A SCHWARZ LEMMA FOR THE MODULUS OF A VECTOR-VALUED ANALYTIC FUNCTION

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Abstract. It is proved that
\[ |\nabla f(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}, \]
where \( f : \mathbb{D} \to \mathbb{B}_k \) is an analytic function from the unit disk \( \mathbb{D} \) into the unit ball \( \mathbb{B}_k \subset \mathbb{C}^k \). Applications to the Lipschitz condition of the modulus of a \( \mathbb{C}^k \)-valued function are given.

1. Introduction

Let \( \mathbb{D} \) denote the open unit disk of the complex plane \( \mathbb{C} \). The classical Schwarz lemma states that if \( f : \mathbb{D} \to \mathbb{D} \) is an analytic function, then
\[ |f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}. \]
This inequality does not hold for analytic functions with values in \( \mathbb{B}_k := \{ w \in \mathbb{C}^k : |w| < 1 \} \), \( k \geq 2 \). For instance, the function \( f(z) = (z, 1)/\sqrt{2} \) satisfies
\[ |f'(0)| = \sqrt{1 - |f(0)|^2} > 1 - |f(0)|^2. \]
However (1.1) can also be written as
\[ |\nabla f(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D} \text{ and } f(z) \neq 0, \]
because scalar-valued analytic functions satisfy
\[ |\nabla f(z)| = |f'(z)|, \quad z \in \mathbb{D} \text{ and } f(z) \neq 0. \]
If we define
\[ |\nabla f(z)| = |f'(z)|, \quad z \in \mathbb{D} \text{ and } f(z) = 0, \]
then the inequality in (1.3) holds for all \( z \in \mathbb{D} \). It turns out that this form of Schwarz’s lemma extends to vector-valued functions. Precisely:
Theorem 1.1. Let \( f : \mathbb{D} \mapsto \mathbb{B}_k \) be an analytic function. Then
\[
|\nabla f(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}.
\]

In the standard way, we deduce the following consequence:

Theorem 1.2. Let \( f : \mathbb{D} \mapsto \mathbb{B}_k \) be an analytic function. Then
\[
|f(z)| - |f(a)| \leq \frac{|z - a|}{1 - \bar{a}z}, \quad a, z \in \mathbb{D}.
\]

Of course, we have to explain the meaning of \(|\nabla g|\), where \( g = |f| \). If \( g(z) \neq 0 \), then \( g \) is \( \mathbb{R} \)-differentiable at \( z \) and therefore \( \nabla g \) is treated as the ordinary gradient.

In the general case we set
\[
|\nabla g(z)| = \limsup_{h \to 0} \frac{|g(z + h) - g(z)|}{|h|}.
\]
If \( g(z) \neq 0 \), this coincides with the ordinary definition of \(|\nabla g|\). If \( g(z) = 0 \), then
\[
|\nabla g(z)| = \limsup_{h \to 0} \frac{|f(z + h)|}{|h|} = |f'(z)| = \left( \sum_{j=1}^{k} |f_j'(z)|^2 \right)^{1/2},
\]
where \( f = (f_1, f_2, \ldots, f_k) \). Let
\[
g = |f| = (|f_1|^2 + |f_2|^2 + \ldots + |f_k|^2)^{1/2}.
\]
Since \( g \) is real-valued, we have \(|\nabla g| = 2|\partial g|\), where
\[
\partial g = \frac{1}{2} \left( \frac{\partial g}{\partial x} - i \frac{\partial g}{\partial y} \right), \quad z = x + iy.
\]
It follows that
\[
|\nabla f(z)| = \frac{1}{|f(z)|} \left| \sum_{j=1}^{k} \bar{f}_j(z) f_j'(z) \right| = \frac{1}{|f(z)|} |(f'(z), f(z))|, \text{ if } f(z) \neq 0.
\]
Here \((z, w) = z_1 \bar{w}_1 + \ldots + z_k \bar{w}_k\).

As a consequence of (11), we have
\[
|f'(0)| \leq 2(1 - |f(0)|).
\]
Applying this to the function \( w \mapsto f(z + \varepsilon w)/M \), where \( 0 < \varepsilon < 1 - |z| \),
\[
M = \sup_{w \in D_\varepsilon(z)} |f(w)|, \quad D_\varepsilon(z) = \{ w : |w - z| < \varepsilon \},
\]
we obtain
\[
|f'(z)| \leq \frac{2}{\varepsilon} \sup_{w \in D_\varepsilon(z)} (|f(w)| - |f(z)|).
\]

The importance of this inequality lies in the fact that, when combined with a classical theorem of Hardy and Littlewood, it yields a simple proof (see [6]) of the
following result of Dyakonov [2]. Let $\Lambda_\alpha(\mathbb{D})$, $0 < \alpha \leq 1$, denote the class of all functions $f : \mathbb{D} \to \mathbb{C}$ for which $|f(z) - f(w)| \leq \text{const.} \times |z - w|^\alpha$ for all $z, w \in \mathbb{D}$.

**Theorem D.** If $f$ is analytic and $0 < \alpha \leq 1$, then $f \in \Lambda_\alpha(\mathbb{D})$ if and only if $|f| \in \Lambda_\alpha(\mathbb{D})$.

Indeed, from the hypothesis $|f(w)| - |f(z)| \leq |z - w|\alpha$ and (1.12) it follows that $|f'(z)| \leq 2(1 - |z|)\alpha - 1$. Now the proof is completed by the theorem of Hardy and Littlewood mentioned earlier:

**Theorem HL.** A function $f$ analytic in $\mathbb{D}$ belongs to $\Lambda_\alpha(\mathbb{D})$ ($0 < \alpha \leq 1$) if and only if

\[(1.13)\quad |f'(z)| \leq \text{const.} \times (1 - |z|)^{\alpha - 1}, \quad z \in \mathbb{D}.
\]

Note also that Theorem D can be viewed as an improvement of the Hardy-Littlewood theorem on conjugate functions (see [3] and [4]).

However, Theorem D does not extend to $\mathbb{C}^k$-valued functions ($k \geq 2$), which implies that the same holds for the relation (1.1) (see Section 2.2). So we have to consider functions with additional properties (see Theorem 3.2 and 3.3).

2. **Proof of Theorem 1.1**

**Lemma 2.1.** If $f : \mathbb{D} \to \mathbb{C}^k$ is analytic and $f(\mathbb{D}) \subset \mathbb{D}$, then

\[(2.1)\quad |f'(0)| \leq \sqrt{1 - |f(0)|^2}.
\]

**Proof.** From the hypothesis $f(\mathbb{D}) \subset \mathbb{D}$ it follows that

\[(2.2)\quad 1 > \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 \, dt
\]

\[\geq |f(0)|^2 + |f'(0)|^2 r^2, \quad 0 < r < 1.
\]

The result follows. \(\square\)

**Proof of Theorem 1.1.** It suffices to prove that

\[(2.3)\quad |\nabla f(0)| \leq 1 - |f(0)|^2
\]

and then to apply this to the functions $f \circ \sigma_a$, where

\[(2.4)\quad \sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad a, z \in \mathbb{D}.
\]

If $f(0) = 0$, then, by Lemma 2.1 we have $|\nabla f(0)| = |f'(0)| \leq 1$, as desired. If $f(0) \neq 0$, then we consider the scalar-valued function $h(z) = (f(z), f(0) / |f(0)|)$. Applying the classical Schwarz’s lemma to $h$, we get

\[|\nabla f(0)| = |(f'(0), f(0) / |f(0)|)| = |h'(0)| \leq 1 - |h(0)|^2 = 1 - |f(0)|^2,
\]

which completes the proof. \(\square\)

**Remark 2.2.** If $k = 2$, then equality in (2.1) is attained for $f(z) = (z, 1)/\sqrt{2}$.

**Remark 2.3.** Applying (2.3) to the function $f \circ \sigma_z$, we get

\[(2.5)\quad |f'(z)| \leq \frac{\sqrt{1 - |f(z)|^2}}{1 - |z|^2}.
\]
3. Lipschitz conditions on the modulus

Theorem D, as noted in the Introduction, does not hold in the class of $C^2$-valued analytic functions. To see this, consider the function $f(z) = ((1 - z)^{\alpha/2}, 1)$, $0 < \alpha \leq 1$. We have

$$|f(z) - f(w)| = \sqrt{|1 - z|^{\alpha} + 1} - \sqrt{|1 - w|^{\alpha} + 1}$$

$$\leq |1 - w|^{\alpha} - |1 - z|^{\alpha}$$

$$\leq |z - w|^{\alpha},$$

while

$$|f(1) - f(r)| = (1 - r)^{\alpha/2}, \quad 0 < r < 1.$$

This shows that in the following result the index $\alpha/2$ is optimal.

**Theorem 3.1.** If $f : \mathbb{D} \mapsto \mathbb{C}^k$, $k \geq 2$, is analytic and if $|f| \in A_\alpha(\mathbb{D})$, $0 < \alpha \leq 1$, then $f \in A_{\alpha/2}(\mathbb{D})$.

This fact is a consequence of (3.1). See [1] for the case of analytic functions with values in a Banach space.

On the other hand, since Theorem HL remains valid for $C^k$-valued functions (with the obvious definition of $A_\alpha$), we have the following consequence of Theorem D, as noted in the Introduction, does not hold in the class of $C^2$-valued analytic functions.

**Theorem 3.2.** Let $f : \mathbb{D} \mapsto \mathbb{C}^k$ be an analytic function such that

$$|f'(z)||f(z)| \leq K|f'(z), f(z)|, \quad z \in \mathbb{D},$$

where $K$ is a constant independent of $z$. Let $0 < \alpha \leq 1$. Then $f \in A_\alpha(\mathbb{D})$ if and only if $|f| \in A_\alpha(\mathbb{D})$.

In [2], Dyakonov proved another important result:

**Theorem D2.** Let $f : \mathbb{D} \mapsto \mathbb{C}$ be analytic on $\mathbb{D}$ and continuous on $\mathbb{D} \cup \partial \mathbb{D}$, let $0 < \alpha < 1$, and let $P_g$ denote the Poisson integral of $g \in L^1(\mathbb{T})$,

$$P_g(z) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} |d\zeta|, \quad z \in \mathbb{D}.$$

If $g = |f|$, then the following conditions are equivalent:

(i) $f \in A_\alpha(\mathbb{D})$.

(ii) The restriction $g|_{\partial \mathbb{D}}$ belongs to $A_\alpha(\partial \mathbb{D})$ and there exists a constant $C$ such that

$$P_g(z) - g(z) \leq C(1 - |z|)^\alpha, \quad \text{for all } z \in \mathbb{D}.$$  

(iii) $g|_{\partial \mathbb{D}}$ belongs to $A_\alpha(\partial \mathbb{D})$ and there exists a constant $C$ such that $g(\zeta) - g(r\zeta) \leq C(1 - r)^\alpha$ for all $\zeta \in \partial \mathbb{D}$ and $0 < r < 1$.

(iv) There exists a constant $C$ such that $g(\zeta) - g(z) \leq C|\zeta - z|^\alpha$ for all $\zeta \in \partial \mathbb{D}$ and $z \in \mathbb{D}$.

Using the approach of the papers [5] and [6], one can prove the following.

**Theorem 3.3.** Theorem D2 remains true if we assume that $f : \mathbb{D} \mapsto \mathbb{C}^k$ ($k \geq 2$) is an analytic function that satisfies (3.1).

Further generalizations can be made by using arbitrary majorants; we omit that discussion here.

Finally, note the following fact:
Proposition 3.4. If \( f \) satisfies (3.1) and \( \varphi : \mathbb{D} \to \mathbb{D} \) is an analytic function, then \( f \circ \varphi \) satisfies (3.1) (with the same constant \( K \)).

References