THE LIMIT OF $\mathbb{F}_p$-BETTI NUMBERS OF A TOWER OF FINITE COVERS WITH AMENABLE FUNDAMENTAL GROUPS

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Abstract. We prove an analogue of the Approximation Theorem of $L^2$-Betti numbers by Betti numbers for arbitrary coefficient fields and virtually torsion-free amenable groups. The limit of Betti numbers is identified as the dimension of some module over the Ore localization of the group ring.

0. Introduction

A residual chain of a group $G$ is a sequence $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ of normal subgroups of finite index such that $\bigcap_{i \geq 0} G_i = \{e\}$. The $n$-th $L^2$-Betti number of any finite free $G$-CW complex $X$ is the limit of the $n$-th Betti numbers of $G_i \setminus X$ normalized by the index $[G : G_i]$ for $i \to \infty$. If we instead consider Betti numbers $b_n(G_i \setminus X; k)$ with respect to a field of characteristic $p > 0$, the questions as to whether the limit exists, what it is, and whether it is independent of the residual chain are completely open for arbitrary residually finite $G$.

For $G = \mathbb{Z}^k$ and every field $k$ Elek showed that $\lim_{i \to \infty} b_n(G_i \setminus X; k)$ exists and expresses it in terms of the entropy of $G$-actions on the Pontrjagin duals of finitely generated $kG$-modules \[4\] – his techniques play an important role in this paper (see Section 1.3). It was observed in \[11\] Theorem 17 that the mere convergence of the right hand side of \[1\] in Theorem 0.2 for every amenable $G$ and every field $k$ follows from a general convergence principle for subadditive functions on amenable groups \[9\] and a theorem by Weiss \[18\].

The main purpose of this paper is to determine the limit $\lim_{i \to \infty} b_n(G_i \setminus X; k)$ in algebraic terms for a large class of amenable groups including virtual torsion-free elementary amenable groups. This makes the limit computable by homological techniques; see e.g., the spectral sequence argument of Example 6.3.

More precisely, the limit will be expressed in terms of the Ore dimension. The group ring $kG$ of a torsionfree amenable group satisfying the zero-divisor conjecture fulfills the Ore condition with respect to the subset $S = kG - \{0\}$ \[12\] Example 8.16 on page 324; we will review the Ore localization in Subsection 1.1.1. The Ore
localization \( S^{-1}kG \) is a skew field containing \( kG \). Therefore the following definition makes sense:

**Definition 0.1** (Ore dimension). Let \( G \) be a torsionfree amenable group such that \( kG \) contains no zero-divisors. The *Ore dimension* of a \( kG \)-module \( M \) is defined by

\[
\dim_{kG}^{\text{Ore}}(M) = \dim_{S^{-1}kG}(S^{-1}kG \otimes_{kG} M).
\]

The following theorem is our main result. We will prove a more general version, including virtually torsionfree groups, in Section 5.

**Theorem 0.2.** Let \( k \) be a field. Let \( G \) be a torsionfree amenable group for which \( kG \) has no zero-divisors\(^1\) Let \( (G_n)_{n \geq 0} \) be a residual chain of \( G \). Then:

1. Consider a finitely presented \( kG \)-module \( M \). Then

\[
\dim_{kG}^{\text{Ore}}(M) = \lim_{n \to \infty} \frac{\dim_k(k \otimes_{kG_n} M)}{[G : G_n]}.
\]

2. Consider a finite free \( kG \)-chain complex \( C_* \). Then we get for all \( i \geq 0 \)

\[
\dim_{kG}^{\text{Ore}}(H_i(C_*)) = \lim_{n \to \infty} \frac{\dim_k(H_i(k \otimes_{kG_n} C_*))}{[G : G_n]}.
\]

3. Let \( X \) be a finite free \( G \)-CW-complex. Then we get for all \( i \geq 0 \)

\[
\dim_{kG}^{\text{Ore}}(H_i(X)) = \lim_{n \to \infty} \frac{\dim_k(H_i(G_n \setminus X;k))}{[G : G_n]}.
\]

**Remark 0.3** (Fields of characteristic zero). Let \( G \) be a group with a residual chain \( (G_n)_{n \geq 0} \), and let \( M \) be a finitely presented \( kG \)-module. Then the Approximation Theorem for \( L^2 \)-Betti numbers says that

\[
\dim_{N(G)}(N(G) \otimes_{kG} M) = \lim_{n \to \infty} \frac{\dim_k(k \otimes_{kG_n} M)}{[G : G_n]},
\]

provided \( k \) is an algebraic number field. Here \( N(G) \) is the group von Neumann algebra and \( \dim_{N(G)} \) is the von Neumann dimension. See [11] for \( k = \mathbb{Q} \) and [3] for the general case.

Let \( k \) be a field of characteristic zero and let \( u = \sum g \in G x_g g \in kG \) be an element.

Let \( F \) be the finitely generated field extension of \( \mathbb{Q} \) given by \( F = \mathbb{Q}(x_g \mid g \in G) \subset k \).

Then \( u \) is already an element in \( FG \). The field \( F \) embeds into \( \mathbb{C} \); since \( F \) is finitely generated, it is a finite algebraic extension of a transcendental extension \( F' \) of \( \mathbb{Q} \).

(S. Theorem 1.1 on p. 356) and \( F' \) has finite transcendence degree over \( \mathbb{Q} \).

Since the transcendence degree of \( \mathbb{C} \) over \( \mathbb{Q} \) is infinite, there exists an embedding \( F' \hookrightarrow \mathbb{C} \) induced by an injection of a transcendence basis of \( F'/\mathbb{Q} \) into a transcendence basis \( \mathbb{C}/\mathbb{Q} \), which extends to \( F \hookrightarrow \mathbb{C} \) because \( \mathbb{C} \) is algebraically closed. This reduces the case of fields of characteristic zero to the case \( k = \mathbb{C} \). In [6] Elek proved (0.4) for amenable \( G \) and \( k = \mathbb{C} \). (see also [4]).

Moreover, if \( G \) is a torsionfree amenable group such that \( CG \) contains no zero-divisors and \( k \) is a field of characteristic zero, then

\[
\dim_{N(G)}(N(G) \otimes_{kG} M) = \dim_{kG}^{\text{Ore}}(M).
\]

\(^1\text{This assumption is satisfied if } G \text{ is torsionfree elementary amenable. See Remark [11].}\)
This follows from [12] Theorem 6.37 on page 259, Theorem 8.29 on page 330, Lemma 10.16 on page 376, and Lemma 10.39 on page 388. In particular, Theorem 0.2 follows for $k$ of characteristic zero. So the interesting new case is the one of a field of prime characteristic.

1. Review of Ore localization and Elek’s dimension function

1.1. Ore localization. We review the Ore localization of rings. For proofs and more information the reader is referred to [17]. Consider a torsionfree group $G$ and a field $k$. Let $S$ be the set of non-zero-divisors of $kG$. This is a multiplicatively closed subset of $kG$ and contains the unit element of $kG$. Suppose that $kG$ satisfies the Kaplansky Conjecture or zero-divisor conjecture, i.e., $S = kG - \{0\}$. Further assume that $S$ satisfies the left Ore condition, i.e., for $r \in kG$ and $s \in S$ there exists $r' \in kG$ and $s' \in S$ with $s'r = r's$. Then we can consider the Ore localization $S^{-1}kG$. Recall that every element in $S^{-1}kG$ is of the form $s^{-1} \cdot r$ for $r \in kG$ and $s \in S$ and that $s_0^{-1} \cdot r_0 = s_1^{-1} \cdot r_1$ holds if and only there exists $u_0, u_1 \in R$ satisfying $u_0 r_0 = u_1 r_1$ and $u_0 s_0 = u_1 s_1$. Addition is given on representatives by $s_0^{-1} r_0 + s_1^{-1} r_1 = t^{-1} (cs_0 r_0 + c_1 r_1)$ for $t = c_0 s_0 = c_1 s_1$. Multiplication is given on representatives by $s_0^{-1} r_0 \cdot s_1^{-1} r_1 = (ts_0)^{-1} cr_1$, where $cs_1 = tr_0$. The zero element is $e^{-1} \cdot 0$, and the unit element is $e^{-1} \cdot e$. The Ore localization $S^{-1}kG$ is a skew field, and the canonical map $kG \to S^{-1}kG$ sending $r$ to $e^{-1} \cdot r$ is injective. The functor $S^{-1}kG \otimes_{kG} -$ is exact.

Remark 1.1 (The Ore condition for group rings). If a torsionfree amenable group $G$ satisfies the Kaplansky Conjecture, i.e., $kG$ contains no zero-divisor, then for $S = kG - \{0\}$ the Ore localization $S^{-1}kG$ exists and is a skew field [12 Example 8.16 on page 324]. Every torsionfree elementary amenable group satisfies the assumptions above for all fields $k$ [7 Theorem 1.2; 10 Theorem 2.3]. If the group $G$ contains the free group of rank two as a subgroup, then the Ore condition is never satisfied for $kG$ [10 Proposition 2.2].

From the previous remark and the discussion above we obtain:

Theorem 1.2. Let $G$ be a torsionfree amenable group such that $kG$ contains no zero-divisors. Then the Ore dimension $\dim_{kG}^{Ore}$ has the following properties:

(i) $\dim_{kG}^{Ore}(kG) = 1$.

(ii) For any short exact sequence of $kG$-modules $0 \to M_0 \to M_1 \to M_2 \to 0$ we get

$$\dim_{kG}^{Ore}(M_1) = \dim_{kG}^{Ore}(M_0) + \dim_{kG}^{Ore}(M_2).$$

1.2. Crossed products, Goldie rings, and the generalized Ore localization. Throughout, let $G$ be a group and let $k$ be a skew field.

Let $R$ be a ring. The notion of a crossed product generalizes the one of a group ring. A crossed product $R \ast G = R \ast_{e, \tau} G$ is determined by maps $e : G \to \text{aut}(R)$ and $\tau : G \times G \to R^\times$ such that, roughly speaking, $e$ is a homomorphism up to the 2-cocycle $\tau$. We refer to the survey [12 10.3.2 on p. 398] for details. If $G$ is an extension of $H$ by $Q$, then the group ring $kG$ is isomorphic to a crossed product $kH \ast Q$. Some results in this paper are formulated for crossed products, although we only need the case of group rings for Theorem 0.2. So the reader may think of group rings most of the time. However, crossed products show up naturally, e.g., in proving that the virtual Ore dimension (5.1) is well defined.
We recall the following definition.

**Definition 1.3.** A ring $R$ is *left Goldie* if there exists $d \in \mathbb{N}$ such that every direct sum of non-zero left ideals of $R$ has at most $d$ summands and the left annihilators $a(x) = \{ r \in R \mid rx = 0 \}$, $x \in R$, satisfy the maximum condition for ascending chains. A ring $R$ is *prime* if for any two ideals $A, B$ in $R$, $AB = 0$ implies $A = 0$ or $B = 0$.

The subgroup of $G$ generated by its finite normal subgroups will be denoted by $\Delta^+(G)$. Then $\Delta^+(G)$ is also the set of elements of finite order which have only finitely many conjugates. We need the following three results:

**Lemma 1.4** ([15 Corollary 5 of Lecture 4]). If $\Delta^+(G) = 1$, then $k \ast G$ is prime.

**Theorem 1.5** ([13 Theorem 4.10 on p. 456]). The set of non-zero-divisors in a prime left Goldie ring satisfies the Ore condition. The Ore localization $S^{-1}R$ is isomorphic to $M_d(D)$ for some $d \in \mathbb{N}$ and some skew field $D$.

**Theorem 1.6.** If $G$ is amenable and $k \ast G$ is a domain, then $k \ast G$ is a prime left Goldie ring. If $G$ is an elementary amenable group such that the orders of the finite subgroups are bounded, then $k \ast G$ is left Goldie.

**Proof.** If $G$ is amenable and $k \ast G$ is a domain, then $k \ast G$ satisfies the Ore condition ([3 Theorem 6.3]). Thus its Ore localization with respect to $S = k \ast G - \{0\}$ is a skew field. By [13 Theorem 4.10 on p. 456] $k \ast G$ is a prime left Goldie ring. The second assertion is taken from [7 Proposition 4.2]. □

Next we extend the definition of Ore dimension to prime left Goldie rings. Let $R$ be such a ring. The functor $S^{-1}R \otimes_R -$ will still be exact ([17 Proposition II.1.4 on page 51]). If $M$ is a left $R$-module, then $S^{-1}R \otimes_R M$ will be a direct sum of $n$ irreducible $S^{-1}R$-modules for some non-negative integer $n$, and then the (generalized) *Ore dimension* of $M$ is defined as

$$\dim_{R,\text{Ore}}^R(M) = \frac{n}{d}.$$ 

Since $S^{-1}R \cong M_d(D)$ (Theorem [14] and $M_d(D)$ decomposes into $d$ copies of the irreducible module $D^d$, we have $\dim_{R,\text{Ore}}^R(R) = 1$.

1.3. **Elek’s dimension function.** Throughout this subsection let $G$ be a finitely generated amenable group. We review Elek’s definition [5] of a dimension function $\dim_{kG}^\text{Elek}$ for finitely generated $kG$-modules.

Fix a finite set of generators and equip $G$ with the associated word metric $d_G$. A *Følner sequence* $(F_n)_{n \geq 0}$ is a sequence of finite subsets of $G$ such that for any fixed $R > 0$ we have

$$\lim_{n \to \infty} \frac{\partial_R F_n}{|F_n|} = 0,$$

where $\partial_R F_n = \{ g \in G \mid d(g, F_k) \leq R \text{ and } d(g, G \setminus F_k) \leq R \}$.

Let $k$ be an arbitrary skew field endowed with the discrete topology and let $\mathbb{N}$ denote the positive integers $\{1, 2, \ldots\}$. Let $n \in \mathbb{N}$. We equip the space of functions $\text{map}(G, k^n) = \prod_{g \in G} k^n$ with the product topology, which is the same as the topology of pointwise convergence. The natural right $G$-action on $\text{map}(G, k^n)$ is defined by

$$(\phi g)(x) = \phi(xg^{-1}) \text{ for } g, x \in G, \phi \in \text{map}(G, k^n).$$
Also $\text{map}(G, k^n)$ is a right $k$-vector space by defining $(\phi k)(x) = \phi(x)k$. For any subset $S \subset G$ and any subset $W \subset \text{map}(G, k^n)$, let

$$W|_S = \{ f : S \to k^n \mid \exists g \in W \text{ with } g|_S = f \}.$$

A right $k$-linear subspace $V \subset \text{map}(G, k^n)$ is called invariant if $V$ is closed and invariant under the right $G$-action.

Elek defines the average dimension $\dim_G^A(V)$ of an invariant subspace $V$ by choosing a Følner sequence $(F_n)_{n \in \mathbb{N}}$ of $G$ and setting

\begin{equation}
\dim_G^A(V) = \limsup_{n \to \infty} \frac{\dim_k(V|_{F_n})}{|F_n|}.
\end{equation}

**Theorem 1.8** ([5] Prop. 7.2 and Prop. 9.2). The sequence in (1.7) converges, and its limit $\dim_G^A(V)$ is independent of the choice of the Følner sequence.

**Remark 1.9.** Elek actually defines $\dim_G^A(V)$ using Følner exhaustions, i.e. increasing Følner sequences $(F_n)_{n \in \mathbb{N}}$ with $\bigcup_{n \in \mathbb{N}} F_n = G$. This makes no difference since the existence of the limit of $(\dim_k(V|_{F_n})/|F_n|)_{n \in \mathbb{N}}$ for arbitrary Følner sequences (and thus its independence of the choice) follows from [9 Theorem 6.1].

Let $M$ be a finitely generated left $kG$-module. The $k$-dual $M^* = \text{hom}_k(M, k)$ (where $M$ and $k$ are viewed as left $k$-modules, and $(\phi m) = \phi(am)$ for $\phi \in M^*$, $a \in k$ and $m \in M$) carries the natural right $G$-action $(\phi g)(m) = \phi(gm)$. The dual of the free left $kG$-module $kG^n$ is canonically isomorphic to $\text{map}(G, k^n)$. Any left $kG$-surjection $f : kG^n \twoheadrightarrow M$ induces a right $kG$-injection $f^* : M^* \rightarrow \text{map}(G, k^n)$ such that $\text{im}(f^*)$ is a $G$-invariant $k$-subspace.

**Definition 1.10** (Elek’s dimension function). Let $M$ be a finitely generated left $kG$-module. Its **dimension in the sense of Elek** is defined by choosing a left $kG$-surjection $f : kG^n \twoheadrightarrow M$ and setting

\begin{equation}
\dim_{kG}^{\text{Elek}}(M) = \dim_G^A(\text{im}(f^*)).
\end{equation}

**Theorem 1.12** (Main properties of Elek’s dimension function). Let $G$ be a finitely generated amenable group. The definition (1.11) of $\dim_{kG}^{\text{Elek}}(M)$ is independent of the choice of the surjection $f$, and $\dim_{kG}^{\text{Elek}}$ has the following properties:

(i) $\dim_{kG}^{\text{Elek}}(kG) = 1$.

(ii) For any short exact sequence of finitely generated $kG$-modules $0 \to M_0 \to M_1 \to M_2 \to 0$ we get

$$\dim_{kG}^{\text{Elek}}(M_1) = \dim_{kG}^{\text{Elek}}(M_0) + \dim_{kG}^{\text{Elek}}(M_2).$$

(iii) If the finitely generated $kG$-module $M$ satisfies $\dim_k^{\text{Elek}}(M) = 0$, then every quotient module $Q$ of $M$ satisfies $\dim_{kG}^{\text{Elek}}(Q) = 0$.

**Proof.** The first two assertions are proved in [5 Theorem 1]. Notice that the third condition does not necessarily follow from additivity since the kernel of the epimorphism $M \to Q$ may not be finitely generated. But the third statement is a direct consequence of the definition of Elek’s dimension.

**Remark 1.13** (The dual of finitely generated $kG$-modules). Identify the left $kG$-module $kG^n$ with the finitely supported functions in $\text{map}(G, k^n)$. Here we view...
map\((G,k^n)\) as a left \(k\)-vector space by \((af)(g) = af(g)\), and the left \(G\)-action is given by \((hf)(g) = f(h^{-1}g)\) for \(h, g \in G\) and \(a \in k\). Let
\[
(\langle \cdot , \cdot \rangle \colon kG^n \times \text{map}(G,k^n))
\]
be the canonical pairing (evaluation) of \(kG^n\) and its dual \(\text{map}(G,k^n)\). If we view an element \(f \in kG^n\) as a finitely supported function \(G \to k^n\) (in \(\text{map}(G,k^n)\)), then the pairing of \(f \in kG^n\) with \(l \in \text{map}(G,k^n)\) is given by
\[
\langle f,l \rangle = \sum_{g \in G} (f(g), l(g)),
\]
where \(\langle \cdot , \cdot \rangle\) denotes the standard inner product in \(k^n\). For a subset \(W \subset kG^n\) let
\[
W^\perp = \{ f \in \text{map}(G,k^n) \mid \langle x,f \rangle = 0 \ \forall x \in W \}.
\]
If \(M\) is a finitely generated \(kG\)-module and \(f : kG^n \to M\) is a left \(kG\)-surjection, then \(f^* : M^\ast \hookrightarrow \text{map}(G,k^n)\) is a right \(kG\)-injection and
\[
\text{im}(f^*) = \ker(f)^\perp \subseteq \text{map}(G,k^n).
\]

2. Approximation for finitely presented \(kG\)-modules for Elek’s dimension function

The main result of this section is

**Theorem 2.1.** Let \(G\) be a finitely generated amenable group. Consider a sequence of normal subgroups of finite index
\[
G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots
\]
such that \(\bigcap_{n \geq 0} G_n = \{1\}\). Then every finitely presented \(kG\)-module \(M\) satisfies
\[
\dim_{kG}^\text{Elek}(M) = \lim_{n \to \infty} \frac{\dim_k(k \otimes_{kG_n} M)}{[G : G_n]}.
\]

Its proof needs some preparation.

Throughout, let \(G\) be a finitely generated amenable group. For any subset \(S \subset G\) let \(k[S]\) be the \(k\)-subspace of \(kG\) generated by \(S \subset kG\). Let \(j[S] : k[S] \to k[G]\) be the inclusion and \(\text{pr}[S] : kG \to k[S]\) be the projection given by
\[
\text{pr}[S](g) = \begin{cases} 
g & \text{if } g \in S, \\
0 & \text{if } g \in G \setminus S. \end{cases}
\]

**Theorem 2.2.** Let \(G\) be a finitely generated amenable group. Let \(M\) be a finitely presented left \(kG\)-module \(M\) with a presentation \(kG^r \xrightarrow{f} kG^s \xrightarrow{\partial} M \to 0\). For every subset \(S \subset G\) we define
\[
M[S] = \text{coker}\left(\text{pr}[S] \circ f \circ j[S] : k[S]^r \to k[S]^s\right).
\]
Let \((F_n)_{n \geq 0}\) be a Følner sequence of \(G\). Then
\[
\dim_{kG}^\text{Elek}(M) = \lim_{n \to \infty} \frac{\dim_k(M[F_n])}{|F_n|}.
\]

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Proof: The map $f$ is given by right multiplication with a matrix $A \in M_{r,s}(kG)$. Viewing $A$ as a map $G \to k^{r \times s}$ it is clear what we mean by the support $\text{supp}(A)$ of $A$. Let $R > 0$ be the diameter of $\text{supp}(A) \cup \text{supp}(A)^{-1}$. Since

$$\lim_{n \to \infty} \frac{|\partial R F_n|}{|F_n|} = 0,$$

it is enough to show that for every $n \geq 1$

$$|\dim_k(M[F_n]) - \dim_k(\text{im}(p^*))[F_n]| \leq s \cdot |\partial R F_n|.

(2.3)

For the definition of inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle$ we refer to Remark 1.13. Define the following $k$-linear subspaces of $\text{map}(F_n, k^s)$:

$$W_n = \{ \phi : F_n \to k^s \mid \langle \text{pr}_n \circ f \circ j_n(x), \phi \rangle = 0 \ \forall x \in k[F_n]^r \} ,

V_n = \{ \phi : F_n \to k^s \mid \exists \tilde{\phi} : G \to k^s \text{ satisfying } \tilde{\phi}|F_n = \phi, \langle f(y), \tilde{\phi} \rangle = 0 \ \forall y \in kG^r \} ,

Z_n = \{ \phi : F_n \to k^s \mid \phi|_{\partial R F_n} = 0 \} .

Since $\dim_k(M[F_n]) = \dim_k(W_n)$ and $\dim_k(\text{im}(p^*)|_{F_n}) = \dim_k(V_n)$, the desired estimate is equivalent to

$$|\dim_k(W_n) - \dim_k(V_n)| \leq s \cdot |\partial R F_n|.

(2.4)

By additivity of $\dim_k$ we obtain that

$$\dim_k(W_n \cap Z_n) \geq \dim_k(W_n) - \dim_k(\text{map}(F_n, k^s)) + \dim_k(Z_n)

\geq \dim_k(W_n) - s \cdot |F_n| + s \cdot (|F_n| - |\partial R F_n|)

= \dim_k(W_n) - s \cdot |\partial R F_n|.

Similarly, we get

$$\dim_k(V_n \cap Z_n) \geq \dim_k(V_n) - s \cdot |\partial R F_n|.

To prove (2.4) it hence suffices to show that

$$W_n \cap Z_n \subset V_n ,

(2.5)

V_n \cap Z_n \subset W_n .

(2.6)

Let $\phi \in W_n \cap Z_n$. Extend $\phi$ by zero to a function $\tilde{\phi} : G \to k^s$. Let $y \in kG^r$. Then we can decompose $y$ as $y = y_0 + y_1$ with $\text{supp}(y_0) \subset F_n$ and $\text{supp}(y_1) \subset G \setminus F_n$. By definition of the radius $R$ it is clear that $\text{supp}(f(y_1)) \subset G \setminus F_n \cup \partial R F_n$. Because of $\phi \in Z_n$ we have $\langle f(y_1), \tilde{\phi} \rangle = 0$. The fact that $\phi \in W_n$ implies that

$$\langle f(y_0), \tilde{\phi} \rangle = \langle \text{pr}_n \circ f \circ j_n(y_0), \phi \rangle = 0 .

So we obtain that $\langle f(y), \tilde{\phi} \rangle = 0$, meaning that $\phi \in V_n$. The proof of (2.6) is similar.

The following theorem is due to Weiss. Its proof can be found in [2, Proposition 5.5].

**Theorem 2.7 (Weiss).** Let $G$ be a countable amenable group. Let $G_n \subset G, n \geq 1$, be a sequence of normal subgroups of finite index with $\bigcap_{n \geq 1} G_n = \{1\}$. Then there exists, for every $R \geq 1$ and every $\epsilon > 0$, an integer $M = M(R, \epsilon) \geq 1$ such that for $n \geq M$ there is a fundamental domain $Q_n \subset G$ of the coset space $G/G_n$ such that

$$\frac{|\partial R Q_n|}{|Q_n|} < \epsilon .$$
Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. According to Theorem 2.7 let \((Q_n)_{n \geq 0}\) be a Følner sequence of \(G\) such that \(Q_n \subset G\) is a fundamental domain for \(G/G_n\). Choose a finite presentation of \(M\):

\[
kG / f \rightarrow kG^n \rightarrow M \rightarrow 0.
\]

Let \(f_n = k[G/G_n] \otimes_{kG} f\). By right-exactness of tensor products we have the exact sequence

\[
k[G/G_n] \rightarrow k[G/G_n]^* \rightarrow k[G/G_n] \otimes_{kG} M \rightarrow 0.
\]

The natural map \(Q_n \subset G \rightarrow G/G_n\) induces an isomorphism \(j_n : k[Q_n] \rightarrow k[G/G_n]\) of \(k\)-vector spaces. The map \(f\) is given by right multiplication \(f = R_A\) with a matrix \(A \in M_{r\times s}(kG)\). Viewing \(A\) as a map \(G \rightarrow k^{r\times s}\), let \(\text{supp}(A)\) be the support of \(A\). Let \(R > 0\) be the diameter of \(\text{supp}(A) \cup \text{supp}(A)^{-1}\) (with respect to the fixed word metric on \(G\)). Then \(f\) restricts to a map

\[
f|_{Q_n \setminus \partial_R Q_n} : k[Q_n \setminus \partial_R Q_n]^* \rightarrow k[Q_n]^*.
\]

Hence there is precisely one \(k\)-linear map \(g\) for which the following diagram of \(k\)-vector spaces commutes:

\[
\begin{array}{ccc}
k[G/G_n]^* & \xrightarrow{f_n} & k[G/G_n]^* \\
\uparrow{j_n|_{Q_n \setminus \partial_R Q_n}} & & \uparrow{j_n|_{Q_n \setminus \partial_R Q_n}} \\
k[Q_n \setminus \partial_R Q_n]^* & \xrightarrow{f|_{Q_n \setminus \partial_R Q_n}} & k[Q_n]^*
\end{array}
\]

One easily verifies that \(g\) is surjective and that

\[
\ker(g) \subset \text{im}(pr \circ j_n^{-1} \circ f_n : k[G/G_n]^* \rightarrow \text{coker}(f|_{Q_n \setminus \partial_R Q_n})).
\]

The map \(pr \circ j_n^{-1} \circ f_n\) descends to a map

\[
pr \circ j_n^{-1} \circ f_n : \text{coker}(j_n|_{Q_n \setminus \partial_R Q_n}) \rightarrow \text{coker}(f|_{Q_n \setminus \partial_R Q_n}).
\]

Note that

\[
\dim_k(\text{coker}(j_n|_{Q_n \setminus \partial_R Q_n})) = r \cdot |\partial_R Q_n|.
\]

Thus,

\[
\dim_k(\text{coker}(f|_{Q_n \setminus \partial_R Q_n})) - \dim_k(k[G/G_n] \otimes_{kG} M) = \dim_k(\ker(g)) \leq r \cdot |\partial_R Q_n|.
\]

By replacing the upper row in diagram (2.8) by

\[
k[Q_n]^* \xrightarrow{pr|_{Q_n} \circ f|_{Q_n}} k[Q_n]^* \rightarrow M[Q_n] \rightarrow 0
\]

and essentially running the same argument as before, we obtain that

\[
\dim_k(\text{coker}(f|_{Q_n \setminus \partial_R Q_n})) - \dim_k(\text{coker}(M[Q_n])) \leq r \cdot |\partial_R Q_n|.
\]

Since

\[
\frac{|\partial_R Q_n|}{|G/G_n|} = \frac{|\partial_R Q_n|}{|Q_n|} \xrightarrow{n \to \infty} 0
\]

we get that

\[
\lim_{n \to \infty} \frac{\dim_k(k[G/G_n] \otimes_{kG} M)}{|G : G_n|} = 0.
\]
exists if and only if
\[ \lim_{n \to \infty} \frac{\dim_k(M[Q_n])}{|Q_n|} \]
e exists, and in this case they are equal. Now the assertion follows from Theorem 2.2. \hfill \Box

3. Comparing dimensions

The main result of this section is

**Theorem 3.1** (Comparing dimensions). Let \( G \) be a group, let \( k \) be a skew field, and let \( k \ast G \) be a crossed product which is prime left Goldie. Let \( \dim \) be any dimension function which assigns to a finitely generated left \( k \ast G \)-module a non-negative real number and satisfies

(i) \( \dim(k \ast G) = 1 \).

(ii) For every short exact sequence \( 0 \to M_0 \to M_1 \to M_2 \to 0 \) of finitely generated left \( k \ast G \)-modules, we get
\[ \dim(M_1) = \dim(M_0) + \dim(M_2). \]

(iii) If the finitely generated left \( k \ast G \)-module \( M \) satisfies \( \dim(M) = 0 \), then every quotient module \( Q \) of \( M \) satisfies \( \dim(Q) = 0 \).

Then for every finitely presented left \( k \ast G \)-module \( M \), we get
\[ \dim(M) = \dim_{k \ast G}(M). \]

**Proof.** Let \( S \) denote the non-zero-divisors of \( k \ast G \). We have to show that for all \( r, s \in \mathbb{N} \) and every \( r \times s \) matrix \( A \) with entries in \( k \ast G \)
\[ \dim_{k \ast G}(\ker (r_A: S^{-1}k \ast G^r \to S^{-1}k \ast G^s)) = \dim(\ker (r_A: k \ast G^r \to k \ast G^s)), \]
where \( r_A \) denotes the module homomorphism given by right multiplication with \( A \).

First note that we may assume that \( r = s \). Indeed if \( r < s \), replace \( A \) with the \( s \times s \) matrix which is \( A \) for the first \( r \) rows and has \( 0 \)'s on the bottom \( s-r \) rows. On the other hand if \( r > s \), replace \( A \) with the \( r \times r \) matrix \( B \) with entries \( (b_{ij}) \) which is \( A \) for the first \( s \) columns and has \( b_{ij} = \delta_{ij} \) if \( i > s \), where \( \delta_{ij} \) is the Kronecker delta.

We will often use the obvious long exact sequence associated to homomorphisms \( f: M_0 \to M_1 \) and \( g: M_1 \to M_2 \):
\[ 0 \to \ker(f) \to \ker(g \circ f) \to \ker(g) \to \ker(f) \to \ker(g \circ f) \to \ker(g) \to 0. \]

We now assume that \( A \) is an \( r \times r \) matrix. Note that equation (3.2) is true if \( A \) is invertible over \( S^{-1}k \ast G \); this is because then \( \ker r_A = 0 \) (whether \( A \) is considered as a matrix over \( k \ast G \) or \( S^{-1}k \ast G \)).

Next observe that if \( U \in M_r(k \ast G) \) which is invertible over \( M_r(S^{-1}k \ast G) \), then equation (3.2) holds for \( A \) if and only if it holds for \( AU \), and also if and only if it holds for \( UA \). This follows from (3.3), \( \ker U = 0 \), \( \dim(\ker U) = \dim_{k \ast G}(\ker U) = 0 \), and in the second case we use the third property of \( \dim \).

We may write \( S^{-1}k \ast G = M_d(D) \) for some \( d \in \mathbb{N} \) and some skew field \( D \). By applying the Morita equivalence from \( M_d(D) \) to \( D \) and back and by doing Gaussian elimination over \( D \) we see that there are invertible matrices \( U, V \in M_{rd}(S^{-1}k \ast G) \) such that \( U \text{ diag}(A, \ldots, A)V = J \), where there are \( d \) \( A \)'s and \( J \) is a matrix of the
form $\text{diag}(1, \ldots, 1, 0, \ldots, 0)$. Now choose $u, v \in S$ such that $uU, vV \in M_{r,q}(k \ast G)$. Then $(uU) \text{diag}(A, \ldots, A)(Vv) = uJv$, and the result follows. 

**Theorem 3.4** (Comparing Elek’s dimension and the Ore dimension). Let $G$ be a finitely generated group and let $k$ be a skew field. Suppose that $kG$ is a prime left Goldie ring. Then for any finitely presented left $kG$-module $M$,

$$\dim_{kG}^{\text{Elek}}(M) = \dim_{kG}^{\text{Ore}}(M).$$

**Proof.** This follows from Theorem 3.1 and Theorem 1.12.

4. **Proof of the main theorem**

**Proof of Theorem 4.2.** In the first step we reduce the claim to the case where $G$ is finitely generated. Consider a finitely presented left $kG$-module $M$. Choose a matrix $A \in M_{r,s}(kG)$ such that $M$ is isomorphic to the cokernel of $r_A : kG^r \to kG^s$. Since $A$ is a finite matrix and each element in $kG$ has finite support, we can find a finitely generated subgroup $H \subseteq G$ such that $A \in M_{r,s}(kH)$. Both $kG$ and $kH$ are prime left Goldie by Lemma 1.4 and Theorem 1.6. Consider the finitely presented $kH$-module $N := \text{coker}(r_A : kH^r \to kH^s)$. Then $M = kG \otimes_{kH} N$. We can also consider the Ore localization $T^{-1}kH$ for $T$ the set of non-zero-divisors of $kH$. Put $H_n = H \cap G_n$. We obtain a residual chain $(H_n)_{n \geq 0}$ of $H$ and have

$$\dim_{kG}^{\text{Ore}}(M) = \dim_{S^{-1}kG}(S^{-1}kG \otimes_{kG} M) = \dim_{S^{-1}kG}(S^{-1}kG \otimes_{kG} kG \otimes_{kH} N) = \dim_{T^{-1}kH}(T^{-1}kH \otimes_{kH} N) = \dim_{kH}^{\text{Ore}}(N).$$

We compute

$$\dim_{k}\left( k \otimes_{kG_n} M \right)_{[G : G_n]} = \frac{\dim_{k}\left( k[G/G_n] \otimes_{kG} M \right)}{[G : G_n]} = \frac{\dim_{k}\left( k[G/G_n] \otimes_{kG} kG \otimes_{kH} N \right)}{[G : G_n]} = \frac{\dim_{k}\left( k[G/G_n] \otimes_{k[H/H_n]} k[H/H_n] \otimes_{kH} N \right)}{[G : G_n]} = \frac{\dim_{k}\left( k[G/G_n : H/H_n] \cdot \dim_{k}\left( k[H/H_n] \otimes_{kH} N \right) \right)}{[G : G_n]} = \frac{\dim_{k}\left( k \otimes_{kH_n} N \right)}{[H : H_n]}.$$

Therefore the claim holds for $M$ over $kG$ if it holds for $N$ over $kH$. Hence we can assume without loss of generality that $G$ is finitely generated.

Now apply Theorem 2.1 and Theorem 3.3.
We obtain from additivity, the exactness of the functor \( S^{-1}kG \otimes_{kG} \), and the right exactness of the functor \( k \otimes_{kG} \), that

\[
\dim_{kG}^{\text{Ore}}(H_i(C_\ast)) = \dim_{kG}^{\text{Ore}}(\text{coker}(c_{i+1})) + \dim_{kG}^{\text{Ore}}(\text{coker}(c_i)) - \dim_{kG}^{\text{Ore}}(C_{i-1}),
\]

\[
\dim_k(H_i(k \otimes_{kG_n} C_\ast)) = \dim_k(k \otimes_{kG_n} \text{coker}(c_{i+1})) + \dim_k(k \otimes_{kG_n} \text{coker}(c_i)) - \dim_k(k \otimes_{kG_n} C_{i-1}).
\]

Hence the claim follows from assertion [i] applied to the finitely presented \( kG \)-modules \( \text{coker}(c_{i+1}), \text{coker}(c_i) \) and \( C_{i-1} \).

Let \( \Delta \) be a crossed product such that \( \dim_{kG}^{\text{Ore}}(\text{coker}(c_{i+1})) = \dim_{kG}^{\text{Ore}}(\text{coker}(c_i)) - \dim_{kG}^{\text{Ore}}(C_{i-1}) \), and write \( \text{res}^{kH}_{kG}(M) \) for the \( kG \)-module obtained from the \( kG \)-module \( M \) by restricting the \( G \)-action to \( H \).

(5.1) \[
\text{vdim}_{kG}^{\text{Ore}}(M) = \frac{\dim_{kG}^{\text{Ore}}(\text{res}^{kH}_{kG} M)}{[G : H]},
\]

where \( \text{res}^{kH}_{kG} M \) is the \( kH \)-module obtained from the \( kG \)-module \( M \) by restricting the \( G \)-action to \( H \).

We have to show that this is independent of the choice of \( H \). Since every subgroup of finite index contains a normal subgroup of finite index, it is enough to show that if \( K \) is a normal subgroup of finite index in \( H \) and \( K \leq H \leq G \) with \( H \) torsion free, then for every \( kH \)-module \( N \),

(5.2) \[
\frac{\dim_{kG}^{\text{Ore}}(\text{res}^{kK}_{kH} N)}{[H : K]} = \dim_{kH}^{\text{Ore}}(N).
\]

Let \( T \) denote the set of non-zero-divisors of \( k * H \) and write \( S = (k * K) \cap T \). Note that \( \Delta^+(K) = 1 \), so \( k * K \) is still a prime left Goldie ring and hence the ring \( S^{-1}k * K \) exists. Then \( S^{-1}k * H \cong (S^{-1}k * K) \ast [H/K] \) and there is a natural ring monomorphism \( \theta : S^{-1}k * H \hookrightarrow T^{-1}k * H \). Since \( S^{-1}k * K \) is a matrix ring over a skew field by Theorem [i], we see that \( (S^{-1}k * K)[H/K] \) is an Artinian ring because \( H/K \) is finite. But every element of \( T \) is a non-zero-divisor in \( (S^{-1}k * K)[H/K] \), and since every non-zero-divisor in an Artinian ring is invertible (compare [ii] Exercise 22 of Chapter 15 on p. 16), we see that every element of \( T \) is invertible in \( S^{-1}k * H \) and we conclude that \( \theta \) is onto and hence is an isomorphism. We deduce that \( \dim_{S^{-1}k * K}(T^{-1}k * H) = [H : K] \) and that the natural map \( S^{-1}N \to T^{-1}N \) induced by \( s^{-1}n \mapsto s^{-1}n \) is an isomorphism. This proves (5.2).

**Theorem 5.3** (Extension to the virtually torsionfree case). Let \( G \) be an amenable group which possesses a subgroup \( E \subseteq G \) of finite index such that \( kE \) is left Goldie and \( \Delta^+(E) = 1 \), and let \( k \) be a skew field. Then assertions [i] [ii] and [iii] of Theorem [12] remain true, provided we replace \( \dim_{kG}^{\text{Ore}} \) by \( \text{vdim}_{kG}^{\text{Ore}} \) everywhere.
Proof. It suffices to prove the claim for assertion (i) since the proof in Theorem 0.2 that it implies the other two assertions applies also in this more general situation. Let \((G_n)_{n \geq 0}\) be a residual chain of \(G\). To prove the result in general, we may assume that \(G\) is finitely generated. Since \(kE\) is left Goldie and \([G : E] < \infty\), the group ring \(kG\) is also left Goldie. Further, every \(kG_n\) is left Goldie. Since \(\Delta^+(G)\) is finite (its order is bounded by \([G : E]\)), there exists \(N \in \mathbb{N}\) such that \(G_N \cap \Delta^+(G) = 1\), and then \(\Delta^+(G_N) = 1\) and \(G_i \subseteq G_N\) for all \(i \geq N\). Set \(H = G_N\) so that \(kH\) is prime by Lemma \[1.4\]. Then for a finitely presented \(kH\)-module \(L\),
\[
\dim_{kH}^\text{Ore}(L) = \lim_{n \to \infty} \dim_k \left( k \otimes_{k[G \cap H]} L \right) / [H : H \cap G_n]
\]
by Theorems \[2.2\] and \[3.1\]. We have \([G : G_n] = [G : H] \cdot [H : H \cap G_n]\) for \(n \geq N\). This implies for every finitely presented \(kG\)-module \(M\),
\[
\mathrm{vdim}_{kG}^\text{Ore}(M) = \frac{\dim_{kH}^\text{Ore}(\mathrm{res}_{kG}^k H M)}{[G : H]} = \lim_{n \to \infty} \frac{\dim_k \left( k \otimes_{k[G \cap H]} \mathrm{res}_{kG}^k H M \right)}{[G : H] \cdot [H : H \cap G_n]} = \lim_{n \to \infty} \frac{\dim_k \left( k \otimes_{kG} M \right)}{[G : G_n]},
\]
\(\square\)

Remark 5.4. Because of Theorem \[1.3\], Theorem 0.2 is true in the case where \(k\) is a skew field and \(G\) is an elementary amenable group in which the orders of the finite subgroups are bounded (clearly \(\Delta^+(G_n) = 1\) for sufficiently large \(n\)). In particular Theorem 0.2 is true for any virtually torsionfree elementary amenable group.

6. Examples

Remark 6.1. Let \((G_n)_{n \geq 0}\) be a residual chain of a group \(G\). Let \(X\) be a finite free \(G\)-CW-complex. Let \(k\) be a field of characteristic \(\text{char}(k)\). For a prime \(p\) denote by \( \mathbb{F}_p \) the field of \(p\) elements. Then we conclude from the universal coefficient theorem that
\[
\dim_k (H_i(G_n \setminus X; k)) = \dim_{\mathbb{Q}} (H_i(G_n \setminus X; \mathbb{Q})) \quad \text{char}(k) = 0,
\]
\[
\dim_k (H_i(G_n \setminus X; k)) = \dim_{\mathbb{F}_p} (H_i(G_n \setminus X; \mathbb{F}_p)) \quad p = \text{char}(k) \neq 0,
\]
\[
\dim_{\mathbb{F}_p} (H_i(G_n \setminus X; \mathbb{F}_p)) \geq \dim_{\mathbb{Q}} (H_i(G_n \setminus X; \mathbb{Q})).
\]
In particular we conclude from Remark \[1.3\] that
\[
\lim_{n \to \infty} \frac{\dim_k (H_i(G_n \setminus X; k))}{[G : G_n]} \geq \lim_{n \to \infty} \frac{\dim_k (H_i(G_n \setminus X; \mathbb{Q}))}{[G : G_n]} = b_i^{(2)}(X; N(G)),
\]
where the latter term denotes the \(i\)-th \(L^2\)-Betti number of \(X\). In particular we get from Theorem 0.2 for a torsionfree amenable group \(G\) with no zero-divisors in \(kG\) that
\[
\dim_{kG}^\text{Ore} (H_i(X; k)) = \lim_{n \to \infty} \frac{\dim_k (H_i(G_n \setminus X; k))}{[G : G_n]} \geq \lim_{n \to \infty} \frac{\dim_k (H_i(G_n \setminus X; \mathbb{Q}))}{[G : G_n]} = b_i^{(2)}(X; N(G)) = \dim_{kG}^\text{Ore} (H_i(X; \mathbb{C})).
\]
This inequality is in general not an equality, as the next example shows.
Example 6.2. Fix an integer \( d \geq 2 \) and a prime number \( p \). Let \( f_p: S^d \to S^d \) be a map of degree \( p \) and denote by \( i: S^d \to S^1 \vee S^d \) the obvious inclusion. Let \( X \) be the finite CW-complex obtained from \( S^1 \vee S^d \) by attaching a \((d + 1)\)-cell with an attaching map \( i \circ f^d: S^d \to S^1 \vee S^d \). Then \( \pi_1(X) = \mathbb{Z} \). Let \( \tilde{X} \) be the universal covering of \( X \) which is a finite free \( \mathbb{Z}\)-CW-complex. Denote by \( X_n \) the covering of \( X \) associated to \( n \cdot \mathbb{Z} \subseteq \mathbb{Z} \). The cellular \( \mathbb{Z}C \)-chain complex of \( \tilde{X} \) is concentrated in dimension \((d + 1), d \) and 1 and 0, the \((d + 1)\)-th differential is multiplication with \( p \) and the first differential is multiplication with \((z - 1)\) for a generator \( z \in \mathbb{Z} \):

\[
0 \to \cdots \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{p} \mathbb{Z}[\mathbb{Z}] \to \cdots \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{z-1} \mathbb{Z}[\mathbb{Z}].
\]

If the characteristic of \( k \) is different from \( p \), one easily checks that \( H_i(C_*) = 0 \)

\[
dim_{k^\text{Ore}}(H_i(\tilde{X}; k)) = 0 \quad \text{for } i \in \{d, d + 1\}.
\]

If \( p \) is the characteristic of \( k \), then \( H_i(C_*) = k\mathbb{Z} \) and

\[
dim_{k^\text{Ore}}(H_i(\tilde{X}; k)) = 1 \quad \text{for } i \in \{d, d + 1\}.
\]

Hence \( \dim_{k^\text{G}}(H_i(\tilde{X}; k)) \) does depend on \( k \) in general.

Example 6.3. Let \( G \) be a torsionfree amenable group such that \( kG \) has no zero-divisors. Let \( S^1 \to X \to B \) be a fibration of connected CW-complexes such that \( X \) has fundamental group \( \pi_1(X) \cong G \) and \( \pi_1(S^1) \to \pi_1(X) \) is injective. Then

\[
\dim_{k^\text{G}}(H_i(\tilde{X}; k)) = 0
\]

for every \( i \geq 0 \).

Let \( S = kG - \{0\} \) and \( S_0 = k\mathbb{Z} - \{0\} \). By looking at the cellular chain complex one directly sees that

\[
H_i(S^1, S_0^{-1}k\mathbb{Z}) = 0 \quad \forall i \geq 0.
\]

Thus \( H_i(S^1, S^{-1}kG) = S^{-1}kG \otimes_{S_0^{-1}k\mathbb{Z}} H_i(S^1, S_0^{-1}k\mathbb{Z}) = 0 \) for every \( i \geq 0 \). The assertion is implied by the Hochschild-Serre spectral sequence that converges to \( H_{p+q}(\tilde{X}, S^{-1}kG) \) and has the \( E^2 \)-term:

\[
E^2_{pq} = H_p(\tilde{B}, H_q(S^1, S^{-1}kG)).
\]

Example 6.5 (Sublinear growth of Betti numbers). Let \( G \) be an infinite amenable group which possesses a subgroup \( H \) of finite index such that \( kH \) is left Goldie and \( \Delta^+(H) = 1 \), e.g., \( G \) is a virtually torsionfree elementary amenable group. Let \( k \) be a field. Let \( (G_n)_{n \geq 0} \) be a residual chain of \( G \). Denote by \( b_i(G/G_n; K) \) the \( i \)-th Betti number of the group \( G/G_n \) with coefficients in \( k \). Then we get for every \( i \geq 0 \)

\[
\lim_{n \to \infty} \frac{b_i(G/G_n; k)}{|G : G_n|} = 0.
\]

For \( i = 0 \) this is obvious. For \( i \geq 1 \) this follows from Theorem 5.3 and \( H_i(EH; k) = H_i(H; k) = 0 \).

References


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