THE ŁOJASIEWICZ EXPONENT
OF A CONTINUOUS SUBANALYTIC FUNCTION
AT AN ISOLATED ZERO

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Abstract. Let $f$ be a continuous subanalytic function defined in a neighborhood of the origin $0 \in \mathbb{R}^n$ such that $f$ has an isolated zero at $0$. We describe the smallest possible exponents $\alpha, \beta, \theta$ for which we have the following estimates:

$$|f(x)| \geq c \|x\|^\alpha, \quad m_f(x) \geq c \|x\|^\beta, \quad m_f(x) \geq c |f(x)|^\theta$$

for $x$ near zero, where $c > 0$ and $m_f(x)$ is the nonsmooth slope of $f$ at $x$. We prove that $\alpha = \beta + 1, \theta = \beta/\alpha$.

In the smooth case, we have $m_f(x) = \|\nabla f(x)\|$, and we therefore retrieve a result of Gwoździewicz, which is a counterpart of the result of Teissier in the complex case.

1. Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a complex analytic function with isolated singularity at the origin $0 \in \mathbb{C}^n$. It is well known that there exist constants $c, r > 0$ and exponents $\beta, \theta$ such that, for all $\|x\| \leq r$,

$$\|\nabla f(x)\| \geq c \|x\|^\beta, \quad \|\nabla f(x)\| \geq c |f(x)|^\theta. \quad (1)$$

Teissier [26] showed that the smallest possible exponents $\beta, \theta$ for (1) are attained along the polar curve of $f$ and satisfy the following relation:

$$\theta = \frac{\beta}{\beta + 1}. \quad (2)$$

The results of the exact formula of the Łojasiewicz exponent $\beta$ (and hence, $\theta$) for weighted homogeneous isolated singularities are in the recent papers by Krasinski, Oleksik and Płoski [16] and by Tan, Yau and Zuo [25] (see also [13], [14]). Estimations of the Łojasiewicz exponent $\beta$ in the general case can be found in [18], [23], [8], [1].

On the other hand, it was shown by Gwoździewicz in [9] that the relation (2) does not necessarily hold for real analytic functions.

Now let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a real analytic function defined in a neighborhood of the origin $0 \in \mathbb{R}^n$. Assume that $f(x) > 0$ for $0 < \|x\| \ll 1$. Then $\nabla f(x)$ is nonzero for $x$ close to the origin. According to the classical Łojasiewicz inequality (see [19],
there exist constants $c, r > 0$ and exponents $\alpha, \beta, \theta$ such that, for all $\|x\| \leq r$,
\[
|f(x)| \geq c\|x\|^\alpha, \quad \|\nabla f(x)\| \geq c\|x\|^\beta, \quad \|\nabla f(x)\| \geq c|f(x)|^\theta.
\]
Gwoździewicz [9] showed that the best exponents $\alpha, \beta, \theta$ for (3) are attained along the polar curve of $f$, and moreover, these exponents satisfy the following:
\[
\alpha = \beta + 1 \quad \text{and} \quad \theta = \frac{\beta}{\beta + 1}.
\]

The aim of this paper is to establish a nonsmooth version of the relation (4) for continuous subanalytic functions (Theorem 3.1). See, for example, [2] for the definition and basic properties of subanalytic functions.

Given a continuous subanalytic mapping $f: (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$, our approach to generalizing the above-mentioned properties relies on a one-sided notion of generalized gradients called subgradients (see, e.g., [24]). Moreover, our technical tool, contrary to [26] and [9] who use polar curves, is based on the notion of tangency varieties (introduced and studied in [10], [11], [12]).

The paper is organized as follows. In Section 2 we recall some basic notions and facts concerning the Łojasiewicz exponent and the subdifferential. The main result and its proof are given in Section 3.

2. Preliminaries

Throughout this work we shall consider the Euclidean vector space $\mathbb{R}^n$ endowed with its canonical scalar product $\langle \cdot, \cdot \rangle$ and we shall denote its associated norm by $\|\cdot\|$.

2.1. The Łojasiewicz exponent. Let us start with the following:

**Definition 2.1.** By the Łojasiewicz exponent $\ell_0(g, h)$ for the inequality $\|g(x)\| \geq c\|h(x)\|^\alpha$ we mean the number
\[
\inf \{\alpha \in \mathbb{R}_+: \exists c, r > 0 \|g(x)\| \geq c\|h(x)\|^\alpha \text{ for all } \|x\| \leq r\}.
\]

We next recall the definition of the order of continuous subanalytic functions (see [9]). Let $g: [0, 1) \to \mathbb{R}$ be a continuous subanalytic function. Here and subsequently we assume that $g \neq 0$ in every neighborhood of zero. Then there exist (see [9] Lemma 3) a nonnegative rational number $\nu$ and a continuous function $g_1: [0, \delta] \to \mathbb{R}$ ($0 < \delta < 1$) such that for all $\tau \in [0, \delta]$, $g_1(\tau) \neq 0$ and $g(\tau) = \tau^\nu g_1(\tau)$. It is obvious that the exponent $\nu$ is uniquely determined by the function $g$ (even by a germ of $g$ at zero). We call this number the order (at zero) of $g$ and will denote it by $\nu(g)$. We extend the notion of order to subanalytic continuous mappings, putting $\nu(\phi) := \nu(\|\phi\|)$ for $\phi: [0, 1) \to \mathbb{R}^n$.

We have the following easy property.

**Lemma 2.1.** Let $g, h: [0, 1) \to \mathbb{R}$ be continuous subanalytic functions nonvanishing in every neighborhood of zero and let $r$ be a positive rational number. Then
\begin{itemize}
  \item[(i)] $\nu(g^r) = r\nu(g), \nu(gh) = \nu(g) + \nu(h)$, and
  \item[(ii)] $\nu(g) \leq \nu(h)$ if and only if there exist $c, \delta > 0$ such that $|g(\tau)| \geq c|h(\tau)|$ for all $\tau \in [0, \delta]$.
\end{itemize}

**Proof.** See [9] Property 2.1.
In the following lemma we reformulate the main result of [3] (see also [2]) in the case of functions with isolated zeros.

**Lemma 2.2.** Let $U$ be a neighborhood of $0 \in \mathbb{R}^n$, and let $g, h : U \to \mathbb{R}$ be continuous subanalytic functions such that $g^{-1}(0) = h^{-1}(0) = \{0\}$. The following statements hold:

(i) There exists a positive constant $c$ such that $\|g(x)\| \geq c\|h(x)\|^\ell_0(g, h)$ in a neighborhood of the origin.

(ii) For every analytic curve $\phi : [0, 1) \to U$, $\phi(0) = 0$, $\phi \neq 0$, we have $\ell_0(g, h) \geq \nu(\phi) / \nu(h \circ \phi)$.

(iii) There exists an analytic curve $\gamma : [0, 1) \to U$, $\gamma(0) = 0$, $\gamma \neq 0$, such that $\ell_0(g, h) = \nu(\phi \circ \gamma) / \nu(h \circ \gamma)$.

**Proof.** See [9, Corollary 2.4].

\[\Box\]

2.2. The nonsmooth slope. The notion of subdifferential, that is, an appropriate multivalued operator playing the role of the usual gradient mapping, is crucial for our considerations.

**Definition 2.2** (see, e.g., [23] Definition 8.3). (i) The Fréchet subdifferential $\partial f(x)$ of a continuous function $f : U \to \mathbb{R}$ at $x \in U$ is given by

\[
\partial f(x) := \left\{ v \in \mathbb{R}^n \mid \liminf_{y \to x, y \neq x} \frac{f(y) - f(x) - \langle v, y - x \rangle}{\|y - x\|} \geq 0 \right\}.
\]

(ii) The limiting subdifferential at $x \in U$, denoted by $\partial f(x)$, is the set of all cluster points of sequences \(\{v^t\}_{t \geq 1}\) such that $v^t \in \partial f(x^t)$ and $(x^t, f(x^t)) \to (x, f(x))$ as $t \to \infty$.

**Remark 2.1.** It is a well-known result of variational analysis that $\partial f(x)$ (and for tot $\partial f(x)$) is not empty in a dense subset of the domain of $f$ (see [24], for example).

**Definition 2.3** (see [5]). Using the limiting subdifferential $\partial f$, we define the nonsmooth slope of $f$ by

\[m_f(x) := \inf\{\|v\| \mid v \in \partial f(x)\}.
\]

By definition, $m_f(x) = +\infty$ whenever $\partial f(x) = \emptyset$.

**Remark 2.2.** (i) If the function $f$ is of class $C^1$, the above notion coincides with the usual concept of the gradient; that is, $\partial f(x) = \partial f(x) = \{\nabla f(x)\}$, and hence $m_f(x) = \|\nabla f(x)\|$.

(ii) Recently, Bolte et al. [6] (see also [17], [1], [9]) proved that if the function $f$ is subanalytic, then the operators $\partial f$ and $\partial f$ and the function $m_f$ are subanalytic. Moreover, the authors have extended the Lojasiewicz gradient inequality to continuous subanalytic functions. Namely (actually their version is more general than the theorem stated below),

**Theorem 2.1** ([5] Theorem 3.1). Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a continuous subanalytic function defined in a neighborhood of $0 \in \mathbb{R}^n$. Then there exist constants $c, r > 0$ and an exponent $\theta \in [0, 1)$ such that

\[\|m_f(x)\| \geq c|f(x)|^\theta \quad \text{for all } \|x\| \leq r.
\]

Note that we have adopted here the following convention: $0^0 = 1$.

The following also is observed by Massey [22].
Corollary 2.1. Let \( G := (g_1, \ldots, g_k): (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^k, 0) \) be a \( C^1 \)-subanalytic mapping defined in a neighborhood of \( 0 \in \mathbb{R}^n \). Then there exist constants \( c, r > 0 \) and an exponent \( \theta \in [0, 1) \) such that
\[
\sqrt{\|\nabla g_1(x)\|^2 + \cdots + \|\nabla g_k(x)\|^2} \geq c\|G(x)\|^\theta \quad \text{for all } \|x\| \leq r.
\]

Proof. Consider the function \( f: (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}, 0) \) defined by
\[
f(x) := \max_{i=1, \ldots, k} |g_i(x)|.
\]
Clearly, \( f \) is a continuous subanalytic function. Moreover, we have (see [24], for example)
\[
\partial f(x) = \left\{ \sum_{i \in I(x)} \lambda_i \nabla g_i(x) \mid \lambda_i \geq 0 \text{ and } \sum_{i \in I(x)} \lambda_i = 1 \right\},
\]
where \( I(x) := \{ i \in \{1, \ldots, 2k \} \mid g_i(x) = f(x) \} \) and \( g_{k+i}(x) := -g_i(x), i = 1, \ldots, k \). Hence, we have, for all \( x \in U \),
\[
m_f(x) = \lambda_i \geq 0, \quad \min_{\sum_{i \in I(x)} \lambda_i = 1} \left\| \sum_{i \in I(x)} \lambda_i \nabla g_i(x) \right\| \\
\leq \left\| \sum_{i \in I(x)} \frac{1}{\# I(x)} \nabla g_i(x) \right\| = \frac{1}{\# I(x)} \left\| \sum_{i \in I(x)} \nabla g_i(x) \right\|.
\]
Applying the Cauchy-Schwarz inequality, we get for some \( c > 0 \) that
\[
m_f(x) \leq c \sqrt{\sum_{i \in I(x)} \|\nabla g_i(x)\|^2} \leq c \sqrt{2 \sum_{i=1}^k \|\nabla g_i(x)\|^2}.
\]
Then the desired inequality follows at once from Theorem 2.1. \( \square \)

2.3. The tangency variety. In this paper, we will replace the polar curve by a subanalytic subset of \( \mathbb{R}^n \) which we call the tangency variety. Precisely, we have the following definition (see also [10], [11], [12]).

Definition 2.4. The tangency variety of a continuous function \( f: U \rightarrow \mathbb{R} \) is defined as follows:
\[
\Gamma(f) := \{ x \in U \mid \exists \mu \in \mathbb{R} \text{ such that } \mu x \in \partial f(x) \}.
\]

Geometrically, in the smooth case, the set \( \Gamma(f) \) consists of all points \( x \in U \) where the level sets of \( f \) are tangent to the sphere in \( \mathbb{R}^n \) centered at the origin and with radius \( \|x\| \).

The following is a simple fact about the tangency variety \( \Gamma(f) \).

Property 2.1. Let \( f: U \rightarrow \mathbb{R} \) be a continuous function. The origin belongs to the closure of the set \( \Gamma(f) \setminus \{0\} \).

Proof. It is sufficient to check that every sphere \( S^{n-1}_r := \{ x \in \mathbb{R}^n \mid \|x\| = r \} \ (0 < r \ll 1) \) has a nonempty intersection with \( \Gamma(f) \). In fact, since \( S^{n-1}_r \) is compact, there exists \( x \in S^{n-1}_r \) such that \( f(x) = \min_{y \in S^{n-1}_r} f(y) \). It follows from Lagrange’s multiplier theorem that \( \mu x \in \partial f(x) \) for some \( \mu \in \mathbb{R} \); that is, \( x \in \Gamma(f) \). \( \square \)
Remark 2.3. If the continuous function $f$ is subanalytic, then the tangency variety $\Gamma(f)$ is a subanalytic set. The argument is standard (see, e.g., [2]). As we shall not use this statement, we leave the proof as an exercise.

3. The result and its proof

The main result of this paper can now be stated as follows.

**Theorem 3.1.** Let $f: U \rightarrow \mathbb{R}$ be a continuous subanalytic function defined in a neighborhood $U$ of the origin $0 \in \mathbb{R}^n$. Assume that $f(x) > 0$ for $0 < \|x\| < 1$. Then there exists an analytic curve $\gamma: [0, 1) \rightarrow \Gamma(f), \gamma(0) = 0$, for which

(i) the Lojasiewicz exponent $\alpha_0$ for the inequality $|f(x)| \geq c|x|^\alpha$ is equal to $\alpha_0 = \nu(f \circ \gamma)/\nu(\gamma)$;

(ii) the Lojasiewicz exponent $\beta_0$ for the inequality $m_f(x) \geq c\|x\|^\beta$ is equal to $\beta_0 = \nu(m_f \circ \gamma)/\nu(\gamma)$;

(iii) the Lojasiewicz exponent $\theta_0$ for the inequality $m_f(x) \geq c|f(x)|^\theta$ is equal to $\theta_0 = \nu(m_f \circ \gamma)/\nu(f \circ \gamma)$.

Moreover $\alpha_0 = \beta_0 + 1, \theta_0 = \beta_0/\alpha_0$.

Now, we give the following example.

**Example 3.1.** Let us consider the continuous subanalytic function $f: (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$, with $f(x, y) := \max\{|x^2 - y^2|, |xy|\}$. Then $f(x, y) > 0$ for $(x, y) \neq (0, 0)$. Let $\phi: [0, 1) \rightarrow \mathbb{R}$ be an analytic curve defined by $\phi(\tau) = (\tau, \tau)$. We have $f(\phi(\tau)) = \tau^2$ and $\lim_{\tau \rightarrow 0} \|\phi(\tau)\|/|\tau| = 1$. Hence, $\ell_0(f, \| \| ) \geq 2$. In fact it is easy to check that $\ell_0(f, \| \| ) = 2$. According to Theorem 3.1 we have $\ell_0(m_f, \| \| ) = 1$ and $\ell_0(m_f, f) = 1/2$.

In what follows let $f: U \rightarrow \mathbb{R}$ be a continuous subanalytic function defined in a neighborhood $U$ of $0 \in \mathbb{R}^n$.

**Remark 3.1.** Let $\phi: [0, 1) \rightarrow U, \tau \mapsto \phi(\tau)$, be an analytic curve. By definition, $f \circ \phi: [0, 1) \rightarrow \mathbb{R}$ is a continuous subanalytic function. Applying the monotonicity lemma (e.g. [27 Theorem 4.1], [7 Theorem 2.1]) to $f \circ \phi$, we deduce that $f \circ \phi$ is absolutely continuous and differentiable (in fact, analytic) in a complement of a finite set.

The proof of Theorem 3.1 will be divided into several steps, which, for convenience, will be called lemmas.

**Lemma 3.1.** Let $\phi: [0, 1) \rightarrow U, \tau \mapsto \phi(\tau)$, be an analytic curve. Then

$$|(f \circ \phi)'(\tau)| \leq m_f(\phi(\tau))\|\phi'(\tau)\|,$$

for all but finitely many $\tau \in [0, 1)$.

**Proof.** Applying the chain rule calculus for the Fréchet subdifferential [23 Theorem 10.6], we get for all but finitely many $\tau \in [0, 1)$ that

$$\hat{\partial}(f \circ \phi)(\tau) = \{(f \circ \phi)'(\tau)\} = \left\{ \frac{d}{d\tau}(f \circ \phi)(\tau) \right\} = \left\{ \langle v, \phi'(\tau) \rangle \mid v \in \hat{\partial}f(\phi(\tau)) \right\},$$

which yields via a standard argument that

$$|(f \circ \phi)'(\tau)| \leq \|v\|\|\phi'(\tau)\| \quad \text{for all } v \in \hat{\partial}f(\phi(\tau)).$$
It now follows by passing to the limit (according to Definition 2.2(ii)) that
\[ |(f \circ \phi)'(\tau)| \leq \|v\|\|\phi'(\tau)\| \quad \text{for all } v \in \partial f(\phi(\tau)), \]
and by taking the infimum over all \( v \in \partial f(\phi(\tau)) \), we obtain
\[ |(f \circ \phi)'(\tau)| \leq m_f(\phi(\tau))\|\phi'(\tau)\| \]
for all \( \tau \) in the complement of a finite set. \( \Box \)

**Lemma 3.2.** Under the hypotheses of Theorem 3.1, the function \( m_f(x) \) has an isolated zero at 0.

**Proof.** Since \( f(x) > 0 \) for \( 0 < \|x\| \ll 1 \), the function \( f \) has a strict local minimum at the origin. Thanks to Lagrange’s multiplier theorem that \( 0 \in \partial f(0) \); in particular, \( m_f(0) = 0 \).

We next claim that the origin is an isolated zero of \( m_f(x) \). By contradiction and using the Curve Selection Lemma (see [15]), there exists an analytic curve \( \phi: [0,1) \to U, \phi(0) = 0, \phi \neq 0 \), such that \( m_f(\phi(\tau)) = 0 \) for all \( \tau \in [0,1) \). By Lemma 3.1 we get for all but finitely many \( \tau \in [0,1) \) that
\[ (f \circ \phi)'(\tau) = 0. \]
It follows that \( f(\phi(\tau)) = f(\phi(0)) = 0 \) for all \( \tau \) in the complement of a finite set, which is a contradiction. \( \Box \)

**Lemma 3.3.** Under the hypothesis of Theorem 3.1, for any analytic curve \( \phi: [0,1) \to U, \phi(0) = 0, \phi \neq 0 \), we have
\begin{enumerate}
  \item \( \ell_0(f, \| \|) \geq \nu(f \circ \phi)/\nu(\phi); \)
  \item \( \ell_0(m_f, \| \|) \geq \nu(m_f \circ \phi)/\nu(\phi); \)
  \item \( \ell_0(m_f, f) \geq \nu(m_f \circ \phi)/\nu(f \circ \phi); \)
  \item \( \nu(f \circ \phi) \geq \nu(m_f \circ \phi) + \nu(\phi). \)
\end{enumerate}

**Proof.** Items (i)-(iii) follow immediately from Lemma 2.2 (iv). According to Lemma 3.1 we have
\[ |(f \circ \phi)'(\tau)| \leq m_f(\phi(\tau))\|\phi'(\tau)\| \]
for all small \( \tau > 0 \). Combining this with Lemma 2.1 we obtain
\[ \nu((f \circ \phi)') \geq \nu(m_f \circ \phi) + \nu(\phi'), \]
and hence
\[ \nu(f \circ \phi) \geq \nu(m_f \circ \phi) + \nu(\phi). \]
\( \Box \)

**Lemma 3.4.** Under the hypothesis of Theorem 3.1, there exists an analytic curve \( \gamma: [0,1) \to \Gamma(f), \gamma(0) = 0, \gamma \neq 0 \), such that the following conditions hold:
\begin{enumerate}
  \item \( \ell_0(f, \| \|) = \nu(f \circ \gamma)/\nu(\gamma); \)
  \item \( \nu(f \circ \gamma) = \nu(m_f \circ \gamma) + \nu(\gamma). \)
\end{enumerate}

**Proof.** Let us consider a ball \( K := \{ x \in \mathbb{R}^n \mid \|x\| \leq r \} \) contained in \( U \), and put
\[ K^* := \{ x \in K \mid f(x) = \min_{\|y\|^2=\|x\|^2} f(y) \}. \]
It is easy to check (see, e.g., [8]) that the set \( K^* \) is subanalytic and that the origin is a cluster point of \( K^* \setminus \{0\} \). By the Curve Selection Lemma (see [15]), there exists an analytic curve \( \gamma: [0,1) \to \mathbb{R}^n \) such that \( \gamma(0) = 0 \) and \( \gamma([0,1)) \subset K^* \setminus \{0\} \). Set
α := ν(f ◦ γ)/ν(∥γ∥). By Lemma 2.1(i), we have ν(f ◦ γ) = ν(∥γ∥α). Thus, by Lemma 2.1(ii), there are positive constants c, δ such that
\[ f(γ(τ)) ≥ c∥γ(τ)∥^α \] for τ ∈ [0, δ].
Since the norm ∥∥ is a continuous function, there exists ε > 0 such that ∥x∥ ≤ ∥γ(δ)∥ for all ∥x∥ ≤ ε. Fix x ∈ K with ∥x∥ ≤ ε. By continuity of the function τ → ∥γ(τ)∥, there exists τ ∈ [0, δ] such that ∥γ(τ)∥ = ∥x∥. Then it follows from the definition of K* that
\[ f(x) ≥ f(γ(τ)) ≥ c∥γ(τ)∥^α = c∥x∥^α. \]
Therefore
\[ f(x) ≥ c∥x∥^α \] for ∥x∥ ≤ ε.
Consequently, ℓ0(f, ∥∥) ≤ α. Then item (i) follows from Lemma 3.3(i).

Let us now establish item (ii). Indeed, the properties of γ imply that
\[ f(γ(τ)) = \min_{∥y∥^2=∥γ(τ)∥^2} f(y) \] for τ ∈ [0, 1).
We therefore deduce from Lagrange’s multiplier theorem that there exists a function μ: [0, 1) → R such that μ(τ)γ(τ) ∈ ∂f(γ(τ)), which means that γ(τ) ∈ Γ(f). By Lemma 2.2 μ ≠ 0. Applying the monotonicity lemma (e.g., 27 Theorem 4.1, 27 Theorem 2.1) to f ◦ φ and the chain rule calculus for the Fréchet subdifferential [24 Theorem 10.6], we get for all but finitely many τ ∈ [0, 1) that
\[ (f ◦ φ)'(τ) = ⟨μ(τ)γ(τ), γ'(τ)⟩ = \frac{μ(τ)}{2}(∥γ(τ)∥^2)' \]
Hence
\[ m_f(γ(τ)) = \inf_{v ∈ ∂f(γ(τ))} ∥v∥ ≤ │μ(τ)∥γ(τ)∥∥ = \frac{2∥(f ◦ φ)'(τ)∥∥γ(τ)∥}{(∥γ(τ)∥^2)'} \]
for all τ in the complement of a finite set. Combining this with Lemma 2.1, we get
\[ ν(m_f ◦ γ) + ν(γ') ≥ ν((f ◦ γ)'), \]
which implies that
\[ ν(m_f ◦ γ) + ν(γ) ≥ ν(f ◦ γ). \]
Then item (ii) follows from Lemma 3.3(iv).

We are now in a position to finish the proof of Theorem 3.1

Proof of Theorem 3.1(i). Let γ: [0, 1) → Γ(f) be an analytic curve from the statement of Lemma 3.2. Then item (i) follows immediately from Lemma 3.3(i).

(ii). By Lemma 3.2 we may assume (shrinking the ball U if necessary) that m_f(x) ≠ 0 for all x ∈ U \ {0}. Thus, by Lemma 2.2 there exists an analytic curve φ: [0, 1) → U, φ(0) = 0, φ ≠ 0, for which β_0 = ν(m_f ◦ φ)/ν(φ). Using Lemma 3.3 we obtain
\[ β_0 = \frac{ν(m_f ◦ φ)}{ν(φ)} ≤ \frac{ν(f ◦ φ)}{ν(φ)} - 1 ≤ α_0 - 1. \]
For the curve γ we have
\[ α_0 = \frac{ν(f ◦ γ)}{ν(γ)} = \frac{ν(m_f ◦ γ)}{ν(γ)} + 1 ≤ β_0 + 1. \]
From these inequalities we get β_0 ≤ α_0 - 1.
By Lemma 2.2 there exists an analytic curve $\psi: [0, 1) \to U, \psi(0) = 0, \psi \neq 0$, for which $\theta_0 = \frac{\nu(m_f \circ \psi)}{\nu(f \circ \psi)}$. Using Lemma 3.3 we have

$$\theta_0 = \frac{\nu(m_f \circ \psi)}{\nu(f \circ \psi)} \leq 1 - \frac{\nu(\psi)}{\nu(f \circ \psi)} \leq 1 - \frac{1}{\alpha_0}.$$

For the curve $\gamma$ we have

$$\theta_0 \geq \frac{\nu(m_f \circ \gamma)}{\nu(f \circ \gamma)} = 1 - \frac{\nu(\gamma)}{\nu(f \circ \gamma)} = 1 - \frac{1}{\alpha_0}.$$

Collecting together these inequalities we obtain

$$\theta_0 = \frac{\nu(m_f \circ \gamma)}{\nu(f \circ \gamma)} = 1 - \frac{1}{\alpha_0} = \frac{\beta_0}{\alpha_0}.$$

We now suppose that $f$ is a $C^1$-subanalytic function. In this case, the definition of the limiting subdifferential implies that $\partial f(x) = \{\nabla f(x)\}$, so that $m_f(x) = \|\nabla f(x)\|$. Combining these facts with Theorem 3.1 we obtain

**Corollary 3.1.** (9 Theorem 1.3). Let $f: U \to \mathbb{R}$ be a $C^1$-subanalytic function defined in a neighborhood $U$ of the origin $0 \in \mathbb{R}^n$. Assume that $f(x) > 0$ for $0 < \|x\| < 1$. Then there exists an analytic curve

$$\gamma: [0, 1) \to \Gamma(f) = \{x \in \mathbb{R}^n \mid \exists \mu \in \mathbb{R} \text{ such that } \mu x = \nabla f(x)\}, \gamma(0) = 0,$$

for which

(i) the Lojasiewicz exponent $\alpha_0$ for the inequality $|f(x)| \geq c\|x\|^\alpha$ is equal to $\alpha_0 = \frac{\nu(f \circ \gamma)}{\nu(\gamma)}$;

(ii) the Lojasiewicz exponent $\beta_0$ for the inequality $\|\nabla f(x)\| \geq c\|x\|^\beta$ is equal to $\beta_0 = \frac{\nu(\nabla f \circ \gamma)}{\nu(\gamma)}$;

(iii) the Lojasiewicz exponent $\theta_0$ for the inequality $\|\nabla f(x)\| \geq c|f(x)|^\theta$ is equal to $\theta_0 = \frac{\nu(\nabla f \circ \gamma)}{\nu(f \circ \gamma)}$.

Moreover $\alpha_0 = \beta_0 + 1, \theta_0 = \beta_0 / \alpha_0$.

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