REMARKS ON THE AREA THEOREM
IN THE THEORY OF UNIVALENT FUNCTIONS

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Abstract. We prove an integral extension of the classical area theorem for
univalent functions. We give an application finding geometric conditions on
the image domain of a univalent function $f$ which imply that $f$ belongs to the
Hardy space $H^p$, $0 < p < \infty$.

1. Introduction and main results

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$. A complex-valued
function defined in $\mathbb{D}$ is said to be univalent if it is analytic and one-to-one there.
We refer to [5] and [8] for the theory of these functions. Throughout the paper, $U$
will stand for the class of all univalent functions in $\mathbb{D}$. The classical area theorem
[5, p. 29], which is a key in the proof of a good number of results in the theory of
univalent functions, can be stated as follows:

**Theorem 1.** If $\varphi(z) = \sum_{n=0}^{\infty} b_n z^n$, $b_0 \neq 0$, is an analytic function in $\mathbb{D}$ such that
the meromorphic function $\varphi(z)/z$ is one-to-one in $\mathbb{D}$, then

$$\sum_{n=1}^{\infty} n^2 |b_n|^2 \leq \sum_{n=0}^{\infty} |b_n|^2 \frac{n}{n+1}$$

or, equivalently,

$$\int_{\mathbb{D}} |z\varphi'(z)|^2 \, dm(z) \leq \int_{\mathbb{D}} |\varphi(z)|^2 \, dm(z).$$

Here, $dm(z) = dx dy$ denotes the usual Lebesgue area measure.

In this paper we generalize Theorem 1 in the following way.

**Theorem 2.** Let $p > 0$. If $\varphi$ is a function as in Theorem 1 and $\varphi(z) \neq 0$ for all $z \in \mathbb{D}$, then

$$\int_{\mathbb{D}} |z|^p |\varphi(z)|^{p-2} |\varphi'(z)|^2 \, dm(z) \leq \int_{\mathbb{D}} |z|^{p-2} |\varphi(z)|^p \, dm(z).$$  \hfill (1.1)
Consequently, we deduce the following theorem on univalent functions.

**Theorem 3.** If $f$ is univalent in $\mathbb{D}$, $f(0) = 0$ and $p > 0$, then
\[ \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| 1 - \frac{zf'(z)}{f(z)} \right|^2 \, dm(z) \leq \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, dm(z), \]

or, equivalently,
\[ \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| \frac{zf'(z)}{f(z)} \right|^2 \, dm(z) \leq 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \Re \left( \frac{zf'(z)}{f(z)} \right) \, dm(z). \]

Given a space $X$ of analytic functions, one of the most interesting problems in the theory of univalent functions is finding geometric conditions on a domain $\Omega$ which imply that $\Omega$ is an $X$-domain; that is, any analytic function $f$ defined on $\mathbb{D}$ with $f(\mathbb{D}) \subset \Omega$ belongs to $X$. This problem has been solved for a good number of spaces, such as the Bloch space, Besov spaces $B^p$, $1 < p < \infty$, (see [4]). However, this is an open problem for $H^p$ spaces (see [4]). We shall use Theorem 3 to find geometric conditions on the image domain of a function $f \in U$ which imply its membership in $H^p$. For simplicity, we shall assume that $0 \in f(\mathbb{D})$.

Given a domain $\Omega \subset \mathbb{C}$ and a point $w$ in $\Omega$, we shall write $d_\Omega(w)$ to denote the (Euclidean) distance from $w$ to the boundary $\partial \Omega$. The following inequalities play an essential role in the proof of our results (see, e.g., [9], Corollary 1.4).

If $\Omega$ is a simply connected proper subdomain of $\mathbb{C}$ and if $F$ is a conformal mapping from $\Delta$ onto $\Omega$, then we have
\[ d_\Omega(F(z)) \leq |F'(z)|(1 - |z|^2) \leq 4d_\Omega(F(z)), \quad z \in \mathbb{D}. \]

The following result is proved in [2] Corollary 7.

**Corollary 1.** Suppose that $1/2 \leq p < \infty$ and that $f \in A^{2p} \cap U$. Set $\Omega = f(\mathbb{D})$ and suppose that $0 \in \Omega$. For $\varepsilon > 0$, set $\Omega_\varepsilon = \{ w \in \Omega : |w| > \varepsilon \}$. If
\[ \int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^{2p}} \, dm(w) < \infty, \]
for all sufficiently small $\varepsilon > 0$, then $f \in H^p$.

Here, we shall prove the following extension of this result.

**Theorem 4.** Suppose that $0 < \beta < 1$, $1 - \frac{\beta}{2} < p < \infty$, and $f \in A^{2p} \cap U$. Set $\Omega = f(\mathbb{D})$ and suppose that $0 \in \Omega$. If
\[ \int_{\Omega_\varepsilon} \frac{d_\Omega(w)^{2p-2}}{|w|^{2p-2 + \beta \varepsilon}} \, dm(w) < \infty, \]
for some $\delta$, $0 < \delta < \frac{1 + p}{2}$ and all sufficiently small $\varepsilon > 0$, then $f \in H^p$.

Moreover, we shall prove that this result is sharp in a certain sense.

The paper is organized as follows. Section 2 is devoted to proving Theorem 2 and Theorem 6 Corollary 4 and some other results are proved in section 6.

2. **Proof of the main results**

The proof of Theorem 2 is based on the following fact due to Prawitz [10]. The proof is borrowed from [7] (see also [6]).
Theorem 5. Let \( f: \mathbb{D} \mapsto \mathbb{C} \) be a univalent function and \( f(0) = 0 \), and let
\[
J_p(r) = J_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^{-p} \, d\theta, \quad p > 0, \quad 0 < r < 1.
\]
Then
\[
2\pi J'_p(r) = -\frac{p}{r} \text{Im} \int_{\Gamma_r} |w|^{-p-2} \omega \, dw
\]
\[
= -(p/r) \int_{\Gamma_r} |w|^{-p-2} (u \, dv - v \, du)
\]
\[
< 0,
\]
for all \( r \in (0, 1) \), where \( \Gamma_r \) is the image under \( f \) of the circle \( \{ \zeta \in \mathbb{C} : |\zeta| = r \} \) and \( \Gamma_r \) is positively oriented.

Proof. We have
\[
2\pi J'_p(r) = -p \int_0^{2\pi} |f(re^{i\theta})|^{-p-2} \text{Re} \{f(re^{i\theta})f'(re^{i\theta})e^{i\theta}\} \, d\theta
\]
\[
= -(p/r) \text{Im} \int_{|\zeta|=r} |f(\zeta)|^{-p-2} \overline{f(\zeta)} f'(\zeta) \, d\zeta
\]
\[
= -(p/r) \int_{\Gamma_r} |w|^{-p-2} \omega \, dw
\]
\[
= -(p/r) \int_{\Gamma_r} |w|^{-p-2} (u \, dv - v \, du),
\]
where \( \Gamma_r \) is the image under \( f \) of the circle \( |\zeta| = r \) and the curve \( \Gamma_r \) is positively oriented. Now we apply Green’s formula to the domain \( \Omega_{r,R} \) bounded by \( \Gamma_r \) and the circle \( |w| = R \), where \( R > \max_{|z|=r} |f(z)| \). Since
\[
\frac{\partial}{\partial u} (|w|^{-p-2}u) - \frac{\partial}{\partial v} (|w|^{-p-2}v) = -p|w|^{-p-2},
\]
we have
\[
\int_{|w|=R} |w|^{-p-2} (u \, dv - v \, du) - \int_{\Gamma_r} |w|^{-p-2} (u \, dv - v \, du)
\]
\[
= -p \int_{\Omega_{r,R}} |w|^{-p-2} \, du \, dv.
\]
The first integral is equal to \( 2\pi R^{-p} \), and therefore
\[
J'_p(r) = -(p/r)R^{-p} - (p^2/2\pi r) \int_{\Omega_{r,R}} |w|^{-p-2} \, du \, dv.
\]
Letting \( R \) tend to \( \infty \) we get
\[
J'_p(r) = -(p^2/2\pi r) \int_{\Omega_r} |w|^{-p-2} \, du \, dv,
\]
where \( \Omega_r \) is the exterior of the curve \( \Gamma_r \). This concludes the proof. \( \Box \)

Proof of Theorem 2. With the hypotheses of Theorem 2 let \( g(z) = \varphi(z)/z \) and \( f(z) = 1/g(z) = z/\varphi(z) \). Let
\[
I_p(r) = I_p(r, g) = \frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^p \, d\theta = J_p(r, f), \quad p > 0.
\]
The function $f$ satisfies the hypotheses of Theorem 5 and therefore
\[ 2\pi I_p'(r) = 2\pi J_p'(r) = -(p/r) \int_{\Gamma_r} |w|^{-p-2}(u \, dv - v \, du) < 0. \]

By the change $w \mapsto 1/w$ we get
\[ 2\pi I_p'(r) = (p/r) \int_{\gamma_r} |w|^{-p-2}(u \, dv - v \, du) < 0, \]
where $\gamma_r$ denotes the curve $w = g(re^{it})$, $0 \leq t \leq 2\pi$, which is a curve of negative orientation. Now we try to parameterize $\gamma_r$ by $w = F(e^{it})$, where $F(e^{it}) \equiv g(re^{it})$; indeed we choose
\[ F(z) = \frac{\bar{z}}{r}\varphi(rz). \]

Then we have
\[ \int_{\gamma_r} |w|^{-p-2}(u \, dv - v \, du) = \text{Im} \int_{|\zeta|=1} |F(\zeta)|^{-p-2} F(\bar{\zeta}) dF(\zeta). \]

Now, we choose a circle $T_\rho$ of radius $\rho$ centered at 0 ($0 < \rho < \frac{1}{2}$) and apply Green’s formula to the annulus $A_\rho := \{ z : \rho \leq |z| \leq 1 \}$ to get
\[ \text{Im} \int_{|\zeta|=1} |F(\zeta)|^{-p-2} F(\bar{\zeta}) dF(\zeta) = p \int_{A_\rho} |F|^{-p-2} J_F \, dm + \text{Im} \int_{T_\rho} |F|^{-p-2} F \, dF, \]
where $J_F(z)$ is the Jacobian of $F$,
\[ J_F(z) = |\partial F/\partial z|^2 - |\partial F/\partial \bar{z}|^2 = |z\varphi'(rz)|^2 - |\varphi(rz)|^2/r^2; \]
here the circles are positively oriented. From the properties of $\varphi$ it follows that
\[ \left| \text{Im} \int_{T_\rho} |F|^{-p-2} F \, dF \right| \leq C \rho^p \]
and
\[ \int_{\mathbb{B}} |F|^{-p-2} J_F \, dm \leq C \int_{\mathbb{D}} |z|^{-p-2} dm(z) < \infty. \]

Hence, by passing to the limit as $\rho \to 0$, we get
\[ \text{Im} \int_{|\zeta|=1} |F(\zeta)|^{-p-2} F(\bar{\zeta}) dF(\zeta) = p \int_{\mathbb{D}} |F|^{-p-2} J_F \, dm. \]

From this, 2.21 and 2.22, it follows that
\[ \int_{\mathbb{B}} |F(z)|^{-p-2} J_F(z) \, dm(z) < 0; \]
that is,
\[ \int_{\mathbb{B}} |z\varphi(rz)|^{-p-2} |z\varphi'(rz)|^2 \, dm(z) < r^{-2} \int_{\mathbb{D}} |z|^{-p-2} \varphi(rz)|^p \, dm(z) \]
\[ = 2\pi r^{-2} \int_0^1 t^{p-1} I_p(rt, \varphi) \, dt, \]
(2.3)
where, as above,
\[ I_p(s, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(se^{i\theta})|^p \, d\theta. \]
As is well known, $I_p(s, \varphi)$ increases with $s$ and so we can apply the monotone convergence theorem to get

$$
\lim_{r \to 1-} \int_0^1 t^{p-1} I_p(rt, \varphi) \, dt = \int_0^1 t^{p-1} \lim_{r \to 1-} I_p(rt, \varphi) \, dt
$$

From this and (2.3), via Fatou’s lemma, we obtain (1.1). This completes the proof of Theorem 2.

Proof of Theorem 3. If $f$ is univalent in $D$, $f(0) = 0$, then let $\varphi(z) = z/f(z)$. Since

$$
\varphi'(z) = \frac{f(z) - zf'(z)}{f(z)^2},
$$

we see, by Theorem 2, that

$$
\int_D |z|^{2p-2} |f(z)|^2 |f(z) - zf'(z)|^2 \, dm(z) \leq \int_D |z|^{p-2} |z| f(z) |dm(z).
$$

The result follows.

Remark 1. Equality is possible in (1.1), (1.2) and (1.3). In the case of (1.1) we take

$$
\varphi(z) = (1 - e^{-i\theta} z)^2, \quad \theta \in [0, 2\pi].
$$

Then

$$
\text{Im} \int_{|\zeta|=1} |F(\zeta)|^{p-2} \overline{F(\zeta)} \, d\overline{F(\zeta)} = 0,
$$

where

$$
F(z) = \overline{z} \varphi(z) = \overline{z}(1 - e^{-i\theta} z)^2.
$$

Now an application of Green’s formula as above shows that

$$
\int_D |F|^{p-2} J_F \, dm = 0,
$$

which implies (1.1).

The above deduction of Theorem 3 from Theorem 2 shows that equality in (1.2) and (1.3) is attained if $f_\theta$ is any rotation of Koebe’s function:

$$
f_\theta(z) = \frac{z}{(1 - e^{-i\theta} z)^2}.
$$

3. Applications

First, we present the following simple but useful lemma.

Lemma 1. If $f$ is univalent in $D$, $f(0) = 0$ and $p > 0$, then

$$
\int_D |z|^{2p} |f(z)|^{-p} |f'(z)|^2 \, dm(z) < \infty.
$$
Proof. Observe that if \( f \in \mathcal{U} \) with \( f(0) = 0 \), then
\[
\int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, dm(z) < \infty \quad \text{for any } p > 0,
\]
which together with Theorem 3 gives that
\[
\int_{\mathbb{D}} |z|^{2p} |f(z)|^{-p} \left| \frac{f'(z)}{f(z)} \right|^2 \, dm(z)
\leq 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \left| 1 - \frac{zf'(z)}{f(z)} \right|^2 \, dm(z) + 2 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, dm(z)
\leq 4 \int_{\mathbb{D}} |z|^{2p-2} |f(z)|^{-p} \, dm(z) < \infty.
\]
This finishes the proof. \( \square \)

Now, we are ready to obtain our result on \( H^p \)-univalent functions.

Proof of Theorem 4. Bearing in mind [2] Theorem 1, it is enough to prove that
\[
(3.1) \quad \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-1} \, dm(z) < \infty.
\]
Take \( \eta > 0 \) such that \{\{|w| < \eta\} \subset \Omega \) and take \( \varepsilon \) with \( 0 < \varepsilon < \eta \) and set \( \mathbb{D}_\varepsilon = f^{-1}(\Omega \varepsilon) \).

By Hölder’s inequality (with conjugate exponents \( \frac{2}{1+\beta} > 1 \) and \( \frac{2}{1-\beta} \)) and Lemma 1, we deduce that
\[
(3.2) \quad \int_{\mathbb{D}_\varepsilon} |f'(z)|^p (1 - |z|^2)^{p-1} \, dm(z)
= \int_{\mathbb{D}_\varepsilon} \left[ (1 - |z|^2)|f'(z)| \right]^{p-1} |f'(z)|^{\beta} |z|^{-2\beta} |f(z)|^{1-\beta+\delta} \left[ f(z)^{-\delta} |z|^{2\delta} \left| \frac{f'(z)}{f(z)} \right|^{1-\beta} \right] \, dm(z)
\leq \left( \int_{\mathbb{D}_\varepsilon} \left[ (1 - |z|^2)|f'(z)| \right]^{2(p-1)} |f'(z)|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\beta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} \, dm(z) \right)^{\frac{1+\beta}{2}}
\times \left( \int_{\mathbb{D}_\varepsilon} |f(z)|^{-\frac{1+\beta}{1+\beta}} \left| \frac{f'(z)}{f(z)} \right|^2 \, dm(z) \right)^{\frac{1-\beta}{1+\beta}}
\leq C \left( \int_{\mathbb{D}_\varepsilon} \left[ (1 - |z|^2)|f'(z)| \right]^{2(p-1)} |f'(z)|^{\frac{2\beta}{1+\beta}} |z|^{-\frac{4\beta}{1+\beta}} |f(z)|^{\frac{2(1-\beta+\delta)}{1+\beta}} \, dm(z) \right)^{\frac{1+\beta}{2}}.
\]

On the other hand, if \( 0 < \delta < \frac{1+p}{2} \), using Hölder’s inequality (with conjugate exponents \( \frac{1+\beta}{\beta} \) and \( 1 + \beta \)), making the change of variable \( w = f(z) \) and bearing in
mind [1,2] and that \( f \in A^{2p} \), we deduce that
\[
\int_{D} \left[ (1 - |z|^2)|f'(z)| \right]^{\frac{2(p-1)}{1+p}} |f(z)|^{\frac{2p}{1+p}} |z|^{\frac{4p}{1+p}} \ dm(z)
\]
\[
= \int_{D} \left[ (1 - |z|^2)|f'(z)| \right]^{\frac{2(p-1)}{1+p}} |f(z)|^{\frac{2p}{1+p}} |z|^{\frac{4p}{1+p}} \ dm(z)
\]
\[
= \left( \int_{D} \left[ (1 - |z|^2)(f'(z))^2 \right]^{\frac{2(p-1)}{1+p}} |f(z)|^{\frac{2p}{1+p}} |z|^{\frac{4p}{1+p}} \ dm(z) \right)^{\frac{p}{1+p}}
\]
\[
\times \left( \int_{D} |z|^{-\frac{2p-2}{p}} |f(z)|^{2p} \ dm(z) \right)^{\frac{1}{1+p}}
\]
\[
\leq C \left( \int_{\Omega_e} \frac{d\Omega(w)}{|w|^{\frac{2p-2}{p} + 2 + \kappa}} \ dm(w) \right)^{\frac{p}{1+p}} < \infty,
\]
which, together with (3.1) and (3.2), finishes the proof. \( \square \)

Moreover, we are able to prove that this result is sharp in the following sense.

**Theorem 6.** If \( 0 < \beta < 1 \) and \( 1 - \frac{\beta}{2} < p < \infty \), then there exists a univalent function \( g \in A^{2p} \setminus H^p \) with \( g(0) = 0 \) and such that, setting \( \Omega = g(D) \) and \( \Omega_e = \{ w \in \Omega : |w| > \varepsilon \} \),
\[
\int_{\Omega_e} \frac{d\Omega(w)}{|w|^{\frac{2p-2}{p} + 2 + \kappa}} \ dm(w) < \infty, \quad \varepsilon > 0,
\]
for every \( \kappa > 0 \).

**Proof of Theorem 5.** We shall follow the lines of the proof of [2, Theorem 8]. Take \( p \in (1/2, \infty) \) and let \( f \) be the function defined in the proof of [2, Theorem 3], that is,
\[
f(z) = \left[ \frac{1}{(1 - z) \log \frac{2e}{1-z}} \right]^{\frac{1}{p}}, \quad z \in D.
\]
Set
\[
g(z) = f(z) - f(0), \quad z \in D.
\]
Then \( g \) is univalent, \( g(0) = 0 \) and \( g \in A^{2p} \setminus H^p \). Finally, we shall see that (3.3) holds.

Take \( \varepsilon > 0 \). Since \( g(0) = 0 \), there exists \( \eta \) with \( 0 < \eta < 1 \) such that
\[
g^{-1}(\Omega_e) \subset \mathbb{D}_\eta = \{ z \in \mathbb{D} : |z| > \eta \}.
\]
We have that
\[
g'(z) = \frac{1}{p(1-z)^{1+\frac{1}{p}}} \left[ \left( \frac{1}{\log \frac{2e}{1-z}} \right)^{\frac{1}{p}} \left( 1 - \frac{1}{\log \frac{2e}{1-z}} \right) \right], \quad z \in \mathbb{D},
\]
and that there exists a positive constant \( C \) such that
\[
|g(z)| \geq C \left[ \frac{1}{(1-z) \log \frac{2e}{1-z}} \right]^{\frac{1}{p}}, \quad z \in \mathbb{D}_\eta.
\]
So, using (1.4) and assuming without loss of generality that \( \kappa < p \left( \frac{2p-2}{\beta} + 1 \right) \), we obtain

\[
\int_{\Omega} \frac{d\Omega(w)}{|w|^\frac{2p-2}{\beta} + \frac{2}{\kappa}} dm(w)
\leq C \int_{D_n} (1 - |z|^2)^{\frac{2p-2}{\beta}} \frac{\left| g'(z) \right|^{\frac{2p-2}{\beta} + 2}}{|g(z)|^{\frac{2p-2}{\beta} + \frac{2}{\kappa}}} dm(z)
\leq C \int_{D_n} (1 - |z|^2)^{\frac{2p-2}{\beta}} \frac{(1 - z) \log \frac{2e}{1-z}^{\frac{2p-2}{\beta} + \frac{2}{\kappa}}}{(1 - z)^{\left(1 + \frac{1}{\beta}\right)(\frac{2p-2}{\beta} + 2)}} \left| \log \frac{2e}{1-z}^{\frac{2p-2}{\beta} + \frac{2}{\kappa}} \right| dm(z)
\times \left| 1 - \frac{1}{\log \frac{2e}{1-z}^{\frac{2p-2}{\beta} + 2}} \right|^{\frac{2p-2}{\beta} + \frac{2}{\kappa}} dm(z)
\leq C \int_{D_n} (1 - |z|^2)^{\frac{2p-2}{\beta}} \frac{\left| \log \frac{2e}{1-z}^{\frac{2p-2}{\beta} + \frac{2}{\kappa}} \right|^\frac{2}{\kappa}}{(1 - z)^{\frac{2p-2}{\beta} + \frac{2}{\kappa}} - \frac{2}{\kappa}} dm(z)
\leq C \int_0^1 (1 - r)^{-\frac{1}{1+\frac{2}{\kappa}}} \left( \log \frac{2e}{1-r}^{\frac{2p-2}{\beta} + \frac{2}{\kappa}} \right) dr < \infty.
\]

This finishes the proof. \( \square \)

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