CLOSED GEODESICS AND VOLUME GROWTH
OF RIEMANNIAN MANIFOLDS

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ABSTRACT. In this paper, we study the relation between the existence of closed geodesics and the volume growth of open Riemannian manifolds with non-negative curvature.

1. Introduction

Let \( M^n \) be an \( n \)-dimensional complete, noncompact Riemannian manifold with sectional curvature \( K_M \geq 0 \). Let

\[
\alpha_M = \lim_{r \to \infty} \frac{Vol(B(p, r))}{\omega_n r^n},
\]

where \( Vol(B(p, r)) \) is the volume of geodesic ball in \( M^n \) with radius \( r \) around \( p \) and \( \omega_n \) denotes the volume of unit ball in \( R^n \). From [6] we know that \( \alpha_M \) is independent of the choice of base point \( p \). By the Bishop-Gromov volume comparison theorem, we have \( 0 \leq \alpha_M \leq 1 \) and \( M^n \) is isometric to \( R^n \) if and only if \( \alpha_M = 1 \).

The main goal of this paper is to prove the following theorem.

Theorem 1.1. Let \( M^n \) be a complete noncompact manifold with nonnegative section curvature. If \( M^n \) contains a closed geodesic, then the volume growth \( \alpha_M = 0 \).

In other words, if \( \alpha_M > 0 \), then \( M^n \) does not contain any closed geodesic.

We may look at Theorem 1.1 in an intuitive manner: To an open manifold with nonnegative section curvature, the closed geodesic will make the manifold shrink.

By the Cheeger-Gromoll soul theorem (see [2]), if the soul of \( M^n \) is not a point, then \( M^n \) must contain at least one closed geodesic. If the soul is one point, \( M^n \) still may have many closed geodesics. The following is a simple example.

Example 1.2. Let \( M^2 = C_+ \cup S^2_+ \) be a cylinder \( C_+ = S^1 \times [0, \infty) = \{(x, y, z) | x^2 + y^2 = 1, z \geq 0 \} \) glued to the lower hemisphere \( S^2_- = \{(x, y, z) | x^2 + y^2 + z^2 = 1, z \leq 0 \} \).

Then the soul of \( M^2 \) is a point, but \( M^2 \) admits infinitely many closed geodesics.

In fact, our theorem is more significant when the soul is one point. In this case, the volume growth gives a sufficient condition of the nonexistence of closed geodesics, which is not a trivial thing.
Remark 1.3. In what follows, we always assume that manifolds are complete non-compact with nonnegative sectional curvature.

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following two lemmas.

Lemma 2.1. Let \( \sigma(t) \) be a closed geodesic of \( M^n \) with canonical parameter of the arc such that \( \sigma(0) = \sigma(b) = p, \sigma'(0) = \sigma'(b) \), where \( b \) is the length of \( \sigma(t) \). For any ray \( \gamma(t) \) starting at \( p \), we have \( \alpha = \angle(\sigma'(0), \gamma'(0)) = \pi/2 \).

Proof. Let \( l \) be the length of \( \sigma(t) \) from \( \sigma(0) \) to \( \sigma(l) \). By the Toponogov comparison theorem \([1]\), we have

\[
t^2 + l^2 - 2tl \cos \alpha \geq d^2(\sigma(l), \gamma(t)).
\]

Thus

\[
\cos \alpha \leq \frac{t^2 + l^2 - d^2(\sigma(l), \gamma(t))}{2tl},
\]

where \( d(.,.) \) is the distance function. Recalling the condition of the Toponogov comparison theorem \([1]\), one only needs \( l < \infty \). Let \( l = b \). Then \( t = d(\sigma(b), \gamma(t)) \).

Thus

\[
\cos \alpha \leq \frac{b}{2t}.
\]

Let \( t \to \infty \). Then

\[
\cos \alpha \leq 0,
\]

so

\[
\alpha \geq \pi/2.
\]

Considering \( \sigma(b-t) \), we obtain

\[
\pi - \alpha \geq \pi/2.
\]

Hence

\[
\alpha = \pi/2.
\]

\[]

Remark 2.2. Lemma 2.1 can also be deduced by analytic methods. For example, see Theorem 1.10 of \([2]\). But our proof is more direct.

The next lemma is due to Ordway, Stephens and Yang \([3]\). It shows that \( \alpha_M \) is determined by “the volume of rays”.

Lemma 2.3. Let \( \Sigma = \{ \nu \in S_p M | exp_p(t\nu) \text{ is a ray, } t \geq 0 \} \). Here \( S_p M \) is the unit sphere in \( T_p M \). Set

\[
C(\Sigma) = \{ q \in M | q = exp_p(t\nu), \nu \in \Sigma, t \geq 0 \}
\]

and

\[
B(\Sigma, r) = B(p, r) \cap C(\Sigma).
\]

Then we have

\[
\alpha_M = \lim_{r \to \infty} \frac{Vol(B(\Sigma, r))}{\omega_n r^n}.
\]

The proof of Lemma 2.3 is based on the Bishop-Gromov volume comparison theorem. For details, one may see \([3]\).

Now we can prove Theorem 1.1.
Proof. If $M^n$ contains a closed geodesic, by Lemma 2.1, we have $\text{mes}(\Sigma) = 0$ (induced measure of the unit sphere). By Fubini’s theorem, for any $r > 0$ we have $\text{mes}(\exp^{-1}(B(\Sigma, r))) = 0$.

Since $\exp$ is $C^\infty$, by Sard’s theorem [3], for any $r > 0$ we have $\text{Vol}(B(\Sigma, r)) = 0$.

Then by Lemma 2.3, we have $\alpha_M = 0$. □

3. An application of Theorem 1.1

Combining with the Cheeger-Gromoll soul theorem (see [2]), we get another proof of Marenich and Toponogov’s beautiful theorem (see [5]):

**Theorem 3.1.** If $\alpha_M > 0$, then $M^n$ is diffeomorphic to $\mathbb{R}^n$.

Proof. If $M^n$ is not diffeomorphic to $\mathbb{R}^n$, by the Cheeger-Gromoll soul theorem, the soul (a totally geodesic submanifold) of $M^n$ is not a point. Then the soul must contain a closed geodesic (since any compact manifold contains at least one closed geodesic [4]). It is also the closed geodesic of $M^n$, which is a contradiction to Theorem 1.1. □

**Remark 3.2.** By a different method, Theorem 3.1 is also a consequence of Perelman’s celebrated flat strip theorem (cf. [7]).

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**References**


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