform exists, whereas its Borel transform does not. In these cases we still have a relation between the Poisson transforms of $\nu$ and $\kappa$ as established in Proposition 3.1.

In order to prove finer statements on the continuity of $\kappa$, we first establish some results for uniformly Hölder continuous $\nu$. Recall the following definition:

**Definition 3.3.** Let $\eta$ be a $\sigma$-finite Borel measure on the real line and $\alpha \geq 0$. $\eta$ is uniformly $\alpha$ Hölder continuous (U$\alpha$H) if for some constant $K$, $\eta(I) \leq K|I|^\alpha$ for any interval $I$.

*Remark* 3.4.
(i) U1H implies absolute continuity.
(ii) Using the Rogers-Taylor decomposition theorem (see Theorem 4.2), there are no *non-trivial* U$\alpha$H measures for $\alpha > 1$.

For $\nu$ U$\alpha$H, Proposition 3.1 implies the following key estimate for the Poisson transform of $\kappa$:

**Proposition 3.2.** *If $\nu$ is UαH, $0 \leq \alpha \leq 1$, then for some constant $C_\alpha$ and all $z \in \mathbb{H}^+$*

$$P_\kappa(z) \leq C_\alpha \left( \frac{|F_\mu(z)|^2}{P_\mu(z)} \right)^{1-\alpha} . \tag{3.5}$$

*In particular, $\int \frac{1}{1+x^2} \mathrm{d}\kappa(x) < \infty$, whence $\kappa$ is a locally finite Borel measure (i.e. finite on compact sets).*

*Proof.* Let $z \in \mathbb{H}^+$. Recasting $P_\nu(z)$ in terms of the Lebesgue-Stilties measure induced by $M_\nu(\mathrm{Re}\,\{z\}; \delta)$, we get

(3.6)

$$\begin{aligned} P_\nu(z) &= \mathrm{Im}\,\{z\} \int_0^{+\infty} \frac{\mathrm{d}M_\nu(\mathrm{Re}\,\{z\};\delta)}{\delta^2 + \mathrm{Im}\,\{z\}^2} \\ &= \mathrm{Im}\,\{z\} \int_0^{+\infty} 2\delta \frac{M_\nu(\mathrm{Re}\,\{z\};\delta)}{\left(\delta^2 + \mathrm{Im}\,\{z\}^2\right)^2} \mathrm{d}\delta \leq \mathrm{Im}\,\{z\} 2K \int_0^{+\infty} \frac{\delta^{\alpha+1}}{\left(\delta^2 + \mathrm{Im}\,\{z\}^2\right)^2} \mathrm{d}\delta \\ &= \frac{\pi\alpha K}{2\sin\left(\frac{\pi\alpha}{2}\right)} \left(\mathrm{Im}\,\{z\}\right)^{\alpha-1} . \end{aligned}$$

Here, the second equality follows, using integration by parts; the last equality is obtained by contour integration. For $\alpha = 0$, the last equality in (3.6) is to be interpreted in the limit $\alpha \to 0$, i.e. $P_\nu(z) \leq K\,\mathrm{Im}\{z\}^{-1}$.

In particular, for $\nu$ UαH, (3.6) establishes $\int \frac{1}{1+x^2} \mathrm{d}\nu(x) < \infty$. Application of Proposition 3.1 hence yields the desired estimate. ☐

Proposition 3.2 reveals the special nature of the case $\alpha = 1$. Then we obtain a *uniform* upper bound for the Poisson transform of $\kappa$ valid in all of $\mathbb{H}^+$. In fact, this implies for $\kappa$ to inherit "full" continuity of the measure $\nu$:

**Theorem 3.5.** *If $\nu$ is U1H, so is $\kappa$.*

*Proof.* Let $0 \leq f$ be continuous of compact support. Using Proposition 3.2,

$$\begin{aligned} \int f(x)\mathrm{d}x &\geq \limsup_{\epsilon\to 0^+} \frac{1}{C_1} \int f(x) P_\kappa(x+i\epsilon)\mathrm{d}x \\ &= \limsup_{\epsilon\to 0^+} \frac{1}{C_1} \int \left( \int f(x) \frac{\epsilon}{(x-y)^2+\epsilon^2} \mathrm{d}x \right) \mathrm{d}\kappa(y) \\ &\geq \frac{1}{C_1} \int \left( \lim_{\epsilon\to 0^+} \int f(x) \frac{\epsilon}{(x-y)^2+\epsilon^2} \mathrm{d}x \right) \mathrm{d}\kappa(y) \\ &= \frac{\pi}{C_1} \int f(y)\mathrm{d}\kappa(y) . \end{aligned} \tag{3.7}$$

Here, the second equality follows from Tonelli, whereas the second inequality uses Fatou's Lemma. Note that $\sigma$-finiteness of $\kappa$ is implied by Proposition 3.2. ☐

Theorem 3.5 in particular implies $\mathrm{d}\kappa(x) \ll \mathrm{d}x$. Spectral averaging now arises as a special case where the density of $\kappa$ can be calculated explicitly.

*Proof of Theorem* 1.1. Since the Poisson transform of the Lebesgue measure $P_{\mathrm{Leb}}(z) = \pi$, all $z \in \mathbb{H}^+$, Theorem 2.2(i) and Proposition 3.1 yield $\mathrm{d}\kappa(x) = \mathrm{d}x$. ☐

## 4. Continuity with respect to Hausdorff measures

In this section we analyze the degree of continuity of $\kappa$ induced by measures $\nu$ with lesser degree of continuity than considered in the previous section. To this end we make the following definitions:

**Definition 4.1.** For $0 \leq \alpha$ let $h^\alpha$ denote the $\alpha$-dimensional Hausdorff measure on $\mathbb{R}$. Let $\eta$ be a Borel measure on the real line.

(1) $\eta$ is called $\alpha$-continuous ($\alpha$c) if $\eta(B) = 0$ whenever $h^\alpha(B) = 0$.
(2) $\eta$ is called $\alpha$-singular if $\eta$ is supported on a set of zero measure $h^\alpha$.

The main tools for proving Theorem 1.2 are the following two results due to Rogers and Taylor [13, 14, 15]:

**Theorem 4.2** (Rogers and Taylor - 1 (see Theorem 67 in [13]))**.** *Let $\eta$ be a $\sigma$-finite Borel measure on $\mathbb{R}$ and $0 \leq \alpha \leq 1$. Consider the sets $T^\alpha_{\eta;0+} := \left\{x : 0 \leq \overline{D}^\alpha_\eta(x) < \infty\right\}$ and $T^\alpha_{\eta;\infty} := \left\{x : \overline{D}^\alpha_\eta(x) = \infty\right\}$. Then $T^\alpha_{\eta;0+}$ and $T^\alpha_{\eta;\infty}$ are Borel measurable and*

(i) *$\eta$ is $\alpha c$ on $T^\alpha_{\eta;0+}$,*
(ii) *$h^\alpha\left(T^\alpha_{\eta;\infty}\right) = 0$ and $\eta$ is $\alpha$-singular on $T^\alpha_{\eta;\infty}$.*

**Theorem 4.3** (Rogers and Taylor - 2 (see Theorem 68 in [13]))**.** *Let $\eta$ be $\sigma$-finite and $\alpha$-continuous, $\alpha \geq 0$. For $\epsilon > 0$, there exist mutually singular measures $\eta_1$ and $\eta_2$ with $\eta = \eta_1 + \eta_2$ such that*

(i) *$\eta_1$ is U$\alpha$H and*
(ii) *$\eta_2(\mathbb{R}) < \epsilon$.*

*Remark* 4.4. Depending on $\overline{D}^\alpha_\eta$, Theorem 4.2 decomposes $\eta$ into an $\alpha$c and an $\alpha$-singular component. It thus generalizes the usual Lebesgue decomposition for $\alpha = 1$. The relevance of the Rogers-Taylor decomposition in spectral theory was pointed out by Last; see [16].

By Theorem 4.3, any $\alpha$-continuous measure is almost U$\alpha$H. Hence, the proof of Theorem 1.2 boils down to establishing the statement for a U$\alpha$H measure $\nu$. To this end we shall use the following lemma, which quantifies the asymptotic growth of $P_\eta$ and $Q_\eta$ near the support of a probability measure $\eta$.

**Lemma 4.5.** *Let $\eta$ be a probability (Borel) measure on $\mathbb{R}$. Then for $x \in \mathbb{R}$ and $\epsilon > 0$*

$$\max\left\{P_\eta(x+i\epsilon), |Q_\eta(x+i\epsilon)|\right\} \leq \frac{2}{\epsilon}\sum_{n=0}^{\infty} 2^{-n} M_\eta(x; 2^{n+1}\epsilon) \ .$$

*Proof.* Let $x \in \mathbb{R}$ and $\epsilon > 0$. Then:

$$|Q_\eta(x+i\epsilon)| \leq \textstyle\sum_{n=1}^\infty \int_{\epsilon 2^n \leq |x-y| \leq \epsilon 2^{n+1}} \dfrac{|x-y|}{(x-y)^2+\epsilon^2} \mathrm{d}\eta(y)$$
$$+ \textstyle\int_{|x-y|\leq 2\epsilon} \dfrac{|x-y|}{(x-y)^2+\epsilon^2}\mathrm{d}\eta(y) \leq \frac{2}{\epsilon}\sum_{n=0}^\infty 2^{-n} M_\eta(x; 2^{n+1}\epsilon) \ .$$

By a similar computation we obtain the same upper bound for $P_\eta(x+i\epsilon)$. □

We note that this result is an extended version of a lemma in [10] for U$\alpha$H $\eta$.

Together with Proposition 3.2, Lemma 4.5 allows us to control the asymptotic behaviour of $P_\kappa(x+i\epsilon)$ as $\epsilon \to 0^+$. We are thus in a position to prove Theorem 1.2.

## 5. Proof of the main theorem (Theorem 1.2)

We shall divide the proof into two steps: Step 1 establishes the statement for $\nu$ U$\alpha$H. Theorem 4.3 then allows us to extend the result to the $\alpha$c case (Step 2).

**Step 1:** Assume $\nu$ to be U$\alpha$H. If $\alpha = 1$, the statement follows directly from Theorem 3.5. Let $\alpha < 1$. We first examine the situation outside the support of the measure $\mu$.

**Proposition 5.1.** *Let* $0 < \alpha < 1$ *and* $\nu$ *U$\alpha$H. Then* $\kappa$ *is* $\alpha$*c outside* supp$\mu$.

*Proof.* Fixing $x \notin \operatorname{supp}\mu$, there exist positive constants $\Gamma_1$ and $\Gamma_2$ such that

$$|F_\mu(x+i\epsilon)|^2 \leq \Gamma_1 \ , \ P_\mu(x+i\epsilon) \geq \Gamma_2 \epsilon \ ,$$

for all $\epsilon > 0$ sufficiently small. Hence by Proposition 3.2 we obtain

$$\epsilon^{1-\alpha} P_\kappa(x+i\epsilon) \leq C_\alpha \epsilon^{1-\alpha} \left( \frac{|F_\mu(z)|^2}{P_\mu(z)} \right)^{1-\alpha} \leq C_\alpha \left( \frac{\Gamma_1}{\Gamma_2} \right)^{1-\alpha} , \tag{5.1}$$

which implies the claim by Theorem 4.2. □

*Remark* 5.1. By Theorem 4.3 (see the argument given in Step 2), the statement of Proposition 5.1 remains valid if $\nu$ is (only) $\alpha$c.

In order to analyze the situation within the support of $\mu$, we first establish the following lemma:

**Lemma 5.2.** *Let* $0 < \alpha < 1$ *and* $\nu$ *U$\alpha$H. Fix* $0 < \beta < 1$*. Then* $\kappa$ *is* $\gamma$*c on the set* $T^{\beta}_{\mu;0+}$*, where*

$$\gamma(\alpha, \beta) = \alpha - 2(1-\beta)(1-\alpha) \tag{5.2}$$

*as long as* $\beta > \max\left\{0, \frac{2-3\alpha}{2(1-\alpha)}\right\}$.

*Proof.* Let $\beta < 1$ be fixed. By Proposition 5.1 the statement is true outside supp$\mu$. Let $x \in \operatorname{supp}\mu$ and assume $\overline{D}^{\beta}_{\mu}(x) < \infty$ so that $M_\mu(x;\delta) \leq \Lambda_x \delta^\beta, \forall \delta > 0$. Thus,

$$\frac{2}{\epsilon} \sum_{n=0}^{\infty} 2^{-n} M_\mu(x; 2^{n+1}\epsilon) \leq \Lambda_x 2^{1+\beta} \epsilon^{\beta-1} \sum_{n=0}^{\infty} 2^{-n(1-\beta)} < \infty \ . \tag{5.3}$$

Note that finiteness of the upper bound in (5.3) requires $\beta < 1$.

Let $\gamma < 1$. Using Proposition 3.2 and Lemma 4.5, estimate (5.3) yields

$$\epsilon^{1-\gamma} P_\kappa(x+i\epsilon) \leq B_{x,\beta} \left( \frac{\epsilon^{2(\beta-1)+\frac{1-\gamma}{1-\alpha}}}{P_\mu(x+i\epsilon)} \right)^{1-\alpha} . \tag{5.4}$$

By Theorem 4.2 and Proposition 2.1, $\kappa$ will be $\gamma$c on the set $\left\{x : \limsup_{\epsilon\to 0^+} \epsilon^{1-\gamma} P_\kappa(x+i\epsilon) < \infty\right\}$. Choose $\gamma$ such that $2(\beta-1) + \frac{1-\gamma}{1-\alpha} = 1$, i.e. $\gamma = \alpha - 2(1-\beta)(1-\alpha)$. Since, $\epsilon^{-1} P_\mu(x+i\epsilon) \to \int \frac{1}{(x-y)^2} \mathrm{d}\mu(y)$ as $\epsilon \to 0^+$ and $\int \frac{1}{(x-y)^2} \mathrm{d}\mu(y) > 0$ for $x \in \operatorname{supp}\mu$, we obtain that $\kappa$ is $\gamma$c on the set $T^{\beta}_{\mu;0+}$ with $\gamma$ determined by (5.2). Finally, $\gamma > 0$ is ensured by requiring $\beta > \max\left\{0, \frac{2-3\alpha}{2(1-\alpha)}\right\}$. □

In summary we now obtain the claim for $\nu$ U$\alpha$H: Let $\delta = \alpha(1-\epsilon)$, $0 < \epsilon < 1$. It suffices to prove the statement for $\epsilon$ sufficiently small. Let $\beta$ be such that $\gamma(\alpha,\beta) = \delta$, i.e. $\beta = 1 - \frac{\alpha}{2(1-\alpha)}\epsilon$. Choosing $\epsilon$ sufficiently small we can ensure that $\beta > \frac{2-3\alpha}{2(1-\alpha)}$, which is required to apply Lemma 5.2.

For such a choice of $\epsilon$ and $\beta$, Lemma 5.2 implies that for any Borel set $B$ with $h^\delta(B) = 0$,

$$\kappa(B) = \int \mu_{\lambda,\mathrm{sing}}(B \cap T_{\mu;\infty}^{\beta})\mathrm{d}\nu(\lambda) \leq \int \mu_{\lambda,\mathrm{sing}}(T_{\mu;\infty}^{1})\mathrm{d}\nu(\lambda) = 0 \ . \tag{5.5}$$

Applying Propositions 2.2 and 2.1, $\mu_{\lambda,\mathrm{sing}}(T_{\mu;\infty}^{1}) = 0$ for $\lambda \neq 0$, which by continuity of $\nu$ implies the last equality in (5.5).

**Step 2:** Let $0 < \alpha < 1$ and $\delta < \alpha$. If $\nu$ is $\alpha$c, then by Theorem 4.3, given $\epsilon > 0$ there are measures $\nu_1 \perp \nu_2$, $\nu = \nu_1 + \nu_2$ such that $\nu_1$ is U$\alpha$H and $\nu_2(\mathbb{R}) < \epsilon$. Let $B \subseteq \mathbb{R}$ be a Borel set with $h^\delta(B) = 0$. Then, $\int \mu_\lambda(B)\mathrm{d}\nu_1(\lambda) = 0$ by Step 1, whence

$$\kappa(B) = \int \mu_\lambda(B)\mathrm{d}\nu_2(\lambda) < \epsilon \ .$$

An analogous argument shows that $\kappa$ is absolutely continuous if $\alpha = 1$, which concludes the proof of Theorem 1.2.

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