**$p$-CONVERGENT SEQUENCES AND BANACH SPACES IN WHICH $p$-COMPACT SETS ARE $q$-COMPACT**

CÁNDIDO PIÑEIRO AND JUAN MANUEL DELGADO

(Communicated by Nigel J. Kalton)

Abstract. We introduce and investigate the notion of $p$-convergence in a Banach space. Among others, a Grothendieck-like result is obtained; namely, a subset of a Banach space is relatively $p$-compact if and only if it is contained in the closed convex hull of a $p$-null sequence. We give a description of the topological dual of the space of all $p$-null sequences which is used to characterize the Banach spaces enjoying the property that every relatively $p$-compact subset is relatively $q$-compact ($1 \leq q < p$). As an application, Banach spaces satisfying that every relatively $p$-compact set lies inside the range of a vector measure of bounded variation are characterized.

1. Introduction

By a well known characterization due to Grothendieck [7] (see, e.g., [9, p. 30]), a subset $A$ of a Banach space $X$ is relatively compact if and only if there exists $(x_n)$ in $c_0(X)$ (the space of norm-null sequences in $X$) such that $A \subset \{ \sum_n a_n x_n : \sum_n |a_n| \leq 1 \}$. Since then, several authors have dealt with stronger forms of compactness, studying sets sitting inside the convex hulls of special types of null sequences. For instance, it was observed in [14] (see also [1]) that if one considers, instead of $c_0(X)$, the space of $q$-summable sequences $\ell_q(X)$, for some fixed $q \geq 1$, then this stronger form of compactness characterizes Reinov’s approximation property of order $p$, $0 < p < 1$. This latter form of compactness was recently further strengthened by Sinha and Karn [15] as follows. Let $1 \leq p \leq \infty$ and let $p'$ be the conjugate index of $p$ (i.e., $1/p + 1/p' = 1$). The $p$-convex hull of a sequence $(x_n) \in \ell_p(X)$ is defined as $p$-co $(x_n) = \{ \sum_n a_n x_n : \sum_n |a_n|^{p'} \leq 1 \}$ (sup $|a_n| \leq 1$ if $p = 1$). A set $A \subset X$ is said to be relatively $p$-compact if there exists $(x_n) \in \ell_p(X)$ ($(x_n) \in c_0(X)$ if $p = \infty$) such that $A \subset p$-co $(x_n)$. (Note that similar notions with $(x_n)$ being a weakly $p$-summable sequence were already considered in [2, p. 51].) Some results concerning this type of relatively compact set have been set in [3].

The aim of this article is to deepen the study of the geometry of Banach spaces related to $p$-compact sets. In this way, the notion of $p$-convergent sequence is introduced in Section 2 and a Grothendieck-like result is obtained; namely, a subset...
of a Banach space is relatively $p$-compact if and only if it is contained in the closed convex hull of a sequence $p$-convergent to zero (Theorem 2.5).

In [3], Theorem 3.14, Serrano and the authors have proved that every infinite dimensional Banach space has relatively compact sets failing to be $q$-compact for every $1 \leq q < \infty$. Section 3 is devoted to find out if this result is also true when we replace compact (= $\infty$-compact) with $p$-compact, $p > q$. In fact, we come to the same conclusion if $p > 2$ (Proposition 3.5): if $1 < q < p \leq 2$, it is shown that every relatively $p$-compact subset of $X$ is relatively $q$-compact if and only if every $q'$-summing operator from $c_0$ to $X^*$ is $p'$-summing ($p'$ and $q'$ are conjugate exponents of $p$ and $q$, respectively). It is convenient to point out that the description of the topological dual of the space of the $p$-null sequences (Proposition 3.1) simplifies the proofs of this section.

Our notation is standard. A Banach space $X$ will be regarded as a subspace of its bidual $X^{**}$ under the canonical embedding $i_X : X \to X^{**}$. We denote the closed unit ball of $X$ by $B_X$. For Banach spaces $X$ and $Y$, the space of all bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X,Y)$. We also need the following operator ideals: $N_p$—$p$-nuclear operators, $\mathcal{Q}N_p$—quasi-$p$-nuclear operators, $Z_p$—$p$-integral operators and $\Pi_p$—$p$-summing operators. We refer to Pietsch’s book [11] for operator ideals (see also [5] by Diestel, Jarchow and Tonge for common operator ideals $\mathcal{N}_p$ and $\Pi_p$, and [10] by Persson and Pietsch for $\mathcal{Q}N_p$).

Let $1 \leq p \leq \infty$. The space of all weakly $p$-summable sequences (respectively, (strongly) $p$-summable sequences) in $X$ is denoted by $\ell^w_p(X)$ (respectively, $\ell_p(X)$) endowed with its natural norm $\|(x_n)\|_p$ (respectively, $\|(x_n)\|_p$). We write $\ell_\infty(X)$ to describe the space of all bounded sequences $(x_n)$ in $X$ with the norm $\|(x_n)\|_\infty$.

Relying on the notion of $p$-compactness, the notion of $p$-compact operator is defined in an obvious way (see [15]): an operator $T \in \mathcal{L}(X,Y)$ is said to be $p$-compact if $T(B_X)$ is relatively $p$-compact in $Y$. The class of all $p$-compact operators between Banach spaces is denoted by $\mathcal{K}_p$. It is shown in [15] that $\mathcal{K}_p$ is an operator ideal. Even more, $\mathcal{K}_p$, equipped with the norm $k_p$ defined by

$$ k_p(T) = \inf \{ \|(y_n)\|_p : (y_n) \in \ell_p(Y) \text{ and } T(B_X) \subset p\text{-co}(y_n) \} $$

for all $T \in \mathcal{K}_p(X,Y)$, is a Banach operator ideal. In order to make the article self-contained, we list some properties related to $p$-compactness:

- If $1 \leq q \leq p \leq \infty$, then every relatively $q$-compact set is $p$-compact.
- An operator $T : X \to Y$ belongs to $\mathcal{K}_p(X,Y)$ (respectively, $\mathcal{Q}N_p(X,Y)$) if and only if $T^*$ belongs to $\mathcal{Q}N_p(Y^*,X^*)$ (respectively, $\mathcal{K}_p(Y^*,X^*)$) [3 Corollary 3.4 and Proposition 3.8].
- A sequence $\hat{x} = (x_n)$ of $X$ is relatively $p$-compact if and only if $U_{\hat{x}} : e_n \in \ell_1 \mapsto x_n \in X$ is $p$-compact, where $(e_n)$ denotes the unit vector basis of $\ell_1$ [3 Proposition 3.5].

2. $p$-CONVERGENCE

Definition 2.1. Let $p \geq 1$. A sequence $(x_n)$ in a Banach space $X$ is said to be $p$-null if, for every $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $(z_k) \in \varepsilon B_{\ell_p}(X)$ such that $x_n \in p\text{-co}(z_k)$ for all $n \geq n_0$.

A sequence $(x_n)$ in $X$ is said to be $p$-convergent if there exists $x \in X$ such that $(x_n - x)$ is $p$-null.
Theorem 2.5. Let \( \gamma \) and the convergence in a Banach space related to compactness.

2.2 Remark

Proposition 2.4.

Proof. Suppose that \( T^* \) is \( \gamma \)-summing and let \( \hat{x} = (x_n) \) be a null sequence in \( X \). Notice that \( U_{T\hat{x}} - U_{T\hat{z}(n)} = T \circ (U_{\hat{x}} - U_{\hat{z}(n)}) \) for each \( n \in \mathbb{N} \). Now, the dual version of [3 Theorem 3.13] ensures that \( T \circ (U_{\hat{x}} - U_{\hat{z}(n)}) \) belongs to \( K_p(\ell_1, Y) \) and

\[
    k_p(T \circ (U_{\hat{x}} - U_{\hat{z}(n)})) \leq \pi_p(T^*) \|U_{\hat{x}} - U_{\hat{z}(n)}\|.
\]

Since \( \|U_{\hat{x}} - U_{\hat{z}(n)}\| \xrightarrow{n \to \infty} 0 \), it follows that \( U_{T\hat{x}} = k_p \lim_n U_{T\hat{z}(n)} \).

The converse is obvious in view of [3 Theorem 3.14].

2.3 Remark. It is clear that \( \gamma \)-summable sequences are \( \gamma \)-null. For an example of a \( \gamma \)-null sequence failing to be \( \gamma \)-summable, consider \( (n^{-1/\gamma} e_n) \) (where \( e_n \) denotes the unit vector basis in \( \ell_2 \)) which is a null sequence in \( \ell_2 \). From the next proposition, that sequence is \( \gamma \)-null in \( c_0 \). Nevertheless, it does not belong to \( \ell_p(c_0) \).

In [3] Theorem 3.14, it is proved that an operator \( T \in \mathcal{L}(X, Y) \) has \( \gamma \)-summing adjoint if and only if \( T \) maps relatively compact sets in \( X \) to relatively \( \gamma \)-compacts sets in \( Y \). The notion of \( \gamma \)-null sequence allows us to establish the sequential version of this result:

Proposition 2.4. Let \( p \geq 1 \). An operator \( T \in \mathcal{L}(X, Y) \) has \( \gamma \)-summing adjoint if and only if \( T \) maps null sequences in \( X \) to \( \gamma \)-null sequences in \( Y \).

Proof. Since every \( \gamma \)-null sequence \( \hat{x} = (x_n) \) has relatively \( \gamma \)-compact rank, it induces an operator \( U_{\hat{x}} : \ell_1 \to X \) defined by \( U_{\hat{x}}(\alpha_n) = \sum_n \alpha_n x_n \) that is \( \gamma \)-compact. It is easy to show that a bounded sequence \( \hat{x} = (x_n) \) is \( \gamma \)-null if and only if \( U_{\hat{x}} = k_p \lim_n U_{\hat{x}(n)} \), where \( \hat{x}(n) = (x_1, \ldots, x_n, 0, \ldots) \).

We denote by \( c_{0,p}(X) \) the vector space of all \( \gamma \)-null sequences in \( X \) endowed with the norm \( k_p(\hat{x}) = k_p(U_{\hat{x}}) \). This space can be identified with a closed subspace of \( K_p(\ell_1, X) \) using a standard argument.

The following result is intended to make clear the analogy between the \( \gamma \)-convergence and the convergence in a Banach space related to compactness.

Theorem 2.5. Let \( p \geq 1 \). A set in a Banach space \( X \) is relatively \( \gamma \)-compact if and only if it is contained in the closed convex hull of a \( \gamma \)-null sequence.

Proof. If \( A \) is relatively \( \gamma \)-compact in \( X \), there exists a sequence \( \hat{z} = (z_n) \in \ell_p(X) \) so that \( A \subset \gamma\co(z_n) \). The sequence \( (z_n) \) induces an operator \( \phi : e_n \to \ell_{p'} \) that satisfies \( (z_n) \in X \) satisfying \( \phi_{\hat{z}}(B_{\ell_{p'}}) = \gamma\co(z_n) \). In order to obtain a factorization of \( \phi_{\hat{z}} \), choose \( (\alpha_n) \searrow 0 \) such that \( (\alpha_n^{-1} z_n) \in \ell_p(X) \). Now, consider the compact operator \( D : \ell_{p'} \to \ell_{p'} \) and the \( \gamma \)-compact operator \( \phi : \ell_{p'} \to X \) defined by \( D(\beta_n) = (\alpha_n \beta_n) \) and \( \phi(\beta_n) = \sum_n \beta_n \alpha_n^{-1} x_n \). It is obvious that the following diagram is commutative:

\[
\begin{array}{ccc}
\ell_{p'} & \xrightarrow{\phi_{\hat{z}}} & X \\
\downarrow D & & \swarrow \phi \\
\ell_{p'} & & \\
\end{array}
\]

As \( D(B_{\ell_{p'}}) \) is relatively compact in \( \ell_{p'} \), we can find a null sequence \( (\gamma_n) \in \ell_{p'} \) with \( D(B_{\ell_{p'}}) \subset \gamma\co(\gamma_n) \). Then

\[
A \subset \phi_{\hat{z}}(B_{\ell_{p'}}) = \phi(D(B_{\ell_{p'}})) \subset \phi(\gamma\co(\gamma_n)) \subset \gamma\co(\phi(\gamma_n)).
\]

Since \( \phi \) is \( \gamma \)-compact, \( \phi^* \) is quasi \( \gamma \)-nuclear [3 Corollary 3.4] and, in particular, \( \gamma \)-summing. So, according to Proposition 2.3, the sequence \( (\phi(\gamma_n)) \) is \( \gamma \)-null in \( X \).
For the converse, just notice that the closed convex hull of relatively $p$-compact sets are $p$-compact.

We finish this section with a characterization of $p$-null sequences in Banach spaces having the $k_p$-approximation property. A Banach space $X$ has the $k_p$-approximation property if, for every Banach space $Y$, the space $\mathcal{F}(Y,X)$ of all finite rank operators is $k_p$-dense in $K_p(Y,X)$ [4]. This property might be considered as a gradation of the classic approximation property. Since the classic approximation property implies the $k_p$-approximation property for all $p \geq 1$, the following result can be applied to a wide class of Banach spaces.

**Proposition 2.6.** If $X$ has the $k_p$-approximation property, then $\hat{x} = (x_n) \in c_0(p)(X)$ if and only if $\hat{x} \in c_0(X)$ and has relatively $p$-compact rank.

*Proof.* Only the “if” part needs to be proved (the converse is straightforward from the definition of $p$-null sequences). So, given $\varepsilon > 0$ and a sequence $(x_n) \in c_0(X)$ with relatively $p$-compact rank, there exists a finite rank operator $S: X \to X$ such that $k_p(U_{\hat{x}} - S \circ U_{\hat{x}}) < \varepsilon/3$ [4 Theorem 2.1]. According to Proposition 2.4, the sequence $S\hat{x} = (Sx_n)$ is $p$-null and this allows us to find $n_0 \in \mathbb{N}$ for which $k_p(U_{S\hat{x}} - U_{S\hat{x}(n)}) < \varepsilon/3$ whenever $n \geq n_0$. Finally, if $P_n: \ell_1 \to \ell_1$ denotes the projection onto the first $n$-th coordinates, we have, for every $n \geq n_0$,

$$k_p(U_{\hat{x}} - U_{\hat{x}(n)}) \leq k_p(U_{\hat{x}} - S \circ U_{\hat{x}}) + k_p(S \circ U_{\hat{x}} - S \circ U_{\hat{x}(n)}) + k_p(S \circ U_{\hat{x}(n)} - U_{\hat{x}(n)})$$

$$\leq k_p(U_{\hat{x}} - S \circ U_{\hat{x}}) + k_p(U_{S\hat{x}} - U_{S\hat{x}(n)}) + k_p(S \circ U_{\hat{x}} - U_{\hat{x}}) \|P_n\|$$

$$< \varepsilon.$$

□

In [4], it is shown that every Banach space has the $k_2$-approximation property. Nevertheless, we do not know whether the hypothesis of the $k_p$-approximation property can be omitted in the previous result for $p \neq 2$.

### 3. On the Equality $\Pi_p(c_0, X^*) = \Pi_q(c_0, X^*)$

In [3], it is proved that, for every $q \geq 1$ and every infinite dimensional Banach space, there exist relatively compact subsets that are not relatively $q$-compact [3 Theorem 3.14]. The objective of this section is to find out if this result is also true when we replace compact (= $\infty$-compact) with $p$-compact, $p > q$. We begin with a description of the dual space $c_0(p)(X)^*$ which will be very useful in several proofs. Recall that the trace functional (denoted by $\text{tr}$) is well defined and continuous on $\mathcal{N}_1(X, X^{**})$ if and only if $X^*$ has the approximation property.

**Proposition 3.1.** Let $X$ be a Banach space and $p \geq 1$. The dual space $c_0(p)(X)^*$ is isometrically isomorphic to $\Pi_p'(c_0, X^*)$.

*Proof.* Given $S \in \Pi_p'(c_0, X^*)$, we consider the linear form $f_S$ on $c_0(p)(X)$ defined by $f_S(\hat{x}) = \text{tr}(U_{\hat{x}} \circ S)$. Since $U_{\hat{x}}$ is $p$-compact, the adjoint $U_{\hat{x}}^*: X^* \to \ell_\infty$ is quasi $p$-nuclear [3 Corollary 3.4]; in particular, $U_{\hat{x}}^*$ is $p$-summing and the following inequalities hold:

$$\pi_p(U_{\hat{x}}^*) \leq \nu_p^2(U_{\hat{x}}^*) \leq k_p(U_{\hat{x}}).$$

Then, the composition $U_{\hat{x}}^* \circ S \in \Pi_1(c_0, \ell_\infty) = \mathcal{N}_1(c_0, \ell_\infty)$ and $\nu_1(U_{\hat{x}}^* \circ S) \leq \pi_p(U_{\hat{x}}^*) \pi_p'(S)$ [17 p. 55]. Since $\ell_1$ has the approximation property, the map $f_S$ is well defined and

$$|\text{tr}(U_{\hat{x}}^* \circ S)| \leq \nu_1(U_{\hat{x}}^* \circ S) \leq k_p(U_{\hat{x}}) \pi_p'(S).$$
Therefore, \( f_S \in c_{0,p}(X)^* \) and \( \|f_S\| \leq \pi_p'(S) \).

On the other hand, if \( f \in c_{0,p}(X)^* \), we are able to define the operator \( S_f : \alpha \in c_0 \to S_f \alpha \in X^* \) so that \( \langle S_f \alpha, x \rangle = f(\alpha \otimes x) \) for all \( x \in X \), where \( \alpha \otimes x := (\alpha_n x) \) if \( \alpha = (\alpha_n) \). To see that \( S_f \in \Pi_p'(c_0, X^*) \), consider \( (\alpha_k) \in \ell_p^m(c_0) \) with \( \|\alpha_k\|_p^m \leq 1 \) and let us prove

\[
(3.1) \quad \left( \sum_k \|S_f \alpha_k\|_{p'} \right)^{1/p'} \leq \|f\|.
\]

Fix \( N \in \mathbb{N}, x_1, \ldots, x_N \in B_X \) and \( (\beta_k)_{k=1}^N \in B_{\ell_p^N} \). We have

\[
\left| \sum_{k=1}^N \langle S_f \alpha_k, x_k \rangle \beta_k \right| = \left| \int f \left( \sum_{k=1}^N \alpha_k \otimes \beta_k x_k \right) \right| \leq \|f\| \left( \sum_{k=1}^N \alpha_k \otimes \beta_k x_k \right).
\]

Set \( \hat{y} = \sum_{k=1}^N \alpha_k \otimes \beta_k x_k \). Notice that \( (\alpha_k^\prime) \in \ell_p' \) for each \( k \in \mathbb{N} \), so we can ensure that \( U_f e_k^\prime = \sum_{k=1}^N \alpha_k^\prime \beta_k x_k \) belongs to the closed and absolutely convex set \( p^\prime\text{-co}(\beta_k x_k)_{k=1}^N \) for each \( n \in \mathbb{N} \) (here, \((e_k^\prime)\) denotes the unit vector basis of \( \ell_1 \)). From this, it is clear that

\[
k_p(\hat{y}) = k_p(U_f) \leq \|\beta_k x_k\|_{k=1}^N \leq 1.
\]

Summing up,

\[
\left| \sum_{k=1}^N \langle S_f \alpha_k, x_k \rangle \beta_k \right| \leq \|f\|
\]

for all \( N \in \mathbb{N}, x_1, \ldots, x_N \in B_X \) and \( (\beta_k)_{k=1}^N \in B_{\ell_p^N} \), from which (3.1) is obtained, using a standard argument. So \( S_f \) is \( p' \)-summing and, moreover, \( \pi_p'(S) \leq \|f\| \).

Now, the functionals \( f_{S_f} \) and \( f \) coincide on finitely supported sequences in \( c_{0,p}(X) \). Indeed, no matter how we choose \( N \in \mathbb{N} \) and \( \hat{x}_0 = (x_1, \ldots, x_N, 0, 0, \ldots) \),

\[
f(\hat{x}_0) = \sum_{k=1}^N f(\hat{e}_k \otimes x_k) = \sum_{k=1}^N \langle S_f e_k, U_{\hat{x}_0} e_k^\prime \rangle\]

\[
= \sum_{k=1}^N \langle (U_{\hat{x}_0}^* \circ S_f) e_k, e_k^\prime \rangle = \text{tr} \left( (U_{\hat{x}_0}^* \circ S_f) \right) = f_S(\hat{x}_0).
\]

Since \( \hat{x} = k_p - \lim_n \hat{x}_n(n) \) for every \( \hat{x} \in c_{0,p}(X) \) (Remark 2.2), it is clear that the functionals \( f_{S_f} \) and \( f \) coincide on \( c_{0,p}(X) \).

The following result turns out to be the key to answer the question posed at the beginning of this section. As usual, if \( X \) and \( Y \) are Banach spaces and \( x^* \in X^* \) and \( y \in Y \) are fixed, \( x^* \otimes y \) denotes the one dimensional operator from \( X \) to \( Y \) defined by \( (x^* \otimes y)(x) = (x^*, x)y \) for all \( x \in X \).

**Theorem 3.2.** Let \( X \) and \( Y \) be Banach spaces and let \( 1 \leq q < p < \infty \). The following statements are equivalent for an operator \( T \in \mathcal{L}(X,Y) \):

a) The operator \( T \) maps relatively \( p \)-compact subsets of \( X \) to relatively \( q \)-compact subsets of \( Y \).

b) The operator \( T^* \circ u \in \Pi_p'(c_0, X^*) \) whenever \( u \in \Pi_q'(c_0, Y^*) \) (\( u \in \mathcal{L}(c_0, Y^*) \), in case \( q = 1 \)).
Proof. a) ⇒ b) It is easy to prove that the operator $V: u \in \mathcal{K}_p(\ell_1, X) \mapsto T \circ u \in \mathcal{K}_q(\ell_1, Y)$ has closed graph. Then there exists a positive constant $C$ such that

$$k_q(T \circ u) \leq C k_p(u) \quad \text{for all } u \in \mathcal{K}_p(\ell_1, X).$$

In view of this, for every $p$-null sequence $\hat{x}$ we have

$$k_q(U_{T\hat{x}} - U_{T_2(\hat{x})}) = k_q(T \circ (U_2 - U_{T_2(n)})) \leq C k_p(U_2 - U_{T_2(n)}).$$

So, from Remark 2.2, $T$ maps $p$-null sequences to $q$-null sequences and, therefore, we can consider the operator $\hat{T}: (x_n) \in c_0, p(X) \mapsto (Tx_n) \in c_0, q(Y)$. It only remains to prove that the adjoint $\hat{T}^*: \Pi_{q'}(c_0, Y^*) \mapsto \Pi_{q'}(c_0, X^*)$ is defined by $\hat{T}^* = T^* \circ u$ for all $u \in \Pi_{q'}(c_0, Y^*)$. To see this, consider $x \in X$, $\beta = (\beta_n) \in c_0$ and $u \in \Pi_{q'}(c_0, Y^*)$ and denote the sequence $(\beta_n x)$ by $\beta \otimes x$. On one hand, we have

$$\langle \hat{T}^*(u), \beta \otimes x \rangle = \text{tr} \left( U_{\beta \otimes x} \circ \hat{T}^* (u) \right) = \text{tr} \left( (x \otimes \beta) \circ \hat{T}^* (u) \right) = \text{tr} \left( (\hat{T}^* (u)^* x) \otimes \beta \right) = \langle \hat{T}^* (u)^* x, \beta \rangle = \langle x, \hat{T}^* (u) \beta \rangle.$$

On the other hand,

$$\langle u, \hat{T} (\beta \otimes x) \rangle = \text{tr} \left( U_{\beta \otimes x} \circ u \right) = \text{tr} \left( (Tx \otimes \beta) \circ u \right) = \text{tr} \left( u^* (Tx) \otimes \beta \right) = \langle u^* (Tx), \beta \rangle = \langle x, (T^* \circ u) \beta \rangle.$$ 

As $\langle \hat{T}^*(u), \beta \otimes x \rangle = \langle u, \hat{T} (\beta \otimes x) \rangle$, it follows that $\langle x, \hat{T}^* (u) \beta \rangle = \langle x, (T^* \circ u) \beta \rangle$ for all $x \in X$ and $\beta \in c_0$, which yields $\hat{T}^* (u) = T^* \circ u$.

b) ⇒ a) It is a standard argument to prove that the operator $u \in \Pi_{q'}(c_0, Y^*) \mapsto T^* \circ u \in \Pi_{q'}(c_0, X^*)$

has closed graph and, therefore, that it is continuous. If $\Phi$ denotes the restriction of this operator to $\mathcal{N}_q(c_0, Y^*)$, then $\Phi^*: c_0, p(X)^{**} \mapsto \Pi_q(Y^*, \ell_\infty)$ is defined by $\Phi^* (\hat{x}) = U_{T\hat{x}} \circ T^*$ for all $\hat{x} \in c_0, p(X)$ (just using a similar argument as in a) ⇒ b) above). Since $U_{T\hat{x}} \circ T^* = U_{T\hat{x}}^*$, the restriction of $\Phi^*$ to $c_0, p(X)$ is the operator

$$\hat{x} \in c_0, p(X) \mapsto U_{T\hat{x}}^* \in \mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty).$$

Now, we will show that $\Phi^*$ maps $c_0, p(X)$ to $\mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty)$. According to [16, Theorem 2.6], the $q$-summing norm and the quasi $q$-nuclear norm of a finite rank operator coincide. So, given $\hat{x} \in c_0, p(X)$, we have for $m > n$:

$$\nu_q^q \left( \Phi^* (\hat{x}(m) - \Phi^* (\hat{x}(n)) \right) = \nu_q^q \left( \Phi^* (\hat{x}(m)) - \Phi^* (\hat{x}(n)) \right) \leq \| \Phi^* \| k_p (\hat{x}(m) - \hat{x}(n)).$$

This proves that $\left( \Phi^* (\hat{x}(n)) \right)$ is a Cauchy sequence in $\mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty)$ from which it is easy to conclude that $\Phi^* (\hat{x}) \in \mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty)$. Hence, we can replace $\Pi_q(Y^*, \ell_\infty)$ with $\mathcal{Q}\mathcal{N}_q(Y^*, \ell_\infty)$ in [22]. In view of [8, Proposition 3.5], this means that $T$ maps $p$-null sequences to sequences with relatively $q$-compact rank. The proof concludes by just invoking Theorem 2.2. 

Let $\mathcal{P}_{p,q}$ $(1 \leq q < p)$ be the class consisting of all Banach spaces with the property that every relatively $p$-compact subset is relatively $q$-compact.
Corollary 3.3. Let $1 \leq q < p < \infty$. The following statements are equivalent for a Banach space $X$:

a) $X \in \mathcal{P}_{p,q}$.

b) $\Pi_{q'}(c_0, X^*) = \Pi_{p'}(c_0, X^*)$ ($\mathcal{L}(c_0, X^*) = \Pi_{p'}(c_0, X^*)$ if $q = 1$).

In [6], the class of Banach spaces $Y$ satisfying

$$
(3.3) \quad \Pi_2(c_0, Y) = \mathcal{L}(c_0, Y)
$$

is deeply studied. Having in mind that $\mathcal{L}(c_0, Y) = \Pi_\infty(c_0, Y)$, Corollary 3.3 reveals that a dual Banach space $Y = X^*$ enjoys (3.3) if and only if relatively 2-compact subsets of $X$ are relatively 1-compact. Related to the ideas of [6], several conditions equivalent to Corollary 3.3(b) are established in the following result, the proof of which is omitted since it is quite similar to the proof in [6] Proposition 2.1.

Proposition 3.4. Let $X$ be a Banach space and let $1 \leq r < s$. The following statements are equivalent:

a) $\Pi_r(Y, X) = \Pi_s(Y, X)$ for every $L_\infty$-space $Y$.

b) $\Pi_r(Y, X) = \Pi_s(Y, X)$ for some infinite dimensional $L_\infty$-space $Y$.

c) There exists a positive constant $C$ such that $\pi_r(u) \leq C \pi_s(u)$ for all $n \in \mathbb{N}$ and $u \in \mathcal{L}(\ell_\infty^n, X)$.

Proposition 3.5. Let $1 \leq q < p$ and let $X$ be a Banach space.

1) If $X \in \mathcal{P}_{p,q}$ and $E$ is a closed subspace of $X$, then $X/E \in \mathcal{P}_{p,q}$.

2) $X^{**} \in \mathcal{P}_{p,q}$ if and only if $X \in \mathcal{P}_{p,q}$.

3) If $2 \leq q < p$, then $\mathcal{P}_{p,q}$ does not contain any infinite dimensional Banach space.

Proof. To prove 1), notice that $u \in \Pi_{q'}(c_0, (X/E)^*) = \Pi_{q'}(c_0, E^\perp)$ if and only if $i_{E^\perp} \circ u \in \Pi_{q'}(c_0, X^*)$, where $i_{E^\perp}$ denotes the inclusion map from $E^\perp$ into $X^*$. By hypothesis and using the injectivity of the ideal of $p'$-summing operators, it follows that $u \in \Pi_{p'}(c_0, (X/E)^*)$.

If $X^{**} \in \mathcal{P}_{p,q}$, then a similar argument allows us to obtain that $X \in \mathcal{P}_{p,q}$. The converse follows easily from Proposition 3.4(c) using the local reflexivity principle.

Let us argue by contradiction to show 3). Suppose there exists an infinite dimensional Banach space $X \in \mathcal{P}_{p,q}$. By virtue of Corollary 3.3 there exists a positive constant $C$ such that

$$
(3.4) \quad \pi_{p'}(u) \leq C \pi_{q'}(u)
$$

for all $u \in \Pi_{q'}(c_0, X^*)$. Now, for every $\gamma = (\gamma_n) \in \ell_{q'}$, consider the operator

$$
D_\gamma : (x_n^*) \in \ell_{q'}^w(X^*) \rightarrow (\gamma_n x_n^*) \in \ell_{p'}(X^*).
$$

To see that $D_\gamma$ is well defined, take $(x_n^*) \in \ell_q^w(X^*)$ and define the operators $A : c_0 \rightarrow \ell_q$ and $B : \ell_q \rightarrow X^*$ by $A(\alpha_n) = (\gamma_n \alpha_n)$ and $B(\beta_n) = \sum \beta_n x_n^\ast$. It is easy to prove that $A$ is $q'$-summing with $\pi_{q'}(A) \leq \|\gamma\|_{q'}$ and $\|B\|_w \leq 1$. Hence, $B \circ A$ is $q'$-summing and, by hypothesis, $p'$-summing. Moreover, in view of (3.4), we have

$$
\left( \sum_n \|\gamma_n x_n^\ast\|_{p'} \right)^{1/p'} = \left( \sum_n \|(B \circ A)e_n\|_{p'} \right)^{1/p'} \leq \pi_{p'}(B \circ A) \leq C \pi_{q'}(B \circ A) \leq C \|\gamma\|_{q'} \|x_n^\ast\|_q^w.
$$
This shows that $D_\gamma$ is well defined and continuous for all $\gamma \in \ell_{q'}$. In particular, the inequality

$$\left(\sum_{n=1}^{N} \|\gamma_n x_n^*\|_{p'}\right)^{1/p'} \leq C \|\gamma\|_{q'} \|(x_n^*)_{n=1}^{N}\|_q$$

holds for all $N \in \mathbb{N}$ and $x_1^*, \ldots, x_N^* \in X^*$. According to Dvoretzky–Rogers’ Theorem [11, p. 38], given $\varepsilon > 0$, for every $N \in \mathbb{N}$ there exist unitary vectors $x_1^*, \ldots, x_N^* \in X^*$ such that $\|(x_n^*)_{n=1}^{N}\|_2 \leq (1 + \varepsilon)$. As $q \geq 2$, applying (3.7) to these vectors, it follows that

$$\left(\sum_{n=1}^{N} |\gamma_n|^{p'}\right)^{1/p'} \leq C \|\gamma\|_{q'}(1 + \varepsilon)$$

for all $N \in \mathbb{N}$. This is a contradiction if we take $\gamma \in \ell_{q'} \setminus \ell_{p'}$.

If $1 \leq q < p \leq 2$, the following proposition shows that there are (infinite dimensional) spaces in which relatively $p$-compact sets are relatively $q$-compact.

**Proposition 3.6.** The following statements are equivalent for a Banach space $X$:

a) $X^*$ has finite cotype.

b) There exist $p, q \in \mathbb{R}$, $1 \leq q < p \leq 2$, such that $X \in \mathcal{P}_{p,q}$.

In addition, if $X^*$ has cotype $s > 2$ (respectively, $s = 2$), then $X \in \mathcal{P}_{p,q}$ for every $1 \leq q < p < s'$ (respectively, $1 \leq q < p \leq 2$).

**Proof.** a) $\Rightarrow$ b) Suppose $X^*$ has finite cotype $s > 2$ (respectively, $s = 2$). According to [5, Theorem 11.14], we have $\Pi_r(e_0, X^*) = \mathcal{L}(e_0, X^*)$ for all $r > s$ (respectively, $\Pi_1(e_0, X^*) = \mathcal{L}(e_0, X^*)$). So, by Corollary 3.3, $X$ belongs to $\mathcal{P}_{r,1}$ (respectively, $X$ belongs to $\mathcal{P}_{2,1}$).

b) $\Rightarrow$ a) By contradiction, if $X^*$ does not have finite cotype, then $X^*$ contains $\ell_\infty^n$ uniformly [5, Theorem 14.1]. Thus, there is a constant $\lambda > 0$ such that, for all $n \in \mathbb{N}$, there exists an isomorphism $J_n$ from $\ell_\infty^n$ onto a finite dimensional subspace $E_n$ of $X^*$ satisfying

$$\|J_n\| \cdot \|J_n^{-1}\| \leq \lambda \quad \text{for all } n \in \mathbb{N}.$$  

Suppose that $p$ and $q > 1$ are real numbers satisfying b) (if $q = 1$, the proof is quite similar). A call to Corollary 3.3 tells us that there exists a constant $C > 0$ such that

$$\pi_{p'}(u) \leq C \pi_{q'}(u) \quad \text{for all } u \in \mathcal{L}(\ell_\infty^n, X^*) \text{ and } n \in \mathbb{N}.$$  

If $I^n_n$ denotes the identity map on $\ell_\infty^n$, (3.6) and (3.7) yield

$$\pi_{p'}(I^n_\infty) \leq \|J_n^{-1}\| \pi_{p'}(J_n) \leq C \|J_n^{-1}\| \pi_{q'}(J_n) \leq C \lambda \pi_{q'}(I^n_\infty).$$

On one hand, we have the estimation

$$n^{1/p'} = \left(\sum_{k=1}^{n} \|I^n_\infty(e_k)\|_{p'}\right)^{1/p'} \leq \pi_{p'}(I^n_\infty),$$

where $(e_n)$ is the canonical vector basis in $\ell_\infty^n$. On the other hand, as $I^n_\infty$ admits the $q'$-nuclear representation $I^n_\infty = \sum_{k=1}^{n} e_k^* \otimes e_k$, it follows that

$$\nu_{q'}(I^n_\infty) \leq \|(e_k^*)_q\|_{q'} \|(e_k)_q\|_{q} \leq n^{1/q'}.$$
Since \( \pi_q'(I^n_p) = \nu_q'(I^n_p) \) [17, p. 49], [128], [3.9] and [3.10] lead us to conclude
\[
n^{1/p'} \leq \pi_q'(I^n_p) \leq C \lambda \pi_q'(I^n_\infty) \leq C \lambda n^{1/q'}
\]
for all \( n \in \mathbb{N} \), which is a contradiction since \( q' > p' \).

\[ \square \]

Remark 3.7. The additional information in the above result cannot be improved for spaces whose dual has cotype \( s > 2 \). Indeed, if \( s' \leq p \leq 2 \), let us show that \( X = \ell_{s'} \notin \mathcal{P}_{p,q} \) for all \( q < p \). First, put \( p = s' \) and suppose, by contradiction, that there exists \( q < s' \) satisfying \( \ell_{s'} \in \mathcal{P}_{p,q} \). In view of Proposition 3.6 \( \ell_{s'} \) belongs to \( \mathcal{P}_{p,1} \); that is, \( \mathcal{L}(\ell_\infty, \ell_s) = \Pi_s(\ell_\infty, \ell_s) \), which contradicts [3, Theorem 7]. Now, for the case \( p > s' \), it suffices to see that \( X = \ell_{s'} \notin \mathcal{P}_{p,q} \) when \( p \) satisfies \( s' \leq q < p \).

Under this assumption, consider a sequence \( (\lambda_n) \in \ell_p \setminus \ell_q \). The rank of the sequence \( (\lambda_n e_n) \) is obviously relatively \( p \)-compact in \( X \). Arguing again by contradiction, if the sequence is relatively \( q \)-compact, then the operator \( U : e_n \in \ell_1 \mapsto \lambda_n e_n \in X \) is \( q \)-compact. Therefore, \( U^* \) is quasi \( q \)-nuclear [3, Corollary 3.4] and, in particular, \( q \)-summing. As \( (e_n^*) \in \ell_q^w(X^*) \subset \ell_p^w(X^*) \), it follows that
\[
\sum_n |\lambda_n|^q = \sum_n \|U^* e_n^*\|_q^q < +\infty,
\]
which is a contradiction (here, \( (e_n^*) \) designs the unit vector basis of \( X^* = \ell_s \)). Finally, notice that this latter fact can be extended to \( \mathcal{L}_{s'} \)-spaces, since every infinite dimensional \( \mathcal{L}_r \)-space has a complemented subspace isomorphic to \( \ell_r \).

Remark 3.8. In general, the classes \( \mathcal{P}_{p,q} \) are not closed under closed subspaces. For an example, consider \( X = \ell_\infty; X^* \) has cotype 2 since it is an \( \mathcal{L}_1 \)-space. Therefore, \( \ell_\infty \) belongs to \( \mathcal{P}_{2,1} \). Nevertheless, \( \ell_1 \) is isometrically isomorphic to a closed subspace of \( \ell_\infty \) but \( \ell_1 \notin \mathcal{P}_{2,1} \) (Proposition 3.6).

We finish with an application of the preceding results to the theory of vector measures. It is well known that only finite dimensional Banach spaces \( X \) have the property that every relatively compact subset of \( X \) lies inside the range of a vector measure of bounded variation [13, Theorem 3.6]. We are going to prove that if we replace compact (= \( \infty \)-compact) with \( p \)-compact \( (p \leq 2) \), then there exist infinite dimensional Banach spaces with such a property. It suffices to deal with \( p \)-null sequences instead of relative \( p \)-compact sets (Theorem 2.3), so the following lemma gains importance for getting our objective:

Lemma ([12, Lemma 2]). Let \( \hat{x} \) be a bounded sequence in a Banach space \( X \).

1) The rank of \( \hat{x} \) lies inside the range of a \( X^{**} \)-valued measure of bounded variation if and only if the operator \( U_{\hat{x}} : \ell_1 \rightarrow X \) is 1-integral.

2) If \( U_{\hat{x}} \) is 1-nuclear, then the rank of \( \hat{x} \) lies inside the range of an \( X \)-valued measure of bounded variation.

Proposition 3.9. Let \( X \) be a Banach space and let \( 1 \leq p < +\infty \). Every relatively \( p \)-compact subset of \( X \) lies inside the range of a vector measure of bounded variation if and only if \( X \) belongs to \( \mathcal{P}_{p,1} \).

Proof. If every relatively \( p \)-compact subset of \( X \) lies inside the range of a vector measure of bounded variation, the previous lemma guarantees that the operator \( \Phi : \hat{x} \in c_{0,p}(X) \mapsto U_{\hat{x}} \in \mathcal{L}_1(\ell_1, X) \) is well defined. A standard argument shows
that $\Phi$ has closed graph and, therefore, is continuous. Actually, $\Phi$ maps $c_0,p(X)$ into $N_1(\ell_1,X)$, since $\hat{x} = k_p - \lim_{n} \hat{x}(n)$ whenever $\hat{x} \in c_0,p(X)$ and using that $N_1(\ell_1,X)$ is isometrically isomorphic to a closed subspace of $\ell_1(\ell_1,X)$ [11, p. 132]. Obviously, this implies that every relatively $p$-compact subset of $X$ is relatively 1-compact.

For the converse, notice that the operator $U_{\hat{x}}$ belongs to $K_1(\ell_1,X)$ whenever $\hat{x}$ is a relatively $p$-compact sequence in $X$. Since $K_1(\ell_1,X)$ can be isometrically identified with $N_1(\ell_1,X)$, the proof is concluded via [12, Lemma 2]. □

In view of this result and Proposition 3.6, for a fixed $p \in [1,2)$ (respectively, $p = 2$), every Banach space $X$ such that $X^*$ has cotype $s > p'$ (respectively, $s = 2$) satisfies that its relatively $p$-compact sets lies in the range of an $X$-valued measure of bounded variation.

ACKNOWLEDGEMENT

The authors wish to thank the referee, whose suggestions enhanced the clarity of this paper.

REFERENCES


Department of Mathematics (Faculty of Experimental Sciences), Campus Universitario de El Carmen, Avenida de las Fuerzas Armadas s/n, 21071 Huelva, Spain

E-mail address: candido@uah.es

Department of Mathematics (Faculty of Experimental Sciences), Campus Universitario de El Carmen, Avenida de las Fuerzas Armadas s/n, 21071 Huelva, Spain

Current address: Departamento de Matemática Aplicada I (Escuela Técnica Superior de Arquitectura), Avenida de Reina Mercedes, 2, 41012 Sevilla, Spain

E-mail address: jmdelga@us.es