THERE IS NO TAME AUTOMORPHISM OF $\mathbb{C}^3$ WITH MULTIDEGREE $(3, 4, 5)$

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Abstract. Let $F = (F_1, \ldots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be any polynomial mapping. The multidegree of $F$, denoted $\text{mdeg } F$, is the sequence of positive integers $(\deg F_1, \ldots, \deg F_n)$. In this paper we address the following problem: for which sequence $(d_1, \ldots, d_n)$ is there an automorphism or a tame automorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\text{mdeg } F = (d_1, \ldots, d_n)$? We prove, among other things, that there is no tame automorphism $F : \mathbb{C}^3 \to \mathbb{C}^3$ with $\text{mdeg } F = (3, 4, 5)$.

1. Introduction

Let $F = (F_1, F_2) : \mathbb{C}^2 \to \mathbb{C}^2$ be any polynomial automorphism. By the Jung and van der Kulk theorem [1, 2] we know that $\deg F_1 | \deg F_2$ or $\deg F_2 | \deg F_1$. On the other hand, if $d_1, d_2$ are positive integers such that $d_1 | d_2$, then $F = \Phi_2 \circ \Phi_1$, where

$$
\Phi_1 : \mathbb{C}^2 \ni (x, y) \mapsto (x + y^{d_1}, y) \in \mathbb{C}^2,
$$

$$
\Phi_2 : \mathbb{C}^2 \ni (u, w) \mapsto (u, w + u^{d_2}) \in \mathbb{C}^2
$$

is an automorphism of $\mathbb{C}^2$ such that $\text{mdeg } F = (d_1, d_2)$. Similarly if $d_2 | d_1$ we can give an appropriate automorphism of $\mathbb{C}^2$. Thus a sequence of positive integers $(d_1, d_2)$ is the multidegree of some polynomial automorphism of $\mathbb{C}^2$ if and only if $d_1 | d_2$ or $d_2 | d_1$.

It seems natural to ask for which sequence $(d_1, \ldots, d_n)$ is there a polynomial automorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\text{mdeg } F = (d_1, \ldots, d_n)$? Also, the question about existence of a tame automorphism $F : \mathbb{C}^n \to \mathbb{C}^n$ with $\text{mdeg } F = (d_1, \ldots, d_n)$ is natural. Recall that a tame automorphism is, by definition, a composition of linear automorphisms and triangular automorphisms, where a triangular automorphism is a mapping of the form

$$
T : \mathbb{C}^n \ni \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ x_2 + f_2(x_1) \\ \vdots \\ x_n + f_n(x_1, \ldots, x_{n-1}) \end{pmatrix} \in \mathbb{C}^n.
$$
We will denote by Tame($\mathbb{C}^n$) the group of all tame automorphisms of $\mathbb{C}^n$. This is, of course, a subgroup of the group Aut($\mathbb{C}^n$) of all polynomial automorphisms of $\mathbb{C}^n$.

It is easy to see that if there is an automorphism (or tame automorphism) $F: \mathbb{C}^n \to \mathbb{C}^n$ such that $mdeg F = (d_1, \ldots, d_n)$, then there is also an automorphism (or tame automorphism) $\tilde{F}: \mathbb{C}^n \to \mathbb{C}^n$ such that $mdeg \tilde{F} = (d_{\sigma(1)}, \ldots, d_{\sigma(n)})$ for any permutation $\sigma$ of the set $\{1, \ldots, n\}$. Thus without loss of generality, we can assume that $d_1 \leq d_2 \leq \ldots \leq d_n$.

2. SOME SIMPLE REMARKS

In this section we make some simple but useful remarks about existence of automorphisms and tame automorphisms with a given multidegree.

**Proposition 2.1.** If for $1 \leq d_1 \leq \ldots \leq d_n$ there is a sequence of integers $1 \leq i_1 < \ldots < i_m \leq n$, with $m < n$, such that there exists an automorphism $G$ of $\mathbb{C}^m$ with $mdeg G = (d_{i_1}, \ldots, d_{i_m})$, then there exists an automorphism $F$ of $\mathbb{C}^n$ with $mdeg F = (d_1, \ldots, d_n)$. Moreover, if $G$ is tame, then an $F$ can also be found that is tame.

**Proof.** Let $j_1, \ldots, j_{n-m} \in \mathbb{N}$ be such that $1 \leq j_1 < \ldots < j_{n-m} \leq n$ and $\{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_{n-m}\} = \{1, \ldots, n\}$. Of course, $\{i_1, \ldots, i_m\} \cap \{j_1, \ldots, j_{n-m}\} = \emptyset$. Consider the mapping $h = (h_1, \ldots, h_n): \mathbb{C}^n \to \mathbb{C}^n$ given by

$$h_k(x_1, \ldots, x_n) = \begin{cases} x_k & \text{for } k \in \{i_1, \ldots, i_m\}, \\ x_k + (x_{i_1})^{d_k} & \text{for } k \in \{j_1, \ldots, j_{n-m}\}. \end{cases}$$

Of course $h$ is an automorphism of $\mathbb{C}^n$ and $\deg h_k = d_k$ for $k \in \{j_1, \ldots, j_{n-m}\}$.

Consider also the mapping $g = (g_1, \ldots, g_n): \mathbb{C}^n \to \mathbb{C}^n$ given by

$$g_k(u_1, \ldots, u_n) = \begin{cases} G_i(u_{i_1}, \ldots, u_{i_m}) & \text{for } k = i_i, \\ u_k & \text{for } k \in \{j_1, \ldots, j_{n-m}\}. \end{cases}$$

It is easy to see that $g$ is an automorphism of $\mathbb{C}^n$ and $\deg g_k = d_k$ for $k \in \{i_1, \ldots, i_m\}$.

Now taking $F = g \circ h$ we obtain an automorphism of $\mathbb{C}^n$ such that $\deg F_i = d_i$ for all $i \in \{1, \ldots, n\}$. $\square$

**Proposition 2.2.** If for a sequence of integers $1 \leq d_1 \leq \ldots \leq d_n$ there is an $i \in \{1, \ldots, n\}$ such that

$$d_i = \sum_{j=1}^{i-1} k_j d_j \quad \text{with} \quad k_j \in \mathbb{N},$$

then there exists a tame automorphism $F$ of $\mathbb{C}^n$ with $mdeg F = (d_1, \ldots, d_n)$.

**Proof.** Define $h = (h_1, \ldots, h_n): \mathbb{C}^n \to \mathbb{C}^n$ and $g = (g_1, \ldots, g_n): \mathbb{C}^n \to \mathbb{C}^n$ by

$$h_k(x_1, \ldots, x_n) = \begin{cases} x_k & \text{for } k = i_i, \\ x_k + x_i^{d_k} & \text{for } k \neq i \end{cases}$$

and

$$g_k(u_1, \ldots, u_n) = \begin{cases} u_k + u_1^{k_1} \cdots u_{i-1}^{k_{i-1}} & \text{for } k = i, \\ u_k & \text{for } k \neq i. \end{cases}$$
It is easy to see that \( h \) and \( g \) are automorphisms of \( \mathbb{C}^n \) such that \( \deg h_k = d_k \) for \( k \neq i \) and \( \deg g_i (h_1, \ldots, h_n) = d_i \). Since also \( g_k (u_1, \ldots, u_n) = u_k \) for \( k \neq i \), it is easy to see that the automorphism \( F = g \circ h \) has \( \deg F_k = d_k \) for all \( k \in \{1, \ldots, n\} \). □

Let us notice that the condition on \( (d_1, \ldots, d_n) \) given in the above proposition is not necessary for the existence of a tame automorphism \( F \) with \( \text{mdeg} F = (d_1, \ldots, d_n) \). For example, Shestakov and Umirbaev gave in [3] an example of \( F \in \text{Tame} (\mathbb{C}^3) \) with \( \text{mdeg} F = (6, 8, 9) \). See also Proposition 5.2 below.

**Corollary 2.3.** If for a sequence of integers \( 1 \leq d_1 \leq \ldots \leq d_n \) we have \( d_1 \leq n - 1 \), then there exists a tame automorphism \( F \) of \( \mathbb{C}^n \) with \( \text{mdeg} F = (d_1, \ldots, d_n) \).

**Proof.** Let \( r_i \in \{0, 1, \ldots, d_i - 1\} \) be such that \( d_i \equiv r_i (\text{mod} d_i) \) for \( i = 2, \ldots, n \). If there is an \( i \in \{2, \ldots, n\} \) such that \( r_i = 0 \), then \( d_i = k d_1 \) for some \( k \in \mathbb{N}\setminus\{0\} \) and by Proposition 2.2 there exists an automorphism \( F \) of \( \mathbb{C}^n \) with the desired properties.

Thus we can assume that \( r_i \neq 0 \) for all \( i = 2, \ldots, n \). Since \( d_1 - 1 < n - 1 \), there are \( i, j \in \{2, \ldots, n\} \), \( i \neq j \), such that \( r_i = r_j \). Without loss of generality we can assume that \( i < j \), so \( d_j = d_i + k d_1 \) for some \( k \in \mathbb{N} \). Then by Proposition 2.2 there exists an automorphism \( F \) of \( \mathbb{C}^n \) with the desired properties. □

### 3. Examples

In this section we give some positive results about existence of tame automorphisms of \( \mathbb{C}^3 \) with a given multidegree \((d_1, d_2, d_3)\). The first one is the following.

**Example 3.1.** For every \( d_2, d_3 \in \mathbb{N} \) with \( 2 \leq d_2 \leq d_3 \), there is a tame automorphism \( F \) of \( \mathbb{C}^3 \) such that

\[
\text{mdeg} F = (2, d_2, d_3).
\]

**Proof.** This is a consequence of Corollary 2.3 □

**Example 3.2.** For any \( d_3 \geq 4 \) such that \( d_3 \neq 5 \) there is a tame automorphism \( F \) of \( \mathbb{C}^3 \) such that

\[
\text{mdeg} F = (3, d_3).
\]

**Proof.** We have

\[
4 = 0 \cdot 3 + 1 \cdot 4
\]

and

\[
d_3 = \begin{cases} 
(2 + k) \cdot 3 + 0 \cdot 4 & \text{for } d_3 = 6 + 3k, \\
(1 + k) \cdot 3 + 1 \cdot 4 & \text{for } d_3 = 7 + 3k, \\
(0 + k) \cdot 3 + 2 \cdot 4 & \text{for } d_3 = 8 + 3k.
\end{cases}
\]

Thus we can apply Proposition 2.2 □

**Example 3.3.** For any \( d_3 \geq 5 \) such that \( d_3 \neq 7 \) there is a tame automorphism \( F \) of \( \mathbb{C}^3 \) such that

\[
\text{mdeg} F = (3, 5, d_3).
\]

**Proof.** We have

\[
5 = 0 \cdot 3 + 1 \cdot 5, \quad 6 = 2 \cdot 3 + 0 \cdot 5
\]

and

\[
d_3 = \begin{cases} 
(1 + k) \cdot 3 + 1 \cdot 5 & \text{for } d_3 = 8 + 3k, \\
(3 + k) \cdot 3 + 0 \cdot 5 & \text{for } d_3 = 9 + 3k, \\
(0 + k) \cdot 3 + 2 \cdot 5 & \text{for } d_3 = 10 + 3k.
\end{cases}
\]
Thus we can apply Proposition 2.2.

Example 3.4. For any $d_3 \geq 5$ such that $d_3 \neq 6, 7, 11$, there is a tame automorphism $F$ of $\mathbb{C}^3$ such that

$$\text{mdeg } F = (4, 5, d_3).$$

Proof. We have

$$5 = 0 \cdot 4 + 1 \cdot 5, \quad 8 = 2 \cdot 4 + 0 \cdot 5,$$

$$9 = 1 \cdot 4 + 1 \cdot 5, \quad 10 = 0 \cdot 4 + 2 \cdot 5$$

and

$$d_3 = \begin{cases} 
(3 + k) \cdot 4 + 0 \cdot 5 & \text{for } d_3 = 12 + 4k, \\
(2 + k) \cdot 4 + 1 \cdot 5 & \text{for } d_3 = 13 + 4k, \\
(1 + k) \cdot 4 + 2 \cdot 5 & \text{for } d_3 = 14 + 4k, \\
(0 + k) \cdot 4 + 3 \cdot 5 & \text{for } d_3 = 15 + 4k.
\end{cases}$$

Thus we can apply Proposition 2.2.

The above examples justify the following question.

Question. Is there any automorphism (or tame automorphism) $F$ of $\mathbb{C}^3$ such that $\text{mdeg } F \in \{(3, 4, 5), (3, 5, 7), (4, 5, 6), (4, 5, 7), (4, 5, 11)\}$?

4. Partial answer

In this section we give a partial answer to the question posed in the last section. Namely, we show the following.

Theorem 4.1. There is no tame automorphism $F = (F_1, F_2, F_3)$ of $\mathbb{C}^3$ such that $\text{mdeg } F = (3, 4, 5)$.

Before we give the proof of Theorem 4.1 we recall some results and notions from the papers of Shestakov and Umirbaev [3, 4].

Definition 4.2 ([3], Definition 1). A pair $f, g \in k[X_1, \ldots, X_n]$ is called *-reduced if

(i) $f, g$ are algebraically independent;
(ii) $\overline{f}, \overline{g}$ are algebraically dependent, where $\overline{h}$ denotes the highest homogeneous part of $h$;
(iii) $\overline{f} \not\in k[\overline{g}]$ and $\overline{g} \not\in k[\overline{f}]$.

Definition 4.3 ([3], Definition 1). Let $f, g \in k[X_1, \ldots, X_n]$ be a *-reduced pair with $\deg f < \deg g$. Put $p = \frac{\deg f}{\gcd(\deg f, \deg g)}$. Then $f, g$ is called a $p$-reduced pair.

Theorem 4.4 ([3], Theorem 2). Let $f, g \in k[X_1, \ldots, X_n]$ be a $p$-reduced pair, and let $G(x, y) \in k[x, y]$ with $\deg y G(x, y) = pq + r$, $0 \leq r < p$. Then

$$\deg G(f, g) \geq q(p \deg g - \deg f) + r \deg g.$$

In the above theorem, $[f, g]$ denotes the Poisson bracket of $f$ and $g$; for us it is only important that

$$\deg [f, g] = 2 + \max_{1 \leq i < j \leq n} \deg \left( \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right)$$

if $f, g$ are algebraically independent, and $\deg [f, g] = 0$ otherwise.
Notice also that the estimate from Theorem 4.1 is true even if the condition (ii) of Definition 4.2 is not satisfied. Indeed, if \( G(x, y) = \sum_{i,j} a_{i,j} x^iy^j \), then by the algebraic independence of \( f \) and \( g \) we have

\[
\deg G(f, g) = \max_{i,j} \deg(a_{i,j} f^i g^j) \geq \deg_y G(x, y) \cdot \deg g
\]

\[
= (pq + r) \deg g \geq q(p \deg g - \deg f - \deg g + \deg[f, g]) + r \deg g.
\]

The last inequality is a consequence of the fact that \( \deg[f, g] \leq 0 \).

We will also use the following theorem.

**Theorem 4.5** ([3], Theorem 3). Let \( F = (F_1, F_2, F_3) \) be a tame automorphism of \( \mathbb{C}^3 \). If \( \deg F_1 + \deg F_2 + \deg F_3 > 3 \) (in other words, if \( F \) is not a linear automorphism), then \( F \) admits either an elementary reduction or a reduction of one of the types I-IV (see [3, Definitions 2-4]).

Let us recall that an automorphism \( F = (F_1, F_2, F_3) \) admits an elementary reduction if there exists a polynomial \( g \in \mathbb{C}[x, y] \) and a permutation \( \sigma \) of \( \{1, 2, 3\} \) such that \( \deg(F_{\sigma(1)} - g(F_{\sigma(2)}, F_{\sigma(3)})) < \deg F_{\sigma(1)} \).

Now we are in a position to prove Theorem 4.1.

**Proof of Theorem 4.1.** Assume that \( F = (F_1, F_2, F_3) \) is an automorphism of \( \mathbb{C}^3 \) such that \( m \deg F = (3, 4, 5) \). We will show that this hypothetical automorphism cannot be tame. First of all, notice that any pair \( F_1, F_3 \) with \( i, j \in \{1, 2, 3\}, i \neq j \), satisfies conditions (i) and (iii) of Definition 4.2. Indeed, this follows from the fact that \( F_1, F_2, F_3 \) are algebraically independent and that \( 3, 4 \notin 5\mathbb{N}, 3, 5 \notin 4\mathbb{N} \) and \( 4, 5 \notin 3\mathbb{N} \). By Theorem 4.5, it is enough to show that \( F \) admits neither reductions of types I-IV nor an elementary reduction.

Assume that \( (F_1, F_2, F_3) \) admits a reduction of type I or II. Then by the definition (see [3, Definitions 2 and 3]), for some \( n \in \mathbb{N} \setminus \{0\} \) and some permutation \( \sigma \) of \( \{1, 2, 3\} \) we have \( \deg F_{\sigma(1)} = 2n \) and \( \deg F_{\sigma(3)} = ns \), where \( s \geq 3 \) is odd. But in the sequence 3, 4, 5 there is only one even number, namely 4. Thus \( 2n = 4, n = 2 \) and then \( n \) is also even, a contradiction.

Now assume that \( (F_1, F_2, F_3) \) admits a reduction of type III or IV. Then by the definition (see [3, Definition 4]), for some \( n \in \mathbb{N} \setminus \{0\} \) and some permutation \( \sigma \) of \( \{1, 2, 3\} \) we have \( \deg F_{\sigma(1)} = 2n \) and either

\( \deg F_{\sigma(2)} = 3n, \quad n < \deg F_{\sigma(3)} \leq \frac{3}{2}n \) \hspace{1cm} (1)

or

\( \frac{5}{2}n < \deg F_{\sigma(2)} \leq 3n, \quad \deg F_{\sigma(3)} = \frac{3}{2}n. \) \hspace{1cm} (2)

Of course, as before, we have \( 2n = 4, n = 2 \). Since \( 3n = 6 \), (1) is impossible, and since \( \frac{5}{2}n = 5, 3n = 6 \) and \( \deg F_{\sigma(2)} \in \mathbb{N} \), (2) is also impossible. Thus we obtain a contradiction.

It remains to show that \( (F_1, F_2, F_3) \) does not admit an elementary reduction.

Assume that \( (F_1, F_2, F_3) \) does not admit an elementary reduction.

\( (F_1, F_2, F_3 - g(F_1, F_2)) \), where \( g \in k[x, y] \), is an elementary reduction of \( (F_1, F_2, F_3) \). Thus, in particular, \( \deg g(F_1, F_2) = 5 \). But this is impossible. Indeed, by Theorem 4.1 we have

\( \deg g(F_1, F_2) \geq q(pm - m - n + \deg[F_1, F_2]) + mr, \) \hspace{1cm} (3)
where \( n = \deg F_1, m = \deg F_2, p = n/\gcd(n, m) \) and \( \deg_y g(x, y) = q p + r \) with \( 0 \leq r < p \). In our case we have \( n = 3, m = 4, p = 3 \). Since \( F_1, F_2 \) are algebraically independent, \( \deg [F_1, F_2] \geq 2 \). Thus (3) can be rewritten as follows:

\[
\deg g(F_1, F_2) \geq q(3 \cdot 4 - 4 - 3 + \deg [F_1, F_2]) + 4r.
\]

Since also \( 3 \cdot 4 - 4 - 3 + \deg [F_1, F_2] = 5 + \deg [F_1, F_2] \geq 7 > 5, q \) must be zero and \( r \) must be not greater than 1. This means that \( g(F_1, F_2) = g_1(F_1) + g_2(F_1) F_2 \) for some \( g_1, g_2 \in k[x] \). Since \( 3 \mathbb{N} \cap (4 + 3 \mathbb{N}) = \emptyset \), we see that \( \deg g(F_1, F_2) = \max \{ 3 \deg g_1, 4 + 3 \deg g_2 \} \). Since \( 5 \notin 3 \mathbb{N} \cup (4 + 3 \mathbb{N}) \), we obtain a contradiction.

Now, assume that

\[
(F_1, F_2 - g(F_1, F_3), F_3),
\]

where \( g \in k[x, y] \), is an elementary reduction of \((F_1, F_2, F_3)\). In this case we have \( \deg g(F_1, F_3) = 4 \). But this is also impossible. Indeed, by Theorem 4.4.1 we have

\[
\deg g(F_1, F_3) \geq q(p m - m - n + \deg [F_1, F_2]) + mr,
\]

where \( n = 3, m = 5, p = 3 \) and \( \deg_y g(x, y) = 3q + r \) with \( 0 \leq r < 3 \). Since \( pm - m - n + \deg [F_1, F_2] = 7 + \deg [F_1, F_2] > 4 \), we see that \( q \) must be zero. Also, \( r \) must be zero, because \( m = 5 > 4 \). Thus \( g(F_1, F_3) = g(F_1) \), and then \( \deg g(F_1, F_3) = 3 \deg g \). This contradicts \( 4 \notin 3 \mathbb{N} \).

Finally, assume that

\[
(F_1 - g(F_2, F_3), F_2, F_3),
\]

where \( g \in k[x, y] \), is an elementary reduction of \((F_1, F_2, F_3)\). As before, we obtain

\[
3 = \deg g(F_2, F_3) \geq q(4 \cdot 5 - 5 - 4 + \deg [F_2, F_3]) + 5r,
\]

where \( \deg_y g(x, y) = 4q + r \) with \( 0 \leq r < 4 \). Then \( q \) and \( r \) must be zero. Thus \( g(F_2, F_3) = g(F_2) \), and then \( \deg g(F_2, F_3) = 4 \deg g \). This contradicts \( 3 \notin 4 \mathbb{N} \).

In a similar way we can show the following theorem.

**Theorem 4.6.** There is no tame automorphism \( F \) of \( \mathbb{C}^3 \) such that

\[
mdeg F \in \{(3, 5, 7), (4, 5, 7), (4, 5, 11)\}.
\]

By the above theorem, Theorem 4.1, Corollary 2.2 and examples from Section 3 we have the following theorem.

**Theorem 4.7.** In the following statements mdeg is considered as a map from the set of all endomorphisms of \( \mathbb{C}^n \) into the set \( \mathbb{N}^n \).

(i) For all integers \( d_3 \geq d_2 \geq 2, (2, d_2, d_3) \in \mdeg(Tame(\mathbb{C}^3)) \).

(ii) If \( d_3 \geq 4 \), then \( (3, 4, d_3) \in \mdeg(Tame(\mathbb{C}^3)) \) if and only if \( d_3 \neq 5 \).

(iii) If \( d_3 \geq 5 \), then \( (3, 5, d_3) \in \mdeg(Tame(\mathbb{C}^3)) \) if and only if \( d_3 \neq 7 \).

(iv) If \( d_3 \geq 5 \) and \( d_3 \neq 6, 7, 11 \), then \( (4, 5, d_3) \in \mdeg(Tame(\mathbb{C}^3)) \).

Notice that (iv) cannot yet be formulated with “if and only if”, as (iii) can, because we do not know whether \( (4, 5, 6) \in \mdeg(Tame(\mathbb{C}^3)) \) or not. The author believes that \( (4, 5, 6) \notin \mdeg(Tame(\mathbb{C}^3)) \), which however seems to be hard to prove.
5. General conjecture

The author believes that for any prime number $p \geq 3$ and $d_3 \geq d_2 \geq p$ the following is true.

**Conjecture 5.1.** $(p, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ if and only if $p|d_2$ or $d_3 \in p\mathbb{N} + d_2\mathbb{N}$.

When $d_1$ is a composite number one cannot expect a similar result, as the following proposition shows.

**Proposition 5.2.** For any number $d_3 \geq 6$ we have $(4, 6, d_1) \in \text{mdeg} \left( \text{Tame} \left( \mathbb{C}^3 \right) \right)$.

**Proof.** If $d_3$ is even, then one can easily see that $d_3 \in 4\mathbb{N} + 6\mathbb{N}$. Thus we can assume that $d_3$ is odd and use the following constructions.

Since $(x + z^4)^3 = z^{12} + 3xz^8 + 3x^2z^4 + x^3$ and $(y + z^6)^2 = z^{12} + 2yz^6 + y^2$, we see that $\text{deg} \left[ (y + z^6)^2 - (x + z^4)^3 \right] = 9$ and so $\text{deg} \left[ (y + z^6)^2 - (x + z^4)^3 \right] (x + z^4)^k = 9 + 4k$ for any $k \in \mathbb{N}$. This means that

$$\text{mdeg}(F_2 \circ F_1) = (4, 6, 9 + 4k),$$

where

$$F_1(x, y, z) = (x + z^4, y + z^6, z),$$

$$F_2(u, v, w) = (u, v, w + (v^2 - u^3) w^k).$$

Since $(x + z^4)^3 = z^{12} + 3xz^8 + 3x^2z^4 + x^3$ and $(y + \frac{3}{2}xz^2 + z^6)^2 = z^{12} + 3xz^8 + 2yz^6 + \frac{9}{4}x^2z^4 + 3yx^2 + y^2$, it follows that $\text{deg} \left[ (y + \frac{3}{2}xz^2 + z^6)^2 - (x + z^4)^3 \right] = 7$ and $\text{deg} \left[ (y + \frac{3}{2}xz^2 + z^6)^2 - (x + z^4)^3 \right] (x + z^4)^k = 7 + 4k$ for any $k \in \mathbb{N}$. Thus we have

$$\text{mdeg}(F_2 \circ F_1) = (4, 6, 7 + 4k),$$

where

$$F_1(x, y, z) = (x + z^4, y + \frac{3}{2}xz^2 + z^6, z),$$

$$F_2(u, v, w) = (u, v, w + (v^2 - u^3) w^k).$$

\[\square\]

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