ON A QUESTION OF D. SHLYAKHTENKO

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Abstract. In this short paper we construct two countable, infinite conjugacy class (ICC) groups which admit free, ergodic, probability measure-preserving orbit equivalent actions but whose group von Neumann algebras are not (stably) isomorphic.

1. Introduction

Two countable, discrete groups Γ and Λ are orbit equivalent if they admit free, probability measure-preserving actions which generate isomorphic equivalence relations. They are W∗-equivalent (or von Neumann equivalent) if their group von Neumann algebras are isomorphic. D. Shlyaktenko noticed that, for all known examples of orbit equivalent groups, their group von Neumann algebras are isomorphic and speculated that this might be the case in general. Subsequently, a number of people have also asked the question of whether orbit equivalence of groups implies von Neumann equivalence ([4],[8]). Motivated by this question, we prove:

Theorem 1. There exist two countable, discrete, infinite conjugacy class groups Γ and Λ which are orbit equivalent but not W∗-equivalent. Moreover, the group von Neumann algebras LΓ and LΛ are not stably isomorphic.

The construction of the groups Γ and Λ is based on the observation that being an infinite conjugacy class group is not an orbit equivalence invariant. Indeed, from [1] we have that Γ0 = S∞, the group of finite permutations of N, is orbit equivalent to Λ0 = Z. This example already shows that there are orbit equivalent groups which are not von Neumann equivalent. Further, notice that the map Γ0 → Γ = (Γ0 × F2)⋆Z turns every pair (Γ0, Λ0) of orbit equivalent groups into a pair (Γ, Λ) of orbit equivalent, infinite conjugacy class groups. Finally, by applying the Kurosh type results for free product von Neumann algebras from [2] we derive that the group von Neumann algebras of Γ and Λ are not stably isomorphic.

2. Proof of theorem

Before proving the theorem, we recall the notion of stable isomorphism of II1 factors. For a II1 factor M and 0 < t ∈ R, the amplification Mt is defined as the isomorphism class of p(Mn(C) ⊗ M)p, where n > t is an integer and p ∈ Mn(C) ⊗ M is a projection of trace t/n. It is well known that this isomorphism class does not
depend on the choices of $n$ and $p$. Then two $\Pi_1$ factors are called stably isomorphic if one of them is isomorphic with an amplification of the other.

Proof. Let $\Gamma_0$ and $\Lambda_0$ be two infinite amenable groups and assume that $\Gamma_0$ is infinite conjugacy class (ICC) while $\Lambda_0$ is abelian. By \cite{3}, $\Gamma_0$ and $\Lambda_0$ are orbit equivalent. Further, by Section 2.2 in \cite{2}, the ICC groups $\Gamma = (\Gamma_0 \times F_2) \ast Z$ and $\Lambda = (\Lambda_0 \times F_2) \ast Z$ are orbit equivalent. We claim that the group von Neumann algebras $M = L\Gamma$ and $N = L\Lambda$ are not stably isomorphic. Let $M_0 = L\Gamma_0$ and $N_0 = L\Lambda_0$ and note that $M = (M_0 \overline{\otimes} LF_2) \ast LZ$ and $N = (N_0 \overline{\otimes} LF_2) \ast LZ$.

We suppose by contradiction that this is not the case. Therefore we can find an isomorphism $\theta : M^t \rightarrow N$, for some $t > 0$. Since $M_0$ is a factor we can view $M_0^t$ as a subfactor of $M^t$. The commutant of $M_0^t$ in $M^t$ is then equal to $LF_2$. Since the latter is a noninjective factor, by the first part of the proof of Theorem 3.3 in \cite{4} we deduce that a corner of $\theta(M_0^t)$ can be embedded into $N_0 \overline{\otimes} LF_2$ inside $N$ in the sense of Popa (Theorem 2.1 in \cite{7}).

This amounts to the existence of two nonzero projections $q_1 \in M_0^t$, $p \in N_0 \overline{\otimes} LF_2$, a nonzero partial isometry $v \in N$ and a unital injective homomorphism $\phi : \theta(q_1 M_0^t q_1) \rightarrow p(N_0 \overline{\otimes} LF_2) p$ such that $xv = \phi(x) v$ for every $x \in \theta(q_1 M_0^t q_1)$. Moreover, $v$ satisfies $vv^* \in \theta(q_1 M_0^t q_1)' \cap \theta(q_1) N \theta(q_1)$ and $v^* v \in \phi(\theta(q_1 M_0^t q_1)') \cap pNp$ and therefore \cite{6} together with the factoriality of $M_0$ implies that $vv^* \in \theta((C q_1) \overline{\otimes} LF_2)$ and $v^* v \in p(N_0 \overline{\otimes} LF_2) p$. Hence $vv^* = \theta(q_1 \otimes q_2)$ for some nonzero projection $q_2 \in LF_2$, and let $u \in N$ be a unitary such that $\theta(q_1 \otimes q_2) u = v$. By combining these relations with $xv = \phi(x) v$ we obtain that

$$u^* \theta((q_1 \otimes q_2)(M_0^t \overline{\otimes} C_1)(q_1 \otimes q_2)) u = v^* \theta(M_0^t) v = (v^* v) \phi(\theta(q_1 M_0^t q_1)) \subset N_0 \overline{\otimes} LF_2.$$

By \cite{3} this further implies that $u^* \theta((q_1 \otimes q_2)(M_0^t \overline{\otimes} LF_2)'(q_1 \otimes q_2)) u \subset N_0 \overline{\otimes} LF_2$. Replacing $\text{Ad}(u^* \circ \theta)$ with $\theta$ and then denoting by $q = q_1 \otimes q_2 \in M_0^t \overline{\otimes} LF_2$ and $r = \theta(q) \in N^t \overline{\otimes} LF_2$ we can therefore assume that $\theta(q(M_0^t \overline{\otimes} LF_2)' q) \subset r(N_0 \overline{\otimes} LF_2)' r$.

Since the center, $Z$, of $r(N_0 \overline{\otimes} LF_2)' r$ is diffuse and $\theta$ is an isomorphism, we derive that $\theta^{-1}(Z)$ is also a diffuse subalgebra which is contained in the relative commutant of $q(M_0^t \overline{\otimes} LF_2)' q$ in $q M^t q$. On the other hand, by \cite{3}, the relative commutant of $q(M_0^t \overline{\otimes} LF_2)' q$ in $q M^t q$ is equal to the center of $q(M_0^t \overline{\otimes} LF_2)' q$, which consists only in the scalars because $M_0$ is a factor. Therefore we have reached a contradiction.

We would like to end this note by mentioning the following open question: Are there any examples of $W^*$-equivalent groups which are not orbit equivalent?

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