INEQUIVALENT MEASURES OF NONCOMPACTNESS
AND THE RADIUS OF THE ESSENTIAL SPECTRUM

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Abstract. The Kuratowski measure of noncompactness \( \alpha \) on an infinite dimensional Banach space \((X, \| \cdot \|)\) assigns to each bounded set \( S \) in \( X \) a nonnegative real number \( \alpha(S) \) by the formula
\[
\alpha(S) = \inf \{ \delta > 0 \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } \text{diam}(S_i) \leq \delta, \text{ for } 1 \leq i \leq n < \infty \}.
\]

In general a map \( \beta \) which assigns to each bounded set \( S \) in \( X \) a nonnegative real number and which shares most of the properties of \( \alpha \) is called a homogeneous measure of noncompactness or homogeneous MNC. Two homogeneous MNC's \( \beta \) and \( \gamma \) on \( X \) are called equivalent if there exist positive constants \( b \) and \( c \) with \( b\beta(S) \leq \gamma(S) \leq c\beta(S) \) for all bounded sets \( S \subset X \). There are many results which prove the equivalence of various homogeneous MNC's. Working with \( X = \ell^p(\mathbb{N}) \) where \( 1 \leq p \leq \infty \), we give the first examples of homogeneous MNC's which are not equivalent.

Further, if \( X \) is any complete, infinite dimensional Banach space and \( L : X \rightarrow X \) is a bounded linear map, one can define \( \rho(L) = \sup \{ |\lambda| \mid \lambda \in \text{ess}(L) \} \), where ess\((L)\) denotes the essential spectrum of \( L \). One can also define \( \beta(L) = \inf \{ \lambda > 0 \mid \beta(\lambda S) \leq \lambda\beta(S) \text{ for every } S \in B(X) \} \).

The formula \( \rho(L) = \lim_{m \to \infty} \beta(L^m)^{1/m} \) is known to be true if \( \beta \) is equivalent to \( \alpha \), the Kuratowski MNC; however, as we show, it is in general false for MNC's which are not equivalent to \( \alpha \). On the other hand, if \( B \) denotes the unit ball in \( X \) and \( \beta \) is any homogeneous MNC, we prove that
\[
\rho(L) = \limsup_{m \to \infty} \beta(L^m B)^{1/m} = \inf \{ \lambda > 0 \mid \lim_{m \to \infty} \lambda^{-m}\beta(L^m B) = 0 \}.
\]

Our motivation for this study comes from questions concerning eigenvectors of linear and nonlinear cone-preserving maps.

If \( (X, d) \) is a complete metric space and \( S \) is a bounded subset of \( X \), then K. Kuratowski [10] has defined \( \alpha(S) \), the Kuratowski measure of noncompactness of \( S \), by
\[
\alpha(S) := \inf \{ \delta > 0 \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } \text{diam}(S_i) \leq \delta, \text{ for } 1 \leq i \leq n < \infty \}.
\]
so-called in fixed point theory and functional analysis. Let us also mention the following Properties (K5), (K6), and (K7) make the Kuratowski MNC a very useful tool here.

Here \( \text{diam}(T) \) denotes the diameter of a set \( T \subset X \), namely

\[
\text{diam}(T) := \sup \{d(x, y) \mid x, y \in T\}.
\]

We shall denote by \( \mathcal{B}(X) \) the collection of all bounded subsets of \( X \). Kuratowski has shown, and it is straightforward to verify, that \( \alpha \) satisfies the following properties:

\begin{align*}
(\text{K1}) & \quad \alpha(S) = 0 \text{ if and only if } S \text{ is compact, for every } S \in \mathcal{B}(X); \\
(\text{K2}) & \quad \alpha(S) \leq \alpha(T) \text{ for every } S, T \in \mathcal{B}(X) \text{ with } S \subset T; \\
(\text{K3}) & \quad \alpha(S \cup \{x_0\}) = \alpha(S) \text{ for every } S \in \mathcal{B}(X) \text{ and } x_0 \in X; \text{ and} \\
(\text{K4}) & \quad \alpha(S) = \alpha(S) \text{ for every } S \in \mathcal{B}(X).
\end{align*}

If \( S \) and \( T \) are subsets of a real or complex Banach space \( (X, \| \cdot \|) \) and \( \lambda \) is a scalar, we shall let \( \text{co}(S) \) denote the convex hull of \( S \), namely the smallest convex set containing \( S \), and we shall write \( S + T := \{s + t \mid s \in S \text{ and } t \in T\} \) and \( \lambda S := \{\lambda s \mid s \in S\} \). G. Darbo \[6\] has observed that, assuming the metric on \( X \) is the usual one obtained from the norm \( \| \cdot \| \), the following properties hold:

\begin{align*}
(\text{K5}) & \quad \alpha(\text{co}(S)) = \alpha(S) \text{ for every } S \in \mathcal{B}(X); \\
(\text{K6}) & \quad \alpha(S + T) \leq \alpha(S) + \alpha(T) \text{ for every } S, T \in \mathcal{B}(X); \text{ and} \\
(\text{K7}) & \quad \alpha(\lambda S) = |\lambda|\alpha(S) \text{ for every } S \in \mathcal{B}(X) \text{ and every scalar } \lambda.
\end{align*}

Properties (K5), (K6), and (K7) make the Kuratowski MNC a very useful tool in fixed point theory and functional analysis. Let us also mention the following so-called set-additivity property, which holds in any metric space:

\begin{align*}
(\text{K8}) & \quad \alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\} \text{ for every } S, T \in \mathcal{B}(X).
\end{align*}

If \( (X, \| \cdot \|) \) is a real or complex Banach space, we shall say that a map \( \beta : \mathcal{B}(X) \to [0, \infty) \) is a homogeneous measure of noncompactness on \( X \) or homogeneous MNC if \( \beta \) satisfies properties (K1)-(K7), with \( \beta \) replacing \( \alpha \) in these conditions. We shall say that \( \beta \) is a homogeneous, set-additive MNC if \( \beta \) satisfies properties (K1)-(K8), with \( \beta \) replacing \( \alpha \) in these conditions. Our terminology differs from some of the literature \[1, 2, 3, 18, \] where a map satisfying properties (K1)-(K6) is simply called an MNC. Of course these properties are not independent. For example, properties (K2), (K6), and (K7) imply property (K4).

If \( \beta \) and \( \gamma \) are homogeneous MNC’s on \( X \), we say that \( \beta \) dominates \( \gamma \) if there exists a number \( c > 0 \) such that \( \gamma(S) \leq c\beta(S) \) for every \( S \in \mathcal{B}(X) \). If \( \beta \) and \( \gamma \) are homogeneous MNC’s on \( X \) such that both \( \beta \) dominates \( \gamma \) and \( \gamma \) dominates \( \beta \), we say that \( \beta \) and \( \gamma \) are equivalent. There are many examples of homogeneous MNC’s (see \[1, 2, 3, 4, 11, 15, 16, 17, 18, \]), but up to now all known examples of homogeneous MNC’s on a given Banach space \( X \) are equivalent. This fact begs the following question.

**Question A.** Does there exist a Banach space \( (X, \| \cdot \|) \) for which there is a homogeneous (possibly set-additive) MNC \( \beta \) on \( X \) which is not equivalent to the Kuratowski MNC \( \alpha \) on \( X \)?

As we shall see below in Theorem \[7\] where a class of inequivalent MNC’s is constructed, Question A is answered in the affirmative.

If \( L : X \to X \) is a bounded linear map and \( \beta \) is a homogeneous MNC on \( X \), one can define

\[
\beta(L) := \inf \{\lambda \geq 0 \mid \beta(\lambda S) \leq \lambda \beta(S) \text{ for every bounded } S \subset X\},
\]

\[
\beta^*(L) := \limsup_{m \to \infty} \beta(L^m)^{1/m},
\]

(1)
where we set $\beta(L) = \infty$ if the set in the first line of (1) is empty. If it is in fact the case that $\beta(L) < \infty$, then one easily shows that
\[
(2) \quad \beta^*(L) = \lim_{m \to \infty} \beta(L^m)^{1/m} = \inf_{m \geq 1} \beta(L^m)^{1/m},
\]
which follows directly from the fact that $\beta(L^{m+n}) \leq \beta(L^m)\beta(L^n) < \infty$ for every $m \geq 1$ and $n \geq 1$. Lemma 4 below implies that if $\beta$ is equivalent to the Kuratowski MNC $\alpha$ on $X$, then there exists a constant $c > 0$, independent of $L$, with $\beta(L) \leq c\lambda \leq c\|L\| < \infty$. Additionally, if $\beta$ is equivalent to $\alpha$, the results of [14] imply that $\beta^*(L) = \rho(L)$, where $\rho(L)$ denotes the radius of the essential spectrum of $L$.

This suggests the following question.

**Question B.** Is it the case that $\beta^*(L) = \rho(L)$ for any homogeneous MNC $\beta$ on $X$, where $\rho(L)$ denotes the radius of the essential spectrum of $L$? If this is not the case, is there an analogous formula for $\rho(L)$ which holds for any homogeneous MNC $\beta$?

For a general homogeneous MNC $\beta$ which is not equivalent to $\alpha$, we shall establish in Theorem 8 below that it may happen that $\beta^*(L) \neq \rho(L)$, and in fact it may happen that $\beta(L^m) = \infty$ for all $m \geq 1$. Elsewhere [14], we shall construct an example for which
\[
\lim_{m \to \infty} \inf_{m \to \infty} \beta(L^m)^{1/m} < \lim_{m \to \infty} \beta(L^m)^{1/m} = \infty.
\]
In such cases $\beta^*(L) = \infty$ while $\rho(L) < \infty$. As will be shown in Theorem 10 below, in place of the quantity $\beta^*(L)$ the appropriate quantity to consider is
\[
(3) \quad \beta^*(L) := \lim_{m \to \infty} \sup_{m \to \infty} \beta(L^m B_1(0))^{1/m} = \inf \{\lambda > 0 \mid \lim_{m \to \infty} \lambda^{-m} \beta(L^m B_1(0)) = 0\},
\]
as it is the case that $\beta^*(L) = \rho(L)$ for every homogeneous MNC $\beta$ and every bounded linear operator $L$ on $X$. We denote
\[
(4) \quad B_r(x) := \{y \in X \mid \|y - x\| < r\}
\]
both here and below.

**Remark.** In order for $\rho(L)$ to be defined above, one needs to have a linear operator on a complex Banach space. Suppose instead that $X$ is a real Banach space, $\beta$ is a homogeneous MNC on $X$, and $L : X \to X$ is a bounded linear map. The complexification $\hat{X}$ of $X$ equals $\{(u, v) | u, v \in X\}$. If one identifies $(u, v)$ with $u + iv$ where $i^2 = -1$, and defines
\[
\|u + iv\| := \sup_{0 \leq \theta \leq 2\pi} \|(\cos \theta)u + (\sin \theta)v\|,
\]
then $\hat{X}$ becomes a complex Banach space. The linear map $L$ then extends to a complex linear map $\hat{L}$ on $\hat{X}$ by $\hat{L}(u + iv) = Lu + iLv$. It is also the case that $\beta$ extends to a homogeneous MNC $\hat{\beta}$ on $\hat{X}$ as follows. For $x = u + iv \in \hat{X}$ define $\text{Re}(x) := u$, and for $\hat{S} \in B(\hat{X})$ define $\text{Re}(\hat{S}) := \{\text{Re}(x) \mid x \in \hat{S}\}$ and set
\[
(5) \quad \hat{\beta}(\hat{S}) := \sup_{0 \leq \theta \leq 2\pi} \beta(\text{Re}(e^{i\theta}\hat{S})).
\]
One can prove that $\hat{\beta}$ is a homogeneous MNC on the complex Banach space $\hat{X}$, that $\hat{\beta}(\hat{L}^m) = \beta(L^m)$, and that $\hat{\beta}(\hat{L}^m \hat{B}_1(0)) = \beta(L^m B_1(0))$, where $\hat{B}_1(0)$ (respectively,
\[ B_1(0) \] denotes the unit ball in \( \hat{X} \) (respectively, \( X \)). It follows that
\[
(6) \quad \hat{\beta}^*(\hat{L}) = \beta^*(L), \quad \hat{\beta}^*(\hat{L}) = \beta^*(L)
\]
both hold. We remark also that if \( \alpha \) denotes the Kuratowski MNC on a real Banach space \( X \) and \( \hat{\alpha} \) denotes its complexification as above, then \( \hat{\alpha} \) is in fact the Kuratowski MNC on \( \hat{X} \). We omit the proofs of these results, which are straightforward for the most part, except for the proof that \( \hat{\alpha} \) is the Kuratowski MNC on \( \hat{X} \); this is given as Proposition 11.

Our interest in Questions A and B and the related issues above arises from the question of the “correct” definition of the “cone essential spectral radius,” denoted \( \rho_C(f) \), for a map \( f : C \to C \). Here \( C \) is a closed cone in a Banach space and \( f \) is a continuous, homogeneous, order-preserving map. This question is, in turn, related to the problem of existence of an eigenvector of \( f \) with eigenvalue equal to \( r_C(f) \), the “cone spectral radius of \( f \),” and to showing that \( \rho_C(f) \leq r_C(f) \); see [11] and [17]. In future work, related to this paper, we shall discuss deficiencies in the definition of \( \rho_C(f) \) in [11], [17], and theorems about existence of eigenvectors of \( f \).

Theorems 7, 8, and 10 are the main results of this paper. In Theorem 7 we shall prove a large class of such inequivalent MNC’s \( \gamma_Y \) on \( Y \) which is not equivalent to the Kuratowski MNC, thereby answering Question A in the affirmative. In fact, we provide a large class of such inequivalent MNC’s \( \gamma_Y \). Much more general results for other spaces are given in [12], but it seems worthwhile to illustrate our approach here in this relatively simple case with a self-contained proof. (In fact we use some ideas from [12] in the example considered in Theorem 8.) In Theorem 8 we study the quantities \( \gamma_Z(\Lambda^m) \) and \( \gamma_Z^\#(\Lambda) \) for homogeneous, set-additive MNC’s \( \gamma_Z \) on \( Z = \ell^p(\mathbb{N} \times \mathbb{N}) \) related to the MNC’s \( \gamma_Y \) of Theorem 7 for a particular shift operator \( \Lambda \) on the space \( Z \). We demonstrate the pathological features of these quantities noted above, in particular that in general \( \gamma_Z^\#(\Lambda) \neq \rho(\Lambda) \), which thereby gives a negative answer to the first part of Question B. In Theorem 10 we prove for a general homogeneous MNC \( \beta \) on a Banach space \( X \) that \( \beta^*(L) \) rather than \( \beta^*(L) \) is the “correct” quantity to consider in studying \( \rho(L) \). In particular we show that \( \beta^*(L) = \rho(L) \) always holds for all bounded linear operators on \( X \), thus providing an affirmative answer to the second part of Question B.

Due to the following result proved in [12], the issue of whether or not a homogeneous MNC satisfies the set-additivity property (K8) is often unimportant.

**Proposition 1** (see [12]). Let \( (X, \| \cdot \|) \) be a Banach space and \( \beta \) a homogeneous MNC on \( X \). For \( S \in \mathcal{B}(X) \), define \( \gamma(S) \) by
\[
(7) \quad \gamma(S) := \inf \{ \max_{1 \leq i \leq n} \beta(S_i) \mid S = \bigcup_{i=1}^{n} S_i \text{ for some } S_i \text{ with } 1 \leq i \leq n < \infty \}.
\]
Then \( \gamma \) is a homogeneous, set-additive MNC on \( X \) with \( \gamma(S) \leq \beta(S) \) for all bounded \( S \subset X \). Moreover, \( \gamma = \beta \) if \( \beta \) itself is a homogeneous, set-additive MNC.

Before presenting our main results we make some fundamental observations.

**Proposition 2.** Let \( (X, \| \cdot \|) \) be a Banach space and \( \beta \) a homogeneous MNC on \( X \). Then the Kuratowski MNC \( \alpha \) dominates \( \beta \).
Proposition 3

Let $\beta = \beta(B_r(0))$, recalling the notation \([4]\). Then homogeneity implies that $\beta(B_r(0)) = rc$. If $S \in \mathcal{B}(X)$ and $d := \alpha(S)$, then given $\varepsilon > 0$, there exists a finite collection of sets $S_1, S_2, \ldots, S_n$ with $S = \bigcup_{i=1}^n S_i$, and with $\text{diam}(S_i) \leq d + \varepsilon$ for $1 \leq i \leq n$. For each $i$ select $x_i \in S_i$ and define $T := \{x_i | 1 \leq i \leq n\}$. Note that $S \subset T + B_{d+\varepsilon}(0)$, so property (K6), along with (K1) and (K2), implies that
\[
\beta(S) \leq \beta(T) + \beta(B_{d+\varepsilon}(0)) = \beta(B_{d+\varepsilon}(0)) = (d + \varepsilon)c.
\]
Since $\varepsilon > 0$ is arbitrary, we conclude that $\beta(S) \leq cd = c\alpha(S)$. \hfill \Box

The next result was obtained independently by Furi and Vignoli in \([7]\) and by Nussbaum in Section A of \([16]\).

Proposition 3 (see \([7]\) and Section A of \([15]\)). Let $(X, \|\|)$ be an infinite dimensional Banach space. If $Q := \{x \in X \mid \|x\| \leq 1\}$ and if $\alpha$ denotes the Kuratowski MNC on $X$, then $\alpha(Q) = 2$.

Lemma 4 below is an easy result; see \([14]\) or Section A of \([16]\). However, as we shall see later, Lemma 4 may fail for general homogeneous MNC’s.

Lemma 4 (see \([14]\) or Section A of \([15]\)). Let $(X_i, \|\|_i)$, for $i = 1, 2$, be Banach spaces, let $\alpha_i$ denote the Kuratowski MNC on $X_i$, and let $L : X_1 \to X_2$ be a bounded linear map. Define
\[
\alpha(L) := \inf \{\lambda \geq 0 \mid \alpha_2(LS) \leq \lambda \alpha_1(S) \text{ for every bounded } S \subset X_1\}.
\]
Then we have $\alpha(L) \leq \|L\|$. Further, if $\beta_i$ is a homogeneous MNC on $X_i$, with $\beta_i$ equivalent to $\alpha_i$ for $i = 1, 2$, then there exists a constant $c > 0$, independent of $L$, such that
\[
\beta_2(LS) \leq c\alpha(L)\beta_1(S) \leq c\|L\|\beta_1(S)
\]
for every $S \in \mathcal{B}(X_1)$.

Our next lemma is true in greater generality (see \([12]\)), but the following version will suffice for our purposes.

Lemma 5. Let $(X_i, \|\|_i)$, for $i = 1, 2$, be Banach spaces, and let $L : X_1 \to X_2$ be a one-one, continuous linear map of $X_1$ onto $X_2$. If $\beta_2$ is a homogeneous MNC on $X_2$, define, for $S \in \mathcal{B}(X_1)$,
\[
\tilde{\beta}_2(S) := \beta_2(LS).
\]
Then $\tilde{\beta}_2$ is a homogeneous MNC on $X_1$, and $\tilde{\beta}_2$ is set-additive if $\beta_2$ is set-additive. If $\alpha_i$ denotes the Kuratowski MNC on $X_i$ and if $\beta_2$ is equivalent to $\alpha_2$, then $\tilde{\beta}_2$ is equivalent to $\alpha_1$.

Proof. The fact that $\tilde{\beta}_2$ is a homogeneous (set-additive) MNC on $X_1$ follows easily from the fact that $L$ is a linear homeomorphism of $X_1$ onto $X_2$. Details are left to the reader.

To see that $\tilde{\beta}_2$ is equivalent to $\alpha_1$ if $\beta_2$ is equivalent to $\alpha_2$, observe that $\tilde{\beta}_2$ is equivalent to $\tilde{\alpha}_2$, where $\tilde{\alpha}_2(S) := \alpha_2(LS)$. Thus it suffices to prove that $\tilde{\alpha}_2$ is equivalent to $\alpha_1$. However, if $S$ is a bounded subset of $X_1$, then Lemma 4 implies that $\alpha_2(LS) \leq \|L\|\alpha_1(S)$ and $\alpha_1(S) = \alpha_1(L^{-1}LS) \leq \|L^{-1}\|\alpha_2(LS)$. This proves that $\tilde{\alpha}_2$ and $\alpha_1$ are equivalent. \hfill \Box

The following lemma will be convenient in establishing Theorem 7
Lemma 6. Let \((X_i, \| \cdot \|_i)\), for \(i = 1, 2\), be Banach spaces, let \(\alpha_i\) denote the Kuratowski MNC on \(X_i\), and let \(L : X_1 \to X_2\) be a one-one, continuous linear map of \(X_1\) onto \(X_2\). Suppose there exists a homogeneous MNC \(\beta_2\) on \(X_2\) which is inequivalent to \(\alpha_2\). Then there exists a homogeneous, set-additive MNC \(\gamma_2\) on \(X_2\) which is inequivalent to \(\alpha_2\). Further, there exists a homogeneous, set-additive MNC \(\gamma_1\) on \(X_1\) which is inequivalent to \(\alpha_1\).

Proof. Proposition 2 implies that \(\alpha_2\) dominates \(\beta_2\), so there must exist a sequence of bounded sets \(S_n \subset X_2\) with \(\alpha_2(S_n) > 0\) and \(\lim_{n \to \infty} \frac{\beta_2(S_n)}{\alpha_2(S_n)} = 0\). Let \(\gamma_2\) be the homogeneous, set-additive MNC derived from \(\beta_2\) as in Proposition 1. Then it is immediate that \(\gamma_2(S) \leq \beta_2(S)\) for all \(S \in \mathcal{B}(X_2)\), and so \(\lim_{n \to \infty} \frac{\gamma_2(S_n)}{\alpha_2(S_n)} = 0\). Define \(\tilde{\gamma}_2\) and \(\tilde{\alpha}_2\) as in Lemma 5 so \(\tilde{\gamma}_2(T) := \gamma_2(LT)\) and \(\tilde{\alpha}_2(T) := \alpha_2(LT)\) for \(T \in \mathcal{B}(X_1)\). Then Lemma 3 implies that \(\tilde{\gamma}_2\) and \(\tilde{\alpha}_2\) are homogeneous, set-additive MNC’s on \(X_1\) and that \(\tilde{\alpha}_2\) is equivalent to \(\alpha_1\), so in particular there exists \(c > 0\) such that \(\tilde{\alpha}_2(T) \leq c\alpha_1(T)\) for every \(T \in \mathcal{B}(X_1)\). If we define \(T_n := L^{-1}S_n\), it follows that

\[
\lim_{n \to \infty} \left(\frac{\tilde{\gamma}_2(T_n)}{\tilde{\alpha}_2(T_n)}\right) \leq c \lim_{n \to \infty} \left(\frac{\tilde{\gamma}_2(T_n)}{\tilde{\alpha}_2(T_n)}\right) = c \lim_{n \to \infty} \left(\frac{\gamma_2(S_n)}{\alpha_2(S_n)}\right) = 0,
\]

so \(\tilde{\gamma}_2\) and \(\alpha_1\) are inequivalent. If we define \(\gamma_1 := \tilde{\gamma}_2\), the proof is complete. \(\square\)

Let \(1 \leq p \leq \infty\) and let \(\mathbb{N}\) denote the natural numbers. We define the Banach space \(Y := \ell^p(\mathbb{N})\) in the usual way: Elements \(y \in Y\) are maps \(y : \mathbb{N} \to \mathbb{C}\) such that \(\|y\|_Y := (\sum_{i=1}^{\infty} |y(i)|^p)^{1/p} < \infty\). As usual, we interpret \(\|y\|_Y := \sup_{i \in \mathbb{N}} |y(i)|\) if \(p = \infty\). (We remark that if we instead take the corresponding real Banach space of maps \(y : \mathbb{N} \to \mathbb{R}\), then the construction below is still valid with the obvious changes.) Similarly, the Banach space \(Z := \ell^p(\mathbb{N} \times \mathbb{N})\) is the set of maps \(z : \mathbb{N} \times \mathbb{N} \to \mathbb{C}\) such that \(\|z\|_Z := (\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |z(i,j)|^p)^{1/p} < \infty\), and again with the corresponding supremum norm if \(p = \infty\). It is well-known that there is a one-one map \(\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) of \(\mathbb{N} \times \mathbb{N}\) onto \(\mathbb{N}\), and that \(\sigma\) induces a linear isometry \(L_\sigma : Y \to Z\) by composition, namely \(L_\sigma y := y \circ \sigma\). We want to prove that there is a homogeneous, set-additive MNC \(\gamma_Y\) on \(Y\) which is inequivalent to the Kuratowski MNC \(\alpha_Y\) on \(Y\). By Lemma 3 it suffices to prove that there exists a homogeneous MNC \(\beta_Z\) on \(Z\) which is inequivalent to the Kuratowski MNC \(\alpha_Z\) on \(Z\).

Theorem 7. Let \(1 \leq p \leq \infty\) and let \(Y\) denote the Banach space \(\ell^p(\mathbb{N})\) with the usual norm. Let \(\alpha_Y\) denote the Kuratowski MNC on \(Y\). Then there exists a homogeneous, set-additive MNC \(\gamma_Y\) on \(Y\) which is inequivalent to \(\alpha_Y\).

Proof. With \(Z = \ell^p(\mathbb{N} \times \mathbb{N})\) and with the norm \(\| \cdot \|_Z\) as above, let \(\alpha_Z\) denote the Kuratowski MNC on \(Z\). By the remarks above, it suffices to prove that there exists a homogeneous MNC \(\beta_Z\) on \(Z\) which is inequivalent to \(\alpha_Z\).

For simplicity, we shall denote \(\alpha_Z\) and \(\beta_Z\) simply by \(\alpha\) and \(\beta\), respectively, and we denote \(\mathcal{B} := \mathcal{B}(Z)\), the set of bounded subsets of \(Z\). Also for simplicity, in what follows we shall assume that \(p < \infty\), as the arguments for \(p = \infty\) are similar.

Let \(a_n\), for \(n \geq 1\), be a nonincreasing sequence of positive reals with \(a_1 \leq 1\) and \(\lim_{n \to \infty} a_n = 0\). Define a Banach space \((\hat{Z}, \| \cdot \|_{\hat{Z}})\) to be the set of maps \(z : \mathbb{N} \times \mathbb{N} \to \mathbb{C}\) such that

\[
\|z\|_{\hat{Z}} := \left(\sum_{i=1}^{\infty} a_i^p \sum_{j=1}^{\infty} |z(i,j)|^p\right)^{1/p} < \infty,
\]
and let $\tilde{\alpha}$ denote the Kuratowski MNC on $\tilde{Z}$. Note that $Z \subset \tilde{Z}$ and that

$$\|z\| \leq \|\tilde{z}\|$$

for all $z \in Z$. For each integer $n \geq 1$ define the linear projection $P_n : Z \rightarrow Z$ by setting $P_n z = x$, where

$$x(i,j) = \begin{cases} z(i,j), & \text{for } 1 \leq i \leq n, \\ 0, & \text{for } i > n. \end{cases}$$

Note also that $P_n : \tilde{Z} \rightarrow \tilde{Z}$ is a projection and that $P_n \tilde{Z} = P_n Z$. It is easy to see that, for all $z \in Z$,

$$\|P_n z\|_Z \leq \|z\|_Z, \quad \|P_n z\|_{\tilde{Z}} \leq \|z\|_{\tilde{Z}}, \quad \|P_n z\|_Z \leq a_n^{-1}\|P_n \tilde{z}\|_{\tilde{Z}},$$

and in fact the second and third inequalities in (9) are valid for every $z \in \tilde{Z}$. Thus by Lemma 4, using (8) and (9), we have that

$$\alpha(S) \leq \alpha(S), \quad \alpha(P_n S) \leq \alpha(S), \quad \tilde{\alpha}(P_n S) \leq \tilde{\alpha}(S),$$

for every $S \in \mathcal{B}$. We now define $\mathcal{A} \subset \mathcal{B}$ by

$$\mathcal{A} := \{S \in \mathcal{B} \mid \lim_{n \rightarrow \infty} \alpha((I - P_n)S) = 0\}.$$  

The reader can easily verify that if $S, T \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, then the sets $\text{co}(S)$, $\lambda S$, $S$, and $S + T$ are all elements of $\mathcal{A}$. Furthermore, if $S \in \mathcal{B}$, then $P_n S \in \mathcal{A}$ for every integer $n \geq 1$.

With these preliminaries we define $\beta : \mathcal{B} \rightarrow [0, \infty)$ by

$$\beta(S) := \inf \{\tilde{\alpha}(A) + \alpha(B) \mid S \subset A + B, \text{ for some } A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}.$$  

We claim that $\beta$ is a homogeneous MNC on $Z$, that $\beta$ is inequivalent to $\alpha$, and that $\beta(S) = \tilde{\alpha}(S)$ for all $S \in \mathcal{A}$.

Observe first that for any $S \in \mathcal{B}$, if we take $A := \{0\}$ and $B := S$ in equation (12), we see that $\beta(S) \leq \alpha(S)$.

If $S \in \mathcal{A}$ and we take $A := S$ and $B := \{0\}$ in (12), we see that $\beta(S) \leq \tilde{\alpha}(S)$. On the other hand, if $S \in \mathcal{A}$ and $S \subset A + B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have, using the first inequality in (10), that

$$\tilde{\alpha}(S) \leq \tilde{\alpha}(A) + \alpha(B) \leq \tilde{\alpha}(A) + \alpha(B),$$

so we obtain from (12) that $\tilde{\alpha}(S) \leq \beta(S)$. We conclude that $\tilde{\alpha}(S) = \beta(S)$ for $S \in \mathcal{A}$, as claimed.

The fact that $\beta$ satisfies property (K2) (with $\beta$ replacing $\alpha$) is obvious. It follows that if $S \in \mathcal{B}$, then $\beta(S) \leq \beta(\text{co}(S))$. On the other hand, given $\varepsilon > 0$, select $A \in \mathcal{A}$ and $B \in \mathcal{B}$ so that $S \subset A + B$ and $\beta(S) \leq \tilde{\alpha}(A) + \alpha(B) < \beta(S) + \varepsilon$. Note that $\text{co}(A) + \text{co}(B)$ is a convex set containing $S$, so $\text{co}(S) \subset \text{co}(A) + \text{co}(B)$. Since $\text{co}(A) \in \mathcal{A}$, we conclude that

$$\beta(\text{co}(S)) \leq \tilde{\alpha}(\text{co}(A)) + \alpha(\text{co}(B)) = \tilde{\alpha}(A) + \alpha(B) < \beta(S) + \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, $\beta(\text{co}(S)) = \beta(S)$. Thus $\beta$ satisfies property (K5).

If $S, T \in \mathcal{B}$ and $\varepsilon > 0$, select $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$ such that $S \subset A_1 + B_1$ and $T \subset A_2 + B_2$, with $\tilde{\alpha}(A_1) + \alpha(B_1) \leq \beta(S) + \varepsilon$ and $\tilde{\alpha}(A_2) + \alpha(B_2) \leq \beta(T) + \varepsilon$. Thus $\beta(\text{co}(S + T)) \leq \beta(\text{co}(S)) + \beta(\text{co}(T)) \leq 2(\beta(S) + \varepsilon)$. Hence $\beta$ satisfies property (K3).
and so property (K3) holds.

Thus

\[ \alpha \] is compact, then

\[ \beta(S + T) \leq \alpha(B) + \alpha(B_1 + B_2) \]

\[ \leq \alpha(A_1) + \alpha(B_1) + \alpha(A_2) + \alpha(B_2) \leq \beta(S) + \beta(T) + 2 \varepsilon. \]

Since \( \varepsilon > 0 \) is arbitrary, we see that \( \beta(S + T) \leq \beta(S) + \beta(T) \), so \( \beta \) satisfies property (K6).

The fact that \( \beta \) satisfies property (K7), namely \( \beta(\lambda S) = |\lambda| \beta(S) \) for all \( S \in \mathcal{B} \) and \( \lambda \in \mathbb{C} \), follows easily from the definition (12) of \( \beta \) and the fact that \( \tilde{\alpha} \) and \( \alpha \) satisfy property (K7). Details are left to the reader.

If \( S \in \mathcal{B} \), property (K2) implies that \( \beta(S) \leq \beta(\tilde{S}) \). On the other hand, we have for any \( \varepsilon > 0 \) that \( \tilde{S} \subseteq S + B_\varepsilon(0) \). Thus from the homogeneity of \( \beta \) and from properties (K2) and (K6), we have that

\[ \beta(\tilde{S}) \leq \beta(S) + \beta(B_\varepsilon(0)) = \beta(S) + \varepsilon \beta(B_1(0)). \]

This shows that \( \beta(\tilde{S}) \leq \beta(S) \) and proves property (K4).

If \( T \in \mathcal{B} \) and \( \overline{T} \) is compact, then \( \beta(T) = 0 \) because \( \beta(T) \leq \alpha(T) = 0 \). If \( \overline{T} \) is compact and \( S \in \mathcal{B} \), we claim that \( \beta(S + T) = \beta(S) \), which certainly implies that property (K3) is satisfied. Property (K2) implies that \( \beta(S) \leq \beta(S + T) \). To see the opposite inequality, select \( x_0 \in S \), define \( \Gamma := (T \cup \{x_0\}) + \{-x_0\} \), and note that \( \Gamma \) is compact and that \( S + T \subseteq S + \Gamma \). Therefore

\[ \beta(S + T) \leq \beta(S + \Gamma) \leq \beta(S) + \beta(\Gamma) \leq \beta(S) + \alpha(\Gamma) = \beta(S), \]

and so property (K3) holds.

Note that we do not claim that \( \beta \) necessarily satisfies property (K8).

We now establish property (K1), which, along with the inequivalence of \( \beta \) and \( \alpha \), is the main point of our construction. First, as noted above, if \( S \in \mathcal{B} \) and \( \tilde{S} \) is compact, then \( \beta(S) = 0 \). Now suppose, conversely, that \( S \in \mathcal{B} \) and \( \beta(S) = 0 \). We have to show that \( \alpha(S) = 0 \), which implies that \( \tilde{S} \) is compact. Given \( \varepsilon > 0 \), equation (12) implies that there exist \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) with \( S \subseteq A + B \) and \( \alpha(A) + \alpha(B) < \varepsilon \). Equation (11) implies that there exists an integer \( N \) with \( \alpha((I - P_N) A) < \varepsilon \). It follows that \( (I - P_N) S \subseteq (I - P_N) A + (I - P_N) B \) and so

\[ \alpha((I - P_N) S) \leq \alpha((I - P_N) A) + \alpha((I - P_N) B) \]

\[ \leq \alpha((I - P_N) A) + \alpha(B) + \alpha(P_N B) \]

\[ \leq \alpha((I - P_N) A) + 2 \alpha(B) < 3 \varepsilon, \]

where the second inequality in (10) has been used. Next, for \( N \) as above, define \( \kappa := a_N \varepsilon \leq \varepsilon \) and select \( A' \in \mathcal{A} \) and \( B' \in \mathcal{B} \) with \( S \subseteq A' + B' \) such that \( \alpha(A') + \alpha(B') < \kappa \). The inequalities in (10) imply that \( \tilde{\alpha}(P_N A') < \kappa \) and \( \alpha(P_N B') < \kappa \), and also \( \alpha(P_N A') \leq a_N^{-1} \tilde{\alpha}(P_N A') \). It follows that

\[ \alpha(P_N S) \leq \alpha(P_N A') + \alpha(P_N B') \]

\[ \leq \frac{\tilde{\alpha}(P_N A')}{a_N} + \alpha(P_N B') < \left( \frac{1}{a_N} + 1 \right) \kappa \leq 2 \varepsilon. \]

Thus

\[ \alpha(S) \leq \alpha((I - P_N) S) + \alpha(P_N S) < 3 \varepsilon + 2 \varepsilon = 5 \varepsilon, \]

and since \( \varepsilon > 0 \) is arbitrary, \( \alpha(S) = 0 \).
Finally, we show that $\beta$ is inequivalent to $\alpha$. For any $n \geq 1$ define
\begin{equation}
Z_n := \{ z \in Z \mid z(i, j) = 0 \text{ for } i \neq n \}, \quad S_n := \{ z \in Z_n \mid \|z\| \leq 1 \}.
\end{equation}
Note that $(Z_n, \|\cdot\|_Z)$ and $(Z_n, \|\cdot\|_Z)$ are infinite dimensional Banach spaces, and in fact $\|z\|_Z = a_n^{-1}\|z\|_Z$ for every $z \in Z_n$. Thus Proposition 3 implies that $\alpha(S_n) = 2$, and also, since $S_n$ is also the closed ball of radius $a_n$ in the space $(Z_n, \|\cdot\|_Z)$, Proposition 3 implies that $\tilde{\alpha}(S_n) = 2a_n$. Further, $S_n \in \mathcal{A}$ and so we have that $\tilde{\alpha}(S_n) = \beta(S_n)$, as noted earlier in this proof. Thus
\begin{equation*}
\lim_{n \to \infty} \left( \frac{\beta(S_n)}{\alpha(S_n)} \right) = \lim_{n \to \infty} a_n = 0,
\end{equation*}
and it follows that $\beta$ and $\alpha$ are inequivalent.

The above theorem suggests the following general question.

Open Question. Is it the case that for any infinite dimensional Banach space $(X, \|\cdot\|)$ there exists a homogeneous (possibly set-additive) MNC $\beta$ which is not equivalent to the Kuratowski MNC $\alpha$ on $X$?

In [12], we provide a partial answer to the above Open Question, by showing that for a large class of Banach spaces of interest in analysis, there does exist a homogeneous, set-additive MNC which is not equivalent to the Kuratowski MNC. In particular, this is verified for general Hilbert spaces; for the Banach spaces $L^p(\Omega, \Sigma, \mu)$, where $(\Omega, \Sigma, \mu)$ is a general measure space and $1 \leq p \leq \infty$; for $C(K)$, where $K$ is a compact Hausdorff space; and for the Sobolev space $W^{m,p}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set. We believe, however, that an answer (positive or negative) to the Open Question is probably difficult and probably will involve techniques beyond those used in [12].

Our next main result studies $\beta(\Lambda^m)$ and $\beta^\#(\Lambda)$ and the corresponding quantities for $\gamma$, for the MNC $\beta = \beta_Z$ in the proof of Theorem 7 and the homogeneous, set-additive MNC $\gamma = \gamma_Z$ derived from $\beta$ by Proposition 1. Recall the definitions and properties (1, 2), of $\beta(\Lambda^m)$ and $\beta^\#(\Lambda)$. We shall take $\Lambda$ to be a particular shift operator.

Theorem 8. With $Z = \ell^p(\mathbb{N} \times \mathbb{N})$, where $1 \leq p \leq \infty$, define $\Lambda : Z \to Z$ by $\Lambda z = x$, where $x(i, j) = z(i+1, j)$ for every $(i, j) \in \mathbb{N} \times \mathbb{N}$. Also fix a nonincreasing sequence $\{a_n\}_{n=1}^\infty$ as in the proof of Theorem 7, with $\beta$ the homogeneous MNC on $Z$ given by equation (12), and $\gamma$ the homogeneous, set-additive MNC derived from $\beta$ as in Proposition 1. Then for every $m \geq 1$,
\begin{equation}
\beta(\Lambda^m) = \gamma(\Lambda^m) = \mu_m := \sup_{n \geq 1} \left( \frac{a_n}{a_{n+m}} \right),
\end{equation}
with the above formula serving as the definition of $\mu_m \in (1, \infty]$.

Remark. It is easily seen that $\|\Lambda^m\|_{\mathcal{L}(Z)} = 1$ for every $m \geq 1$, so $\alpha(\Lambda^m) \leq 1$ by Lemma 4, where $\alpha$ is the Kuratowski MNC on $Z$. (Here and below we let $\|\cdot\|_{\mathcal{L}(X)}$ denote the operator norm associated to a space $X$.) In fact one easily sees that $\alpha(\Lambda^m) = 1$ for every $m$, and so by earlier remarks we have that $\alpha^\#(\Lambda) = \rho(\Lambda) = 1$.

Proof of Theorem 8. Let $m \geq 1$ be an integer which will be fixed for the remainder of the proof. Generally, we shall use the notation and constructions from the proof of Theorem 7 assuming as well that $p < \infty$. 

Let \( S \in \mathcal{B} \) with \( S = \bigcup_{i=1}^{n} S_i \) for some \( S_i \) where \( n < \infty \). Then \( \Lambda^m S = \bigcup_{i=1}^{n} \Lambda^m S_i \) and so

\[
\gamma(\Lambda^m S) \leq \max_{1 \leq i \leq n} \beta(\Lambda^m S_i) \leq \beta(\Lambda^m) \max_{1 \leq i \leq n} \beta(S_i)
\]

from the definition (7) of \( \gamma \) and from Lemma 4. As the above inequalities are valid for every \( S_i \), it follows that \( \gamma(\Lambda^m S) \leq \beta(\Lambda^m) \gamma(S) \) and thus \( \gamma(\Lambda^m) \leq \beta(\Lambda^m) \).

Next suppose that \( S \in \mathcal{A} \), again with \( S = \bigcup_{i=1}^{n} S_i \) for some \( S_i \). Then \( S_i \in \mathcal{A} \) for each \( i \), and \( \beta(S) = \alpha(S) \) and \( \beta(S_i) = \alpha(S_i) \), as noted in the proof of Theorem 7.

Thus

\[
\beta(S) = \alpha(S) = \max_{1 \leq i \leq n} \alpha(S_i) = \max_{1 \leq i \leq n} \beta(S_i)
\]

from the set-additivity of \( \alpha \), and this implies that \( \gamma(S) = \beta(S) \).

Now recall the set \( S_n \subset Z \) as in (15) and the fact, noted in the proof of Theorem 7, that \( \beta(S_n) = 2a_n \). Certainly \( S_n \in \mathcal{A} \), and so also \( \gamma(S_n) = 2a_n \). Observing that \( \Lambda^m S_{n+m} = S_n \) for every \( n \geq 1 \), we have that \( \gamma(\Lambda^m S_{n+m}) = (\frac{a_n}{a_{n+m}}) \gamma(S_{n+m}) \) and therefore \( \gamma(\Lambda^m) \geq \frac{a_n}{a_{n+m}} \). Taking the supremum over \( n \geq 1 \), we conclude that \( \gamma(\Lambda^m) \geq \mu_m \).

It remains to prove that \( \beta(\Lambda^m) \leq \mu_m \). If \( \mu_m = \infty \) we are done, so assume for the remainder of the proof that \( \mu_m < \infty \).

Recall the Banach space \((\tilde{Z}, \| \cdot \| \tilde{Z})\) in the proof of Theorem 7. For any \( z \in \tilde{Z} \) we have that

\[
\| \Lambda^m z \| \tilde{Z} = \left( \sum_{i=m+1}^{\infty} a_{i-m}^2 \sum_{j=1}^{\infty} |z(i,j)|^p \right)^{1/p} 
\leq \left( \sum_{i=m+1}^{\infty} \mu_i^p a_i^p \sum_{j=1}^{\infty} |z(i,j)|^p \right)^{1/p} \leq \mu_m \| z \| \tilde{Z},
\]

and it follows that \( \Lambda^m \tilde{Z} \subset \tilde{Z} \) and \( \| \Lambda^m \| \tilde{Z}(\tilde{Z}) \leq \mu_m \). On the other hand, let \( n > m \) and take any \( z \in Z_n \), with \( Z_n \) as in (13). Then \( \Lambda^m z \in Z_{n-m} \) and so \( z, \Lambda^m z \in \tilde{Z} \) with

\[
\| \Lambda^m z \| \tilde{Z} = a_{n-m} \left( \sum_{j=1}^{\infty} |z(n,j)|^p \right)^{1/p} = \left( \frac{a_{n-m}}{a_n} \right) \| z \| \tilde{Z},
\]

It follows that \( \| \Lambda^m \| \tilde{Z}(\tilde{Z}) \geq \mu_m \) and thus

\[
(14) \quad \| \Lambda^m \| \tilde{Z}(\tilde{Z}) = \mu_m.
\]

Now take any \( S \in \mathcal{B} \) and \( \varepsilon > 0 \). Then there exist \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) so that \( S \subset A + B \) and

\[
\beta(S) \leq \alpha(A) + \alpha(B) < \beta(S) + \varepsilon,
\]

by the definition (12) of \( \beta \). The reader can verify that \( \alpha((I - P_n)\Lambda^m A) = \alpha((I - P_{n+m})A) \), which implies that \( \Lambda^m A \in \mathcal{A} \). We have \( \Lambda^m S \subset \Lambda^m A + \Lambda^m B \) and also \( \mu_m \geq 1 \), so it follows from Lemma 4 from (14), and because \( \| \Lambda^m \| \tilde{Z}(Z) = 1 \)

\[
\beta(\Lambda^m S) \leq \alpha(\Lambda^m A) + \alpha(\Lambda^m B) \leq \| \Lambda^m \| \tilde{Z}(Z) \alpha(A) + \| \Lambda^m \| \tilde{Z}(Z) \alpha(B) = \mu_m \tilde{\alpha}(A) + \tilde{\alpha}(B) \leq \mu_m (\beta(S) + \varepsilon).
\]

We conclude that \( \beta(\Lambda^m) \leq \mu_m \), as desired; hence \( \beta(\Lambda^m) = \gamma(\Lambda^m) = \mu_m \), as claimed. \qed
Remark. Any value $s \in (1, \infty]$ for the quantity $\beta^*(\Lambda)$ can be obtained by a suitable choice of the sequence $\{a_n\}_{n=1}^{n=\infty}$ in the above construction. If $s \in (1, \infty)$, then taking $a_n = s^{-n}$ gives $\beta(\lambda^n) = \mu_m = s^m$, and hence $\beta^*(\Lambda) = s$. If $s = \infty$, then taking, for example, $a_n = n^{-n}$ gives $\beta(\lambda^n) = \mu_m = \infty$ for every $m$, and hence $\beta^*(\Lambda) = \infty$.

While the above construction has been carried out for the space $\ell^p(N \times N)$, where $1 \leq p \leq \infty$, with the aid of results in [12] analogs of Theorem [8] can be proved for a variety of infinite dimensional Banach spaces which arise naturally in analysis.

We return again to the general case. Let $(X, \| \cdot \|)$ be any complex, infinite dimensional Banach space, $\beta$ an arbitrary homogeneous MNC on $X$, and $L : X \to X$ any bounded linear map. There are several inequivalent definitions of $\text{ess}(L)$, the essential spectrum of $L$, and all definitions actually apply when $L : \mathcal{D}(L) \subset X \to X$ is closed and densely defined. For example, F.E. Browder [5] defines $\text{ess}(L)$ to be the set of $\lambda \in \mathbb{C}$ such that (a) $\lambda$ is an accumulation point of $\sigma(L)$, the spectrum of $L$, or that (b) $\mathcal{R}(\lambda I - L)$, the range of $\lambda I - L$, is not closed, or that (c) $\bigcup_{i=1}^{n} \mathcal{N}(\lambda I - L)$ is infinite dimensional, where $\mathcal{N}(B)$ denotes the null space of a linear map $B$. Another possible definition is $\text{ess}(L) = \{ \lambda \in \mathbb{C} \mid \lambda I - L \text{ is not Fredholm of index 0} \}$. F. Wolf [19] defines $\text{ess}(L) = \{ \lambda \in \mathbb{C} \mid \lambda I - L \text{ is not Fredholm} \}$, and T. Kato [9] defines $\text{ess}(L) = \{ \lambda \in \mathbb{C} \mid \lambda I - L \text{ is not semi-Fredholm} \}$. Simple examples involving shift operators on $l^2(\mathbb{N})$ show that these definitions are not equivalent. However, by using classical results of Gohberg and Krein [8] and index theory for semi-Fredholm operators (see [9]), one can prove that for all definitions, $\text{ess}(L)$ is nonempty and that

$$\rho(L) := \sup \{ |\lambda| \mid \lambda \in \text{ess}(L) \}$$

is the same for all definitions of $\text{ess}(L)$. If $|\lambda| < \rho(L)$ and $\lambda \in \sigma(L)$, then $\lambda$ is an eigenvalue of $L$ of finite algebraic multiplicity, $\lambda$ is an isolated point of $\sigma(L)$, and $\lambda I - L$ is Fredholm of index 0.

Now let $\alpha$ denote the Kuratowski MNC on $X$ and define $\eta$, the ball measure of noncompactness on $X$, by

$$\eta(S) := \inf \{ r > 0 \mid S \subset \bigcup_{i=1}^{n} B_r(x_i) \text{ for some } x_i \in X, \text{ for } 1 \leq n < \infty \},$$

with $B_r(x)$ as in (4). It is well-known that $\eta$ is a homogeneous, set-additive MNC and that

$$\alpha(S) \leq \eta(S) \leq \alpha(S)$$

for every $S \in \mathcal{B}(X)$. If $L : X \to X$ is a bounded linear map, it is also known (see Lemma 1 in [14]) that

$$\eta(L^m) = \eta(L^mB_1(0)).$$

It follows from equations (10) and (17) and earlier remarks that

$$\rho(L) = \eta^*(L) = \lim_{m \to \infty} \eta(L^mB_1(0))^{1/m} = \lim_{m \to \infty} \alpha(L^mB_1(0))^{1/m} = \alpha^*(L),$$

where $\rho(L)$ is as in (15) and where we recall that $\beta^*(L)$, for any homogeneous MNC $\beta$, is given by (3). As any such $\beta$ is dominated by $\alpha$ by Proposition [2] it follows from (18) that

$$\beta^*(L) \leq \alpha^*(L) = \rho(L).$$
We claim that $\beta^*(L) = \rho(L)$. To prove this we shall use an old result of Yood [20] and some facts about semi-Fredholm operators (see [9]). In the following lemma, recall that a map $f$ from a topological space $U$ to a topological space $V$ is called proper if $f^{-1}(K)$ is compact (possibly empty) for every compact $K \subset V$.

**Lemma 9** (Yood [20]). Let $X$ and $Y$ be Banach spaces (real or complex) and $L : X \rightarrow Y$ a bounded linear map. Then the map $L|S : S \rightarrow Y$ is proper for every closed, bounded $S \subset X$ if and only if $\mathcal{N}(L)$, the null space of $L$, is finite dimensional, and $\mathcal{R}(L)$, the range of $L$, is closed.

**Theorem 10.** Let $X$ be a complex Banach space, $L : X \rightarrow X$ a bounded linear map, and $\beta$ any homogeneous MNC on $X$. Then

$$\beta^*(L) = \rho(L),$$

where $\beta^*(L)$ is given by equation (3) and $\rho(L)$ by equation (15). If instead $X$ is a real Banach space, then

$$\beta^*(L) = \rho(\hat{L}),$$

where $\hat{L} : \hat{X} \rightarrow \hat{X}$ is the complexification of $L$ and $\hat{X}$ is the complexification of $X$.

**Proof.** First suppose that $X$ is a complex Banach space. Let $r > 0$ and $|\lambda| > \beta^*(L)$, and denote $L_{\lambda} := \lambda^{-1}L$. Then by equation (3),

$$\lim_{m \rightarrow \infty} \beta(L^m_{\lambda}B_r(0)) = \lim_{m \rightarrow \infty} r\beta(L^m_{|\lambda|}B_r(0)) = 0. \tag{20}$$

Let $Q_r := \overline{B_r(0)}$. We claim that $(I-L_{\lambda})|Q_r$ is proper, equivalently, that $(\lambda I - L)|Q_r$ is proper. As $r > 0$ is arbitrary, this implies that $(\lambda I - L)|S$ is proper for every closed, bounded $S \subset X$. To prove our claim, let $K \subset X$ be compact and let $T := \{x \in Q_r | (I-L_{\lambda})x \in K\}$. The set $T$ is closed, by continuity. If $x \in T$, then $x = L_{\lambda}y \pm z$ for some $y \in K$, and it follows for all $m \geq 1$ that $x = L^{m}_{\lambda}x + \sum_{i=0}^{m-1} L^i_{\lambda}z$.

This implies that

$$T \subset L^m_{\lambda}T + \left( \sum_{i=0}^{m-1} L^i_{\lambda} \right)K \subset L^m_{\lambda}Q_r + K_m, \tag{22}$$

where $K_m := (\sum_{i=0}^{m-1} L^i_{\lambda})K$ is compact. It follows from (22) that

$$\beta(T) \leq \beta(L^m_{\lambda}Q_r) + \beta(K_m) = \beta(L^m_{\lambda}Q_r) \leq \beta(L^m_{\lambda}B_r(0)) = \beta(L^m_{\lambda}B_r(0)),$$

and with (21) it follows that $\beta(T) = 0$. Thus $T$ is compact. Yood’s lemma now implies that $\mathcal{N}(\lambda I - L)$ is finite dimensional and $\mathcal{R}(\lambda I - L)$ is closed, that is, $\lambda I - L$ is a semi-Fredholm operator with index $i(\lambda I - L) := \dim(\mathcal{N}(\lambda I - L)) - \codim(\mathcal{R}(\lambda I - L)) < \infty$. Moreover, the value of $i(\lambda I - L)$ is independent of such a $\lambda$ due to the continuity of the index of semi-Fredholm operators. As $\lambda I - L$ is invertible for $|\lambda| > \|L\|$, this value is $i(\lambda I - L) = 0$. Thus $\lambda I - L$ is Fredholm of index 0 for all $\lambda$ with $|\lambda| > \beta^*(L)$. Using Wolf’s definition of $\text{ess}(L)$ we have that $\rho(L) \leq \beta^*(L)$; thus $\rho(L) = \beta^*(L)$ from (19).

If $X$ is a real Banach space, then (20) follows from (6) and the surrounding remark.

Lastly, we prove the following result, which was discussed in a remark above.
Proposition 11. Let $X$ be a real Banach space, let $\alpha$ denote the Kuratowski MNC on $X$, and let $\hat{\alpha}$ denote its complexification, as in (5). Then $\hat{\alpha}$ is also the Kuratowski MNC on $\hat{X}$.

Proof. With $\hat{\alpha}$ denoting the complexification of $\alpha$ as in the statement of the proposition, let $\overline{\alpha}$ denote the Kuratowski MNC on $\hat{X}$. We must show that $\hat{\alpha} = \overline{\alpha}$. First observe that if $\hat{S} \subset \hat{X}$ is any bounded set, then $\text{diam}(e^{i\theta} \hat{S}) = \text{diam}(\hat{S})$ and $\text{diam}(\text{Re}(\hat{S})) \leq \text{diam}(\hat{S})$; hence

$$\text{diam}(\text{Re}(e^{i\theta} \hat{S})) \leq \text{diam}(\hat{S}),$$

for any $\theta \in \mathbb{R}$. Now with such an $\hat{S}$ fixed, denote $\overline{\alpha} = \overline{\alpha}(\hat{S})$ and let $\varepsilon > 0$. Then $\hat{S} = \bigcup_{j=1}^{n} \hat{S}_j$ for some sets $\hat{S}_1, \hat{S}_2, \ldots, \hat{S}_n \subset \hat{X}$, each with $\text{diam}(\hat{S}_j) \leq \overline{\alpha} + \varepsilon$. For any $\theta \in \mathbb{R}$ we have that $\text{Re}(e^{i\theta} \hat{S}) = \bigcup_{j=1}^{n} \text{Re}(e^{i\theta} \hat{S}_j)$, and as $\text{diam}(\text{Re}(e^{i\theta} \hat{S}_j)) \leq \overline{\alpha} + \varepsilon$, it follows that $\alpha(\text{Re}(e^{i\theta} \hat{S})) \leq \overline{\alpha} + \varepsilon$. Taking the supremum over $\theta$ and letting $\varepsilon \to 0$ now gives $\hat{\alpha}(\hat{S}) \leq \overline{\alpha}(\hat{S})$.

Now denote $\hat{a} = \hat{\alpha}(\hat{S})$. Then $\alpha(\text{Re}(e^{i\theta} \hat{S})) \leq \hat{a}$ for every $\theta$. Fix $m > 0$ and let $\theta_k = \frac{2\pi k}{m}$ for $1 \leq k \leq m$. Also fix $\varepsilon > 0$. Then for each $k$ in the above range there exist sets $S_{k,j} \subset X$ for $1 \leq j \leq n_k < \infty$ such that $\text{Re}(e^{i\theta_k} \hat{S}) = \bigcup_{j=1}^{n_k} S_{k,j}$ with

$$\text{diam}(S_{k,j}) \leq \alpha(\text{Re}(e^{i\theta_k} \hat{S})) + \varepsilon \leq \hat{a} + \varepsilon.$$ 

Now define $\tilde{S}_{k,j} = \{ x \in \hat{S} \mid \text{Re}(e^{i\theta_k} x) \in S_{k,j} \}$, so clearly $\hat{S} = \bigcup_{k=1}^{m} \tilde{S}_{k,j}$ for every $k$. Now consider all sequences $\sigma = (j_1, j_2, \ldots, j_m)$ where $1 \leq j_k \leq n_k$, and for each such $\sigma$ let $\tilde{T}_\sigma = \bigcap_{k=1}^{m} \tilde{S}_{k,j_k}$. Then $\hat{S} = \bigcup_{\sigma} \tilde{T}_\sigma$, where the union is taken over all possible such sequences $\sigma$, of which there are finitely many. We wish to obtain an upper bound for the diameter of $\tilde{T}_\sigma$ for each $\sigma$. Fixing $\sigma = (j_1, j_2, \ldots, j_m)$, let $x, y \in \tilde{T}_\sigma$. For any $k$ with $1 \leq k \leq m$ we have that $x, y \in \tilde{S}_{k,j_k}$, and therefore $\text{Re}(e^{i\theta_k} x), \text{Re}(e^{i\theta_k} y) \in S_{k,j_k}$. Thus

$$\| \text{Re}(e^{i\theta_k} (x - y)) \| = \| \text{Re}(e^{i\theta_k} x) - \text{Re}(e^{i\theta_k} y) \| \leq \text{diam}(S_{k,j_k}) \leq \hat{a} + \varepsilon.$$ 

Denoting $x - y = u + iv$, where $u, v \in X$, this can be written as

$$\| (\cos \theta_k)u - (\sin \theta_k)v \| \leq \hat{a} + \varepsilon.$$ 

Now for any $\theta \in [0, 2\pi]$, there exists $k$ such that $|\theta - \theta_k| \leq \frac{2\pi}{m}$. Then

$$\| (\cos \theta)u - (\sin \theta)v \| \leq \| (\cos \theta_k)u - (\sin \theta_k)v \| + \| (\cos \theta - \cos \theta_k)u - (\sin \theta - \sin \theta_k)v \| \leq \| (\cos \theta_k)u - (\sin \theta_k)v \| + \frac{2\pi}{m} \| u \| + \frac{2\pi}{m} \| v \| \leq \hat{a} + \varepsilon + \frac{4\pi}{m} \| x - y \|.$$ 

Taking the supremum over $\theta$ in the first term above gives $\| u - iv \|$, and upon noting that $\| u - iv \| = \| u + iv \| = \| x - y \|$ we obtain

$$\| x - y \| \leq \hat{a} + \varepsilon + \frac{4\pi}{m} \| x - y \| \leq \hat{a} + \varepsilon + \frac{4\pi}{m} \text{diam}(\hat{S}).$$ 

As $x, y \in T_\sigma$ are arbitrary, this gives an upper bound for $\text{diam}(T_\sigma)$ and thus an upper bound

$$\overline{\alpha}(\hat{S}) \leq \hat{a} + \varepsilon + \frac{4\pi}{m} \text{diam}(\hat{S})$$.
for the Kuratowski MNC of $\mathcal{S}$. As $\varepsilon$ and $m$ are arbitrary, it follows that $\overline{\sigma}(\mathcal{S}) \leq \widehat{a} = \widehat{a}(\mathcal{S})$. With this, the proposition is proved. \qed

References


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