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CHARACTERIZATION OF SUBDIAGONAL ALGEBRAS

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ABSTRACT. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. We show that $\mathcal{A}$ has L^p-factorization ($1 \leq p < \infty$) if and only if $\mathcal{A}$ is a subdiagonal algebra. Also, we obtain some characterizations of subdiagonal algebras.

0. INTRODUCTION

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. In [1], Arveson introduced the notion of finite, maximal, subdiagonal algebras $\mathcal{A}$ of $\mathcal{M}$, as noncommutative analogues of weak-* Dirichlet algebras. Subsequently several authors studied the (noncommutative) $\mathcal{H}^p$-spaces associated with such algebras ([9, 11, 12, 13, 14, 16, 17, 18]). Arveson [1] proved a Szegö type factorization theorem. Some extensions can be found in [8, 18], and [15]. Labuschagne [10] proved a non-commutative version of Szegö's theorem. In the recent arcticles [3, 4, 5], among other things, Blecher and Labuschagne studied tracial subalgebras of $\mathcal{M}$ and gave several characterizations of subdiagonal algebras. They proved that if a tracial subalgebra $\mathcal{A}$ has L^∞-factorization, then $\mathcal{A}$ is a subdiagonal algebra. We will consider the L^p-factorization ($0 < p < \infty$) property of tracial subalgebras. This paper is organized as follows. Section 1 contains some preliminary definitions. In section 2, we prove that if a tracial subalgebra $\mathcal{A}$ has L^p-factorization ($1 \leq p < \infty$), then $\mathcal{A}$ is a subdiagonal algebra. In section 3, we consider tracial subalgebras, which satisfy L^2-density. We show that if a tracial subalgebra $\mathcal{A}$ has L^p-factorization ($0 < p < 1$) and satisfies L^2-density, then $\mathcal{A}$ is a subdiagonal algebra.

1. PRELIMINARIES

Throughout this paper, we denote by $\mathcal{M}$ a finite von Neumann algebra on a Hilbert space $\mathcal{H}$ with a faithful normal tracial state τ. For $0 < p < \infty$ we denote by $L^p(\mathcal{M})$ the usual noncommutative L^p-space associated with $(\mathcal{M}, \tau)$. Recall that $L^\infty(\mathcal{M}) = \mathcal{M}$, equipped with the operator norm. It is well-known that $L^p(\mathcal{M})$ is a Banach space under $\|.\|_p$ ($1 \leq p \leq \infty$) satisfying all the expected properties such as duality of $L^p(\mathcal{M})$ and $L^q(\mathcal{M})$ for $\frac{1}{p} + \frac{1}{q} = 1$ (see [7, 20]).

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For a subset K of $L^p(\mathcal{M})$, put $J(K) = \{x^* : x \in K\}$, $K^{-1} = \{x : x, x^{-1} \in K\}$, $K^+ = \{x : x \geq 0, x \in K\}$, and $[K]_p$ the closed linear span of K in $L^p(\mathcal{M})$. (Here $[K]_\infty$ is the weak* closure of K.)

Given a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$, an expectation $\mathcal{E} : \mathcal{M} \to \mathcal{N}$ is defined to be a positive linear map which preserves the identity and satisfies $\mathcal{E}(xy) = x\mathcal{E}(y)$ for all $x \in \mathcal{N}$ and $y \in \mathcal{M}$. Since $\mathcal{E}$ is positive it is Hermitian; i.e. $\mathcal{E}(x)^* = \mathcal{E}(x^*)$ for all $x \in \mathcal{M}$. Hence $\mathcal{E}(yx) = \mathcal{E}(y)x$ for all $x \in \mathcal{N}$ and $y \in \mathcal{M}$. For a complete study of $\mathcal{E}$, we refer to [1, 12].

Definition 1.1. Let $\mathcal{A}$ be a w*-closed unital subalgebra of $\mathcal{M}$. If there exists a linear projection $\mathcal{E}$ from $\mathcal{A}$ onto $\mathcal{D} = \mathcal{A} \cap J(\mathcal{A})$ such that

(i) $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y), \ \forall\, x, y \in \mathcal{A}$;
(ii) $\tau \circ \mathcal{E} = \tau$,

then $\mathcal{A}$ is called a tracial subalgebra of $\mathcal{M}$.

Definition 1.2. Let $\mathcal{A}$ be a w*-closed unital subalgebra of $\mathcal{M}$, and let $\mathcal{E}$ be a faithful, normal expectation from $\mathcal{M}$ onto the diagonal von Neumann algebra $\mathcal{D} = \mathcal{A} \cap J(\mathcal{A})$. Then $\mathcal{A}$ is a finite subdiagonal subalgebra of $\mathcal{M}$ with respect to $\mathcal{E}$ if:

(i) $\mathcal{A} + J(\mathcal{A})$ is w*-dense in $\mathcal{M}$;
(ii) $\mathcal{E}(xy) = \mathcal{E}(x)\mathcal{E}(y), \ \forall\, x, y \in \mathcal{A}$;
(iii) $\tau \circ \mathcal{E} = \tau$.

By Theorem 5.6 of [3], $\mathcal{E}$ is precisely the restriction to $\mathcal{A}$ of the unique faithful normal conditional expectation Φ from $\mathcal{M}$ onto $\mathcal{D}$ such that $\tau = \tau \circ \Phi$. Hence we may continue to write Φ as $\mathcal{E}$, and we call this extension the conditional expectation onto $\mathcal{D}$. It is well-known that $\mathcal{E}$ extends to a contractive projection from $L_p(\mathcal{M})$ onto $L_p(\mathcal{D})$ for every $1 \leq p \leq \infty$. The extension will still be denoted by $\mathcal{E}$.

Let $\mathcal{A}_0 = \mathcal{A} \cap \ker(\mathcal{E})$. We say that $\mathcal{A}$ is τ-maximal if

$$\mathcal{A} = \{x \in \mathcal{M} : \ \tau(xy) = 0, \ \forall y \in \mathcal{A}_0\}. \tag{1.1}$$

2. Factorization

Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. For $0 < p < \infty$, we write $\mathcal{A}_p$ for $[\mathcal{A}]_p \cap \mathcal{M}$.

Lemma 2.1. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. Then $\mathcal{A}_p$ is a tracial subalgebra of $\mathcal{M}$ for $1 \leq p < \infty$.*

Proof. The proof is the same as that of Theorem 4.4 in [4]. We only need to prove the following:

$$\mathcal{E}(ab) = \mathcal{E}(a)\mathcal{E}(b), \ \forall a, b \in \mathcal{A}_p\,.$$

Indeed, let $b \in \mathcal{A}_p$. Then there exists a sequence $\{b_n\}$ in $\mathcal{A}$ such that $b_n \to b$ in $L_p(\mathcal{M})$. So for all $a \in \mathcal{A}$ we have $ab_n \in \mathcal{A}$, and $ab_n \to ab$ in $L_p(M)$. Thus

$$\mathcal{E}(ab) = \lim_{n\to\infty} \mathcal{E}(ab_n) = \lim_{n\to\infty} \mathcal{E}(a)\mathcal{E}(b_n) = \mathcal{E}(a)\mathcal{E}(b).$$

Hence, by what we just proved,

$$\mathcal{E}(ab) = \lim_{n\to\infty} \mathcal{E}(ab_n) = \lim_{n\to\infty} \mathcal{E}(a)\mathcal{E}(b_n) = \mathcal{E}(a)\mathcal{E}(b), \ \forall a \in \mathcal{A}_p.$$

□

Definition 2.2. Let $0 < p \leq \infty$, and let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. We say that $\mathcal{A}$ has L^p-factorization, if for all $x \in L^p(\mathcal{M})^{-1}$, there is a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}]_p^{-1}$ such that $x = ua$.

Proposition 2.3. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. Consider the following conditions:*

(i) *For some $1 \leq p_0 < \infty$, $\mathcal{A}$ has L^{p_0}-factorization.*
(ii) *For some $0 < p_0 < 1$, $\mathcal{A}$ has L^{p_0}-factorization and $\mathcal{A}_{p_0}$ is a tracial subalgebra of $\mathcal{M}$.*

Then either one of the conditions (i) *or* (ii) *implies that $\mathcal{A}$ has L^p-factorization for all $0 < p < p_0$.*

Proof. Let $x \in \mathcal{M}^{-1} \subset L^{p_0}(\mathcal{M})^{-1}$. Then there is a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}]_{p_0}^{-1}$ such that $x = ua$, since $\mathcal{A}$ has L^{p_0}-factorization. So $a \in \mathcal{A}_{p_0}^{-1}$, and therefore $\mathcal{A}_{p_0}$ has L^∞-factorization. By Theorem 1.1 of [4], $\mathcal{A}_{p_0}$ is a subdiagonal subalgebra of $\mathcal{M}$. Let $x \in L^p(\mathcal{M})^{-1}$. By Theorem 3.1 of [2], there is a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}_{p_0}]_p^{-1}$ such that $x = ua$. On the other hand, we have

$$[\mathcal{A}]_p \subset [\mathcal{A}_{p_0}]_p \subset [[\mathcal{A}]_{p_0}]_p = [\mathcal{A}]_p.$$

Thus $a \in [\mathcal{A}]_p^{-1}$, and so $\mathcal{A}$ has L^p-factorization. □

Theorem 2.4. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$.*
(ii) *For all $0 < p \leq \infty$, $\mathcal{A}$ has L^p-factorization.*
(iii) *For some $1 \leq p \leq \infty$, $\mathcal{A}$ has L^p-factorization.*

Proof. (i)⇒(ii) follows by Theorem 3.1 of [2] (also see Theorem 4.2.1 of [1], Corollary 4.13 of [5] and the remark following Theorem 8.1 of [15]).

(ii)⇒(iii) is clear.

(iii)⇒(i). When $p = \infty$, it is (b) of Theorem 1.1 of [4]. From the proof of Proposition 2.3, we know that $\mathcal{A}_p$ is a subdiagonal subalgebra of $\mathcal{M}$. We claim that $\mathcal{A}_p = \mathcal{A}$. Indeed, it suffices to show that if $y \in L^1(\mathcal{M})$ and $\tau(ya) = 0$ for each $a \in \mathcal{A}$, then $\tau(ya) = 0$ for each $a \in \mathcal{A}_p$. Let $y = v|y|$ be the polar decomposition of y, and $v_1 \in \mathcal{M}$ such that $|y| = v_1(|y| + I)$. Notice that $\mathcal{A}_p$ is a subdiagonal subalgebra and $|y|+I \in L^1(\mathcal{M})$, $(|y|+I)^{-1} \in \mathcal{M}$. Hence, there is a unitary $u \in \mathcal{M}$ and $b \in [\mathcal{A}_p]_1$ such that $|y| + I = ub$, $b^{-1} \in \mathcal{A}_p$. So $y = wb$, where $w = vv_1u \in \mathcal{M}$. On the other hand,

$$[\mathcal{A}]_1 = [b^{-1}b\mathcal{A}]_1 \subset [b^{-1}[\mathcal{A}_p]_1]_1 \subset [\mathcal{A}_p]_1 = [\mathcal{A}]_1.$$

Since $\tau(wba) = \tau(ya) = 0$ for each $a \in \mathcal{A}$, it follows that $\tau(wa) = 0$ for each $a \in [\mathcal{A}_p]_1$. By τ-maximality of $\mathcal{A}_p$ we obtain $w \in (\mathcal{A}_p)_0$. Hence $y \in [(\mathcal{A}_p)_0]_1$, or $\tau(ya) = 0$ for each $a \in \mathcal{A}_p$. □

Theorem 2.5. *Let $\mathcal{A}$ be a τ-maximal tracial subalgebra of $\mathcal{M}$. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$.*
(ii) *For some $0 < p < 1$, $\mathcal{A}$ has L^p-factorization and $\mathcal{A}_p$ is a tracial subalgebra of $\mathcal{M}$.*

Proof. We need only prove (ii)⇒(i). It is clear that $\mathcal{A}_p$ is a subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{A} \subset \mathcal{A}_p$. If $b \in \mathcal{A}_p$, then $\tau(ba) = \tau(\mathcal{E}(ba)) = \tau(\mathcal{E}(b)\mathcal{E}(a)) = 0$ for each $a \in \mathcal{A}_0$. By τ-maximality of $\mathcal{A}$ we have $b \in \mathcal{A}$. Thus $\mathcal{A} = \mathcal{A}_p$. □

Proposition 2.6. *Let $0 < p < 1$, and let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$. If $\mathcal{A}$ has L^{p_0}-factorization, and $\mathcal{A}_{p_0}$ is a tracial subalgebra of $\mathcal{M}$ for some $0 < p_0 < 1$, then*

$$\|\mathcal{E}(a)\|_p \leq \|a\|_p \quad \forall\, a \in \mathcal{A}. \tag{2.1}$$

Consequently, $\mathcal{E}$ extends to a contractive projection from $[\mathcal{A}]_p$ onto $L^p(\mathcal{D})$. The extension will still be denoted by $\mathcal{E}$.

Proof. Since $\mathcal{A}_{p_0}$ is a subdiagonal subalgebra of $\mathcal{M}$, Theorem 2.1 of [2] implies that

$$\|\mathcal{E}(a)\|_p \leq \|a\|_p \quad \forall\, a \in \mathcal{A}_{p_0}.$$

From this follows (2.1). □

Corollary 2.7. *With the same assumption as in Proposition* 2.6*, we have that $\mathcal{E}(ab) = \mathcal{E}(a)\mathcal{E}(b)$ for $a \in [\mathcal{A}]_p$ and $b \in [\mathcal{A}]_q$ for $0 < p, q \leq \infty$.*

Proof. Note that $ab \in [\mathcal{A}]_r$ for any $a \in [\mathcal{A}]_p$ and $b \in [\mathcal{A}]_q$, where $1/r = 1/p + 1/q$. Thus $\mathcal{E}(ab)$ is well defined. Then the corollary follows immediately from the multiplicativity of $\mathcal{E}$ on $\mathcal{A}$ and (2.1). □

3. L^2-density

Definition 3.1. We say that a tracial subalgebra $\mathcal{A}$ of $\mathcal{M}$ satisfies L^2-density if $\mathcal{A} + J(\mathcal{A})$ is dense in $L^2(\mathcal{M})$ in the usual Hilbert space norm on that space.

It is clear that $\mathcal{A} = \mathcal{A}_0 + \mathcal{D}$, $\mathcal{A} + J(\mathcal{A}) = \mathcal{A}_0 + \mathcal{D} + J(\mathcal{A}_0)$, $[\mathcal{A}]_2 = [\mathcal{A}_0]_2 + [\mathcal{D}]_2$, and

$$[\mathcal{A} + J(\mathcal{A})]_2 = [\mathcal{A}_0]_2 + [\mathcal{D}]_2 + [J(\mathcal{A}_0)]_2.$$

Hence, if $\mathcal{A}$ is an L^2-dense tracial subalgebra of $\mathcal{M}$, then we have

$$L^2(\mathcal{M}) = [\mathcal{A}_0]_2 \oplus L^2(\mathcal{D}) \oplus [J(\mathcal{A}_0)]_2. \tag{3.1}$$

Lemma 3.2. *Let $\mathcal{A}$ be a τ-maximal tracial subalgebra of $\mathcal{M}$ that satisfies L^2-density. Then we have*

(i) $\mathcal{A} = ([J(\mathcal{A}_0)]_1)^\perp, \quad \mathcal{A}_0 = ([J(\mathcal{A})]_1)^\perp$;
(ii) $[\mathcal{A}]_1 = {}^\perp J(\mathcal{A}_0), \quad [\mathcal{A}_0]_1 = {}^\perp J(\mathcal{A})$;
(iii) $[\mathcal{A}]_2 = L^2(\mathcal{M}) \cap [\mathcal{A}]_1, \quad [\mathcal{A}_0]_2 = L^2(\mathcal{M}) \cap [\mathcal{A}_0]_1$.

Proof. (i) Assume $x \in \mathcal{A}$. If $y^* \in [J(\mathcal{A}_0)]_1$, then there exist $\{a_n\} \subset \mathcal{A}_0$ such that $a_n \to y$ in $L^1(\mathcal{M})$. Hence, $\tau(xy) = \lim_{n\to\infty} \tau(xa_n) = 0$, so $\mathcal{A} \subset ([J(\mathcal{A}_0)]_1)^\perp$. Conversely, we take $x \in \mathcal{M},\ x \in ([J(\mathcal{A}_0)]_1)^\perp$; i.e., $\tau(xa) = 0$ for all a in $\mathcal{A}_0$. Since $\mathcal{A}$ is τ-maximal, $x \in \mathcal{A}$. Similarly, we have $\mathcal{A}_0 = ([J(\mathcal{A})]_1)^\perp$.

(ii) It is clear that $[\mathcal{A}]_1 \subset {}^\perp J(\mathcal{A}_0)$. Let $x \in L^1(\mathcal{M})$ and $x \in {}^\perp J(\mathcal{A}_0)$. Assume $x \notin [\mathcal{A}]_1$. Then there exists a $y \in \mathcal{M}$ such that $\tau(y^*x) = 1$ and $y \in ([\mathcal{A}]_1)^\perp$. By (i) we obtain that $y \in J(\mathcal{A}_0)$. Hence, $\tau(y^*x) = 0$, and this is a contradiction. Similarly we can prove $[\mathcal{A}_0]_1 = {}^\perp J(\mathcal{A})$.

(iii) It is obvious that $[\mathcal{A}]_2 \subseteq L^2(\mathcal{M}) \cap [\mathcal{A}]_1$. To prove the converse inclusion, let $a \in L^2(\mathcal{M}) \cap [\mathcal{A}]_1$. Then $a \perp J(\mathcal{A}_0)$. From this it follows that $a \perp [J(\mathcal{A})]_2$. By (3.1), $a \in [\mathcal{A}]_2$. $[\mathcal{A}_0]_2 = L^2(\mathcal{M}) \cap [\mathcal{A}_0]_1$ follows from the continuity of $\mathcal{E}$ on $[\mathcal{A}]_2$. □

We say that $\xi \in L^2(\mathcal{M})$ is a right wandering vector if $\tau(\xi^*\xi b) = 0$ for all $b \in \mathcal{A}_0$.

Lemma 3.3. *Let $\mathcal{A}$ be a τ-maximal tracial subalgebra of $\mathcal{M}$ that satisfies L^2-density. If $|\xi| \in L^2(\mathcal{D})$ for all right wandering vectors ξ, then $\mathcal{A}$ has L^2-factorization.*

The proof of Lemma 3.3 is the same as that of Theorem 3.1 of [2]. We omit the details and refer the reader to [2].

Theorem 3.4. *Let $\mathcal{A}$ be a τ-maximal tracial subalgebra of $\mathcal{M}$ that satisfies L^2-density. Then the following conditions are equivalent:*

(i) *If $x \in [\mathcal{A}]_1^+$, then $x^{\frac{1}{2}} \in H^2(\mathcal{A})$.*
(ii) *If ξ is right wandering vector, then $|\xi| \in L^2(\mathcal{D})$.*
(iii) *$\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$.*

Proof. (ii)$\Rightarrow$(iii) follows from Lemma 3.3 and Theorem 2.4.

(iii) $\Rightarrow$(i) is trivial.

(i)$\Rightarrow$(ii). Let ξ be a right wandering vector. Then

$$\tau(\xi^*\xi b) = 0, \ \forall b \in \mathcal{A}_0.$$

By (ii) of Lemma 3.2 we have $\xi^*\xi \in [\mathcal{A}]_1$, and $|\xi| \in [\mathcal{A}]_2$. Hence,

$$\tau(|\xi|b) = 0, \ \forall b \in \mathcal{A}_0 + J(\mathcal{A}_0).$$

So by the L^2-density of $\mathcal{A}$, it follows that $|\xi| \in L^2(\mathcal{D})$. □

Theorem 3.5. *Let $\mathcal{A}$ be a tracial subalgebra of $\mathcal{M}$ that satisfies L^2-density. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$.*
(ii) *For some $0 < p < 1$, $\mathcal{A}$ has L^p-factorization and $\mathcal{A}_p$ is a tracial subalgebra of $\mathcal{M}$.*

Proof. (ii)$\Rightarrow$(i). Since $\mathcal{A}_p$ is a subdiagonal subalgebra of $\mathcal{M}$,

$$L^2(\mathcal{M}) = [\mathcal{A}_p]_2 \oplus [J((\mathcal{A}_p)_0)]_2.$$

So from

$$L^2(\mathcal{M}) = [\mathcal{A}]_2 \oplus [J(\mathcal{A}_0)]_2, \ [\mathcal{A}]_2 \subset [\mathcal{A}_p]_2, \ [J(\mathcal{A}_0)]_2 \subset [J((\mathcal{A}_p)_0)]_2,$$

it follows that $[\mathcal{A}]_2 = [\mathcal{A}_p]_2$. If $x \in L^2(\mathcal{M})^{-1}$, then there is a unitary $u \in \mathcal{M}$ and $a \in [\mathcal{A}_p]_2^{-1} = [\mathcal{A}]_2^{-1}$ such that $x = ua$, since $\mathcal{A}_p$ is a subdiagonal subalgebra. Hence, $\mathcal{A}$ has L^2-factorization. By Theorem 2.4 we obtain the desired result. □

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