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FLAG VARIETIES AS EQUIVARIANT COMPACTIFICATIONS OF $\mathbb{G}_a^n$

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ABSTRACT. Let G be a semisimple affine algebraic group and P a parabolic subgroup of G. We classify all flag varieties G/P which admit an action of the commutative unipotent group $\mathbb{G}_a^n$ with an open orbit.

INTRODUCTION

Let G be a connected semisimple affine algebraic group of adjoint type over an algebraically closed field of characteristic zero and let P be a parabolic subgroup of G. The homogeneous space G/P is called a (generalized) flag variety. Recall that G/P is complete and the action of the unipotent radical P_u^- of the opposite parabolic subgroup P^- on G/P by left multiplication is generically transitive. The open orbit $\mathcal{O}$ of this action is called the big Schubert cell on G/P. Since $\mathcal{O}$ is isomorphic to the affine space $\mathbb{A}^n$, where $n = \dim G/P$, every flag variety may be regarded as a compactification of an affine space.

Notice that the affine space $\mathbb{A}^n$ has a structure of a vector group or, equivalently, of the commutative unipotent affine algebraic group $\mathbb{G}_a^n$. We say that a complete variety X of dimension n is an equivariant compactification of the group $\mathbb{G}_a^n$ if there exists a regular action $\mathbb{G}_a^n \times X \to X$ with a dense open orbit. A systematic study of equivariant compactifications of the group $\mathbb{G}_a^n$ was initiated by B. Hassett and Yu. Tschinkel in [4]; see also [10] and [1].

In this note we address the question whether a flag variety G/P may be realized as an equivariant compactification of $\mathbb{G}_a^n$. Clearly, this is the case when the group P_u^-, or, equivalently, the group P_u is commutative. It is a classical result that the connected component $\widetilde{G}$ of the automorphism group of the variety G/P is a semisimple group of adjoint type, and $G/P = \widetilde{G}/Q$ for some parabolic subgroup $Q \subset \widetilde{G}$. In most cases the group $\widetilde{G}$ coincides with G, and all exceptions are well known; see [6], [7, Theorem 7.1], [12, page 118], [3, Section 2]. If $\widetilde{G} \neq G$, we say that $(\widetilde{G}, Q)$ is the covering pair of the exceptional pair (G, P). For a simple group G, the exceptional pairs are $(\mathrm{PSp}(2r), P_1)$, $(\mathrm{SO}(2r+1), P_r)$ and (G_2, P_1) with the covering pairs $(\mathrm{PSL}(2r), P_1)$, $(\mathrm{PSO}(2r+2), P_{r+1})$ and $(\mathrm{SO}(7), P_1)$ respectively, where PH

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denotes the quotient of the group H by its center and P_i is the maximal parabolic subgroup associated with the ith simple root. It turns out that for a simple group G the condition $\widetilde{G} \neq G$ implies that the unipotent radical Q_u is commutative and P_u is not. In particular, in this case G/P is an equivariant compactification of $\mathbb{G}_a^n$. Our main result states that these are the only possible cases.

Theorem 1. *Let G be a connected semisimple group of adjoint type and P a parabolic subgroup of G. Then the flag variety G/P is an equivariant compactification of $\mathbb{G}_a^n$ if and only if for every pair $(G^{(i)}, P^{(i)})$, where $G^{(i)}$ is a simple component of G and $P^{(i)} = G^{(i)} \cap P$, one of the following conditions holds:*

(1) *The unipotent radical $P_u^{(i)}$ is commutative.*
(2) *The pair $(G^{(i)}, P^{(i)})$ is exceptional.*

For the convenience of the reader, we list all pairs (G, P), where G is a simple group (up to local isomorphism) and P is a parabolic subgroup with a commutative unipotent radical:

$$(\mathrm{SL}(r+1),\, P_i),\ i = 1, \ldots, r; \quad (\mathrm{SO}(2r+1),\, P_1); \quad (\mathrm{Sp}(2r),\, P_r);$$
$$(\mathrm{SO}(2r),\, P_i),\ i = 1, r-1, r; \quad (E_6,\, P_i),\ i = 1, 6; \quad (E_7,\, P_7);$$

see [9, Section 2]. The simple roots $\{\alpha_1, \ldots, \alpha_r\}$ are indexed as in [2, Planches I-IX]. Note that the unipotent radical of P_i is commutative if and only if the simple root α_i occurs in the highest root ρ with coefficient 1; see [9, Lemma 2.2]. Another equivalent condition is that the fundamental weight ω_i of the dual group $G^\vee$ is minuscule; i.e., the weight system of the simple $G^\vee$-module $V(\omega_i)$ with the highest weight ω_i coincides with the orbit $W\omega_i$ of the Weyl group W.

1. Proof of Theorem 1

If the unipotent radical P_u^- is commutative, then the action of P_u^- on G/P by left multiplication is the desired generically transitive $\mathbb{G}_a^n$-action; see, for example, [5, pp. 22-24]. The same arguments work when for the connected component $\widetilde{G}$ of the automorphism group $\mathrm{Aut}(G/P)$ one has $G/P = \widetilde{G}/Q$ and the unipotent radical Q_u^- is commutative. Since

$$G/P \cong G^{(1)}/P^{(1)} \times \ldots \times G^{(k)}/P^{(k)},$$

where $G^{(1)}, \ldots, G^{(k)}$ are the simple components of the group G, the group $\widetilde{G}$ is isomorphic to the direct product $\widetilde{G^{(1)}} \times \ldots \times \widetilde{G^{(k)}}$; cf. [8, Chapter 4]. Moreover, $Q_u \cong Q_u^{(1)} \times \ldots \times Q_u^{(k)}$ with $Q^{(i)} = \widetilde{G^{(i)}} \cap Q$, Thus the group Q_u^- is commutative if and only if for every pair $(G^{(i)}, P^{(i)})$ either $P_u^{(i)}$ is commutative or the pair $(G^{(i)}, P^{(i)})$ is exceptional.

Conversely, assume that G/P admits a generically transitive $\mathbb{G}_a^n$-action. One may identify $\mathbb{G}_a^n$ with a commutative unipotent subgroup H of $\widetilde{G}$, and the flag variety G/P with $\widetilde{G}/Q$, where Q is a parabolic subgroup of $\widetilde{G}$.

Let $T \subset B$ be a maximal torus and a Borel subgroup of the group $\widetilde{G}$ such that $B \subseteq Q$. Consider the root system Φ of the tangent algebra $\mathfrak{g} = \mathrm{Lie}(\widetilde{G})$ defined by the torus T, its decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots associated with B, the set of simple roots $\Delta \subseteq \Phi^+$, $\Delta = \{\alpha_1, \ldots, \alpha_r\}$, and the root decomposition

$$\mathfrak{g} \;=\; \bigoplus_{\beta\in\Phi^-} \mathfrak{g}_\beta \;\oplus \mathfrak{t} \;\oplus \bigoplus_{\beta\in\Phi^+} \mathfrak{g}_\beta,$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ is a Cartan subalgebra in $\mathfrak{g}$ and

$$\mathfrak{g}_\beta = \{x \in \mathfrak{g} : [y, x] = \beta(y)x \text{ for all } y \in \mathfrak{t}\}$$

is the root subspace. Set $\mathfrak{q} = \mathrm{Lie}(Q)$ and $\Delta_Q = \{\alpha \in \Delta : \mathfrak{g}_{-\alpha} \not\subseteq \mathfrak{q}\}$. For every root $\beta = a_1\alpha_1 + \ldots + a_r\alpha_r$ define $\deg(\beta) = \sum_{\alpha_i\in\Delta_Q} a_i$. This gives a $\mathbb{Z}$-grading on the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k, \quad \text{where} \quad \mathfrak{t} \subseteq \mathfrak{g}_0 \quad \text{and} \quad \mathfrak{g}_\beta \subseteq \mathfrak{g}_k \quad \text{with} \quad k = \deg(\beta).$$

In particular,

$$\mathfrak{q} \;=\; \bigoplus_{k\geq 0} \mathfrak{g}_k \quad \text{and} \quad \mathfrak{q}_u^- \;=\; \bigoplus_{k<0} \mathfrak{g}_k.$$

Assume that the unipotent radical Q_u^- is not commutative, and consider $\mathfrak{g}_\beta \subseteq [\mathfrak{q}_u^-, \mathfrak{q}_u^-]$. For every $x \in \mathfrak{g}_\beta \setminus \{0\}$ there exist $z' \in \mathfrak{g}_{\beta'} \subseteq \mathfrak{q}_u^-$ and $z'' \in \mathfrak{g}_{\beta''} \subseteq \mathfrak{q}_u^-$ such that $x = [z', z'']$. In this case $\deg(z') > \deg(x)$ and $\deg(z'') > \deg(x)$.

Since the subgroup H acts on $\widetilde{G}/Q$ with an open orbit, one may conjugate H and assume that the H-orbit of the point eQ is open in $\widetilde{G}/Q$. This implies that $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{h}$, where $\mathfrak{h} = \mathrm{Lie}(H)$. On the other hand, $\mathfrak{g} = \mathfrak{q} \oplus \mathfrak{q}_u^-$. So every element $y \in \mathfrak{h}$ may be (uniquely) written as $y = y_1 + y_2$, where $y_1 \in \mathfrak{q}$, $y_2 \in \mathfrak{q}_u^-$, and the linear map $\mathfrak{h} \to \mathfrak{q}_u^-$, $y \mapsto y_2$, is bijective. Take the elements $y, y', y'' \in \mathfrak{h}$ with $y_2 = x$, $y_2' = z'$, $y_2'' = z''$. Since the subgroup H is commutative, one has $[y', y''] = 0$. Thus

$$[y_1' + y_2', y_1'' + y_2''] \;=\; [y_1', y_1''] + [y_2', y_1''] + [y_1', y_2''] + [y_2', y_2''] \;=\; 0.$$

But

$$[y_2', y_2''] = x \quad \text{and} \quad [y_2', y_1''] + [y_1', y_2''] + [y_2', y_2''] \in \bigoplus_{k>\deg(x)} \mathfrak{g}_k.$$

This contradiction shows that the group Q_u^- is commutative. As we have seen, the latter condition means that for every pair $(G^{(i)}, P^{(i)})$ either the unipotent radical $P_u^{(i)}$ is commutative or the pair $(G^{(i)}, P^{(i)})$ is exceptional. The proof of Theorem 1 is completed.

2. Concluding remarks

If a flag variety G/P is an equivariant compactification of $\mathbb{G}_a^n$, then it is natural to ask for a classification of all generically transitive $\mathbb{G}_a^n$-actions on G/P up to equivariant isomorphism. Consider the projective space $\mathbb{P}^n \cong \mathrm{SL}(n+1)/P_1$. In [4], a correspondence between equivalence classes of generically transitive $\mathbb{G}_a^n$-actions on $\mathbb{P}^n$ and isomorphism classes of local (associative, commutative) algebras of dimension $n+1$ was established. This correspondence together with classification results from [11] yields that for $n \geq 6$ the number of equivalence classes of generically transitive $\mathbb{G}_a^n$-actions on $\mathbb{P}^n$ is infinite; see [4, Section 3]. On the contrary, a generically transitive $\mathbb{G}_a^n$-action on the nondegenerate projective quadric $Q_n \;\cong\; \mathrm{SO}(n+2)/P_1$ is unique [10, Theorem 4]. It would be interesting to study the same problem for the Grassmannians $\mathrm{Gr}(k, r+1) \cong \mathrm{SL}(r+1)/P_k$, where $2 \leq k \leq r-1$.

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