ON ADDITIVE COMPLEMENTS. II

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(Communicated by Matthew A. Papanikolas)

Abstract. Two infinite sequences $A$ and $B$ of non-negative integers are called additive complements if their sum contains all sufficiently large integers. Let $A(x)$ and $B(x)$ be the counting functions of $A$ and $B$, and let
\[ \limsup_{x \to \infty} \frac{A(x)B(x)}{x} = \alpha(A,B). \]
Recently, the authors [Proceedings of the American Mathematical Society 138 (2010), 1923-1927] proved that for additive complements $A$ and $B$, if $\alpha(A,B) < 5/4$ or $\alpha(A,B) > 2$, then $A(x)B(x) - x \to +\infty$ as $x \to \infty$. In this paper, we prove that for any $\epsilon > 0$ there exist additive complements $A$ and $B$ with $2 - \epsilon < \alpha(A,B) < 2$ and $A(x)B(x) - x = 1$ for infinitely many positive integers $x$.

1. Introduction

Two infinite sequences $A$ and $B$ of non-negative integers are called additive complements if their sum contains all sufficiently large integers.

Let $A(x)$ and $B(x)$ be the counting functions of $A$ and $B$. Namely,
\[ A(x) = \sum_{a \leq x \in A} 1 \quad \text{and} \quad B(x) = \sum_{b \leq x \in B} 1. \]

For the construction of additive complements $A$ and $B$ with $A(x)B(x) \sim x$, one may refer to [6].

In 1964, Danzer [1] conjectured that for additive complements $A$ and $B$, if
\[ \limsup_{x \to \infty} \frac{A(x)B(x)}{x} \leq 1, \]
then
\[ A(x)B(x) - x \to +\infty \quad \text{as} \quad x \to +\infty. \]
(See also [2, 4, p. 75] and [5].) In [7], Sárközy and Szemerédi proved this conjecture. Recently, the authors [3] proved that

\[ \Box \]
Theorem A. For additive complements \(A\) and \(B\), if
\[
\limsup_{x \to \infty} \frac{A(x)B(x)}{x} > 2 \text{ or } \limsup_{x \to \infty} \frac{A(x)B(x)}{x} < \frac{5}{4},
\]
then \(A(x)B(x) - x \to +\infty\) as \(x \to \infty\).

Once we conjectured that if \(A\) and \(B\) are additive complements, then \(A(x)B(x) - x \to +\infty\) as \(x \to +\infty\). After paper [3] went to print, we found a counterexample, so we removed the conjecture from [3] when we checked the proofs. In this paper, as a complement to [3], we prove the following result.

**Theorem 1.** For any integer \(a\) with \(a \geq 2\), there exist additive complements \(A\) and \(B\) such that \(A(x)B(x) - x = 1\) for infinitely many positive integers \(x\) and
\[
\limsup_{x \to \infty} \frac{A(x)B(x)}{x} = \frac{2a + 2}{a + 2}.
\]

2. Proofs of Theorem [1]

We will use the following lemmas. Their proofs are clear.

**Lemma 1.** For any integers \(a_1, a_2, b_1, b_2\) and \(u\), if \(a_1b_2 - a_2b_1 \geq 0\) and \(a_2x + b_2 > 0\) for all \(0 \leq x \leq u\), then for all \(0 \leq x \leq u\) we have
\[
\frac{a_1x + b_1}{a_2x + b_2} \leq \frac{a_1u + b_1}{a_2u + b_2}.
\]

**Lemma 2.** For any integers \(a_1, a_2, b_1, b_2\) and \(v\), if \(a_2 > 0\), \(a_1b_2 - a_2b_1 \geq 0\) and \(a_2v + b_2 > 0\), then
\[
\frac{a_1v + b_1}{a_2v + b_2} \leq \frac{a_1}{a_2},
\]

**Proof of Theorem [1]** For any integer \(a\) with \(a \geq 2\), assume that
\[
A_j = \{ \sum_{i=0}^{a-1} \varepsilon_i a^{2i+j} \mid \varepsilon_i = 0, 1, \ldots, a-1 \}, \quad j = 0, 1,
\]
where the summation is a finite sum. It is clear that \(A_0\) and \(A_1\) are additive complements.

For \(x_k = a^{2k} - 1\) we have
\[
A_0(x_k) = A_1(x_k) = a^k, \quad A_0(x_k)A_1(x_k) - x_k = 1.
\]

Now we prove that
\[
\limsup_{x \to \infty} \frac{A_0(x)A_1(x)}{x} = \frac{2a + 2}{a + 2}.
\]

For \(x_{l,j} = a^{2i+j} + (a-1) \sum_{i=0}^{l-1} a^{2i+j} (j = 0, 1)\), we have \(A_j(x_{l,j}) = 2a^l, A_{1-j}(x_{l,j}) = a^{l+j}\). Then
\[
\frac{A_0(x_{l,j})A_1(x_{l,j})}{x_{l,j}} = \frac{2}{\left(1 + \frac{1}{a^{2l+1}}\right) - \frac{1}{a+1} \frac{1}{a^2}} \to \frac{2a + 2}{a + 2}, \quad l \to \infty.
\]

Now it suffices to prove that for any positive integers \(x \neq x_{l,j}\) we have
\[
\frac{A_0(x)A_1(x)}{x} \leq \frac{2a + 2}{a + 2}.
\]

If a positive integer \(x \notin A_0 \cup A_1\), then
\[
\frac{A_0(x)A_1(x)}{x} = \frac{A_0(x-1)A_1(x-1)}{x} \leq \frac{A_0(x-1)A_1(x-1)}{x-1}.
\]
So we may assume that \( x \in A_0 \cup A_1 \). Hence, there exist integers \( i, j \) with \( j \in \{0, 1\} \) and \( x = \epsilon_i a^j + \epsilon_{i+1} a^{j+1} + \cdots + \epsilon_{i+l} a^{j+l} \), where \( \epsilon_{2i+j} > 0 \) and \( \epsilon_{2i+1+j} = 0 \) for all \( i \). Then \( A_{1-j}(x) = a^{j+1} \). If \( \epsilon_{2i+j} \geq 2 \), then, by \( A_j(x) \leq \epsilon_{2i+j} a^j + a^j \), we have

\[
\frac{A_j(x)A_{1-j}(x)}{x} \leq \frac{a^{2j+1} \epsilon_{2i+j} + a^{2j+1}}{x} \leq \frac{a^{2j+1} \epsilon_{2i+j} + a^{2j+1}}{a^{2j+1} \epsilon_{2i+j}} \leq \frac{3}{2} \leq \frac{2a+2}{a+2}.
\]

So we only need to consider \( \epsilon_{2i+j} = 1 \). Since \( x \neq x_{l,j} \), we may assume that

\[
x = \sum_{i=0}^{m-1} \epsilon_{2i+j} a^{2i+j} + (a - 1) \sum_{i=m}^{l-1} a^{2i+j} + a^{2i+j},
\]

where \( 0 < m \leq l \) and \( \epsilon_{2m+j-2} < a - 1 \) (if \( m = l \), then the second sum is zero). Thus

\[
A_j(x) \leq a^j + (a - 1) \sum_{i=m}^{l-1} a^i + \epsilon_{2m+j-2} a^{m-1} + a^{m-1} = 2a^j - a^m + (\epsilon_{2m+j-2} + 1)a^{m-1}.
\]

Hence by Lemmas 1 \((x = \epsilon_{2m+j-2})\) and 2 \((v = a^{2l+j})\) we have

\[
\frac{A_{1-j}(x)A_j(x)}{x} \leq \frac{a^{l+j}(2a^j - a^m + (\epsilon_{2m+j-2} + 1)a^{m-1})}{a^{2l+j} + (a - 1)(a^{2m+j} + \cdots + a^{2l+j}) + \epsilon_{2m+j-2}a^{2m+j-2}}
\]

\[
= \frac{2a^{2l+j} - a^{m+l+j} + (\epsilon_{2m+j-2} + 1)a^{m+l+j-1}}{(1 + \frac{1}{a+1})a^{2l+j} - \frac{1}{a+1}a^{2m+j} + \epsilon_{2m+j-2}a^{2m+j-2}}
\]

\[
\leq \frac{2a^{2l+j} - a^{m+l+j} + (a - 2 + 1)a^{m+l+j-1}}{(1 + \frac{1}{a+1})a^{2l+j} - \frac{1}{a+1}a^{2m+j} + (a - 2)a^{2m+j-2}}
\]

\[
\leq \frac{2a + 2}{a + 2}.
\]

This completes the proof of Theorem 1.