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ON SPACES OF COMPACT OPERATORS ON $C(K, X)$ SPACES

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ABSTRACT. This paper concerns the spaces of compact operators $\mathcal{K}(E, F)$, where E and F are Banach spaces $C([1, \xi], X)$ of all continuous X-valued functions defined on the interval of ordinals $[1, \xi]$ and equipped with the supremun norm. We provide sufficient conditions on X, Y, α, β, ξ and η, with $\omega \leq \alpha \leq \beta < \omega_1$ for the following equivalence:

(a) $\mathcal{K}(C([1, \xi], X), C([1, \alpha], Y))$ is isomorphic to $\mathcal{K}(C([1, \eta], X), C([1, \beta], Y))$,
(b) $\beta < \alpha^\omega$.

In this way, we unify and extend results due to Bessaga and Pełczyński (1960) and C. Samuel (2009). Our result covers the case of the classical spaces $X = l_p$ and $Y = l_q$, with $1 < p, q < \infty$.

1. INTRODUCTION

We use standard Banach space theory terminology and notions as can be found in [12] and [13]. Let X be a Banach space and K a compact Hausdorff space. By $C(K, X)$ we denote the Banach space of all continuous X-valued functions defined on K and equipped with the supremum norm. This space will be denoted by $C(K)$ in the case $X = \mathbb{R}$. Given Banach spaces X and Y, $\mathcal{K}(X, Y)$ denotes the Banach space of compact operators from X to Y. We write $X \sim Y$ when the Banach spaces X and Y are isomorphic and $X \hookrightarrow Y$ when Y has a subspace isomorphic to X. Let α be an ordinal number. By $[1, \alpha]$ we denote the interval of ordinals $\{\xi : 1 \leq \xi \leq \alpha\}$ endowed with the order topology.

The present paper is devoted to the isomorphic classifications of spaces of compact operators from $C([1, \xi], X)$ to $C([1, \eta], Y)$ spaces, with $\eta < \omega_1$. A fundamental result along these lines is the classical Bessaga-Pełczyński 1960 isomorphic classification of $C([1, \alpha])$ spaces, with $\alpha < \omega_1$ [1, Theorem 4.1], that is:

Theorem 1.1. *Suppose that $\omega \leq \alpha \leq \beta < \omega_1$. Then we have*

$$C([1, \alpha]) \sim C([1, \beta]) \Longleftrightarrow \beta < \alpha^\omega.$$

Very recently C. Samuel classified the $\mathcal{K}(C([1, \alpha]), C([1, \alpha]))$ spaces, with $\alpha < \omega_1$ [15, Theorem 3.3] by proving:

Theorem 1.2. *Suppose that $\omega \leq \alpha \leq \beta < \omega_1$. Then we have*

$$\mathcal{K}(C([1, \alpha]), C([1, \alpha])) \sim \mathcal{K}(C([1, \beta]), C([1, \beta])) \Longleftrightarrow \beta < \alpha^\omega.$$

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The main aim of this short work is to unify and extend these results as follows.

Theorem 1.3. *Let α, β, ξ and η be ordinals with $\omega \leq \alpha \leq \beta < \omega_1$, $\xi\eta < \omega$ or else ξ and η of the same cardinality, X a Banach space such that X^* is weakly sequentially complete and has the approximation property and Y a Banach space which has an unconditional basis and contains no subspace isomorphic to l_1. Then*

$$\mathcal{K}(C([1,\xi],X), C([1,\alpha],Y)) \sim \mathcal{K}(C([1,\eta],X), C([1,\beta],Y)) \Longleftrightarrow \beta < \alpha^\omega.$$

Observe that Theorem 1.1 is the case $\xi = \eta = 1$ and $X = Y = \mathbb{R}$ of Theorem 1.3. Moreover, Theorem 1.2 is the case $\alpha = \xi$, $\beta = \eta$ and $X = Y = \mathbb{R}$ of Theorem 1.3. Notice also that as an immediate consequence of Theorem 1.3 and Theorem 1.1 we get the following cancellation law:

Corollary 1.4. *Let α, β, ξ and η be ordinals with $\omega \leq \alpha \leq \beta < \omega_1$, $\xi\eta < \omega$ or else ξ and η of the same cardinality. Then for every $1 < p, q < \infty$ we have*

$$\mathcal{K}(C([1,\xi],l_p), C([1,\alpha],l_q)) \sim \mathcal{K}(C([1,\eta],l_p), C([1,\beta],l_q)) \Longleftrightarrow C([1,\alpha]) \sim C([1,\beta]).$$

This corollary gives a partial answer to [8, Problem 4.2.3]. We stress that the statement of Corollary 1.4 is also true for $1 < p < \infty$ and $q = 1$; see [8, Remark 4.1.3] for $\xi < \omega$ and [9, Remark 1.7] for $\xi \geq \omega$. Nevertheless the isomorphic classification of the $\mathcal{K}(l_1, l_q^\alpha)$ spaces, with $\omega \leq \alpha < \omega_1$ and $1 \leq q < \infty$, remains an open question; see [8, Problem 4.2.2]. In order to prove Theorem 1.3 we need some preliminary results.

2. Preliminary results

From now on following [1] the $C([1,\alpha],X)$ spaces will be denoted by X^α. One of the key steps in proving Theorem 1.3 is the following proposition. We will denote by $X \hat{\hat{\otimes}} Y$ the injective tensor product of the Banach spaces X and Y.

Proposition 2.1. *Let X be a weakly sequentially complete Banach space and Y a Banach space which has an unconditional basis and contains no subspace isomorphic to l_1. Then for every set Γ we have*

$$\mathbb{R}^{\omega^\omega} \not\hookrightarrow l_1(\Gamma, X) \hat{\hat{\otimes}} Y^\omega.$$

Proof. Suppose for contradiction that $l_1(\Gamma, X) \hat{\hat{\otimes}} Y^\omega$ has a subspace isomorphic to $\mathbb{R}^{\omega^\omega}$. Since $\mathbb{R}^{\omega^\omega}$ is separable, it is easily seen that

$$\mathbb{R}^{\omega^\omega} \hookrightarrow l_1(\mathbb{N}, X) \hat{\hat{\otimes}} Y^\omega. \tag{1}$$

Assume that Y has an unconditional basis and contains no subspace isomorphic to l_1. Then Y^ω does not contain a subspace isomorphic to l_1 (see for instance [7, Theorem 2.3]) and has an unconditional basis. So by a well known result of James [12, Theorem 1.c.9] Y^ω has a shrinking unconditional basis. Moreover, by [5, Corollary 1] $l_1(\mathbb{N}, X)$ is weakly sequentially complete. Therefore according to [5, Theorem 3] $l_1(\mathbb{N}, X) \hat{\hat{\otimes}} Y^\omega$ has the property (u) introduced by Pełczyński in [14, Definition 1]. Since this property is inherited by closed subspaces ([14, page 252] or [13, Proposition 1.c.3]), we would conclude by (1) that $\mathbb{R}^{\omega^\omega}$ has the property (u), which is absurd by an unpublished result of Pełczyński; see [10, pages 210-211] and [11, Proposition 5.3]. □

The next proposition is inspired by [15, Theorem 3.2].

Proposition 2.2. *Let $\omega^\omega \le \xi \le \eta < \omega_1$ be two ordinals and X a Banach space such that X^ω contains no subspace isomorphic to $\mathbb{R}^{\omega^{\omega}}$. If $\mathbb{R}^\eta \hookrightarrow X^\xi$, then $\mathbb{R}^\eta \hookrightarrow \mathbb{R}^\xi$.*

Proof. We introduce two sets of ordinals:

$$I_1 = \{\theta : \omega^\omega \le \theta < \omega_1, \ \ \mathbb{R}^\theta \not\hookrightarrow \mathbb{R}^\gamma, \ \forall \gamma < \theta\},$$
$$I_2 = \{\theta : \omega^\omega \le \theta < \omega_1, \ \ \mathbb{R}^\theta \not\hookrightarrow X^\gamma, \ \forall \gamma < \theta\}.$$

First of all we will prove that $I_1 = I_2$. Clearly $I_2 \subset I_1$. Observe that by Theorem 1.1 and our hypothesis, we deduce that $\omega^\omega \in I_2$. Now, assume that I_2 is a proper subset of I_1. Let α_1 be the least element of $I_1 \backslash I_2$. We have $\omega^\omega < \alpha_1$. Since $\alpha_1 \notin I_2$, there exists an ordinal $\gamma_1 < \alpha_1$ such that $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\gamma_1}$.

Let $\alpha_2 = \min\{\gamma, \ \omega^\omega \le \gamma < \alpha_1 : \mathbb{R}^{\alpha_1} \hookrightarrow X^\gamma\}$. We have $\alpha_2 \le \gamma_1$. Now, we will show that $\alpha_2 \in I_1$. If this is not the case, there exists an ordinal $\gamma_2 < \alpha_2$ such that $\mathbb{R}^{\alpha_2} \hookrightarrow \mathbb{R}^{\gamma_2}$. Therefore $X^{\alpha_2} \hookrightarrow X^{\gamma_2}$. Consequently, $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\gamma_2}$. This is a contradiction to the definition of α_2.

So $\alpha_2 \in I_1$ and since $\alpha_2 < \alpha_1$, it follows from the definition of α_1 that $\alpha_2 \in I_2$. That is, $\mathbb{R}^{\alpha_2} \not\hookrightarrow X^\gamma$, $\forall \gamma < \alpha_2$. Thus by [6, Lemma 3.3], we conclude that $\mathbb{R}^{\alpha_2^\omega} \not\hookrightarrow X^{\alpha_2}$.

On the other hand, note that if $\alpha_1 < {\alpha_2}^\omega$, then by Theorem 1.1, $\mathbb{R}^{\alpha_1} \sim \mathbb{R}^{\alpha_2}$, which is absurd by the definition of α_1. Consequently ${\alpha_2}^\omega \le \alpha_1$ and $\mathbb{R}^{\alpha_2^\omega} \hookrightarrow \mathbb{R}^{\alpha_1}$. Furthermore, by the definition of α_2, $\mathbb{R}^{\alpha_1} \hookrightarrow X^{\alpha_2}$. Therefore $\mathbb{R}^{\alpha_2^\omega} \hookrightarrow X^{\alpha_2}$, in contradiction to what we have proved above. Hence $I_1 = I_2$.

Next, to complete the proof of the proposition, suppose that $\mathbb{R}^\eta \not\hookrightarrow \mathbb{R}^\xi$ and let $\xi_1 = \min\{\theta : \ \mathbb{R}^\eta \hookrightarrow \mathbb{R}^\theta\}$. Hence $\xi < \xi_1 \le \eta$ and $\mathbb{R}^{\xi_1} \not\hookrightarrow \mathbb{R}^\gamma$, $\forall \gamma < \xi_1$. In particular, $\xi_1 \in I_1 = I_2$, which is absurd, because $\mathbb{R}^{\xi_1} \hookrightarrow \mathbb{R}^\eta \hookrightarrow X^\xi$. □

3. Proof of Theorem 1.3

Initially notice that if γ and θ are ordinals with $\gamma < \omega$ and $\omega \le \theta < \omega_1$, then by Theorem 1.1, $\mathbb{R}^{\theta\gamma} \sim \mathbb{R}^\theta$. Therefore by [3, Proposition 5.3] we deduce that

$$\mathcal{K}(X^\gamma, Y^\theta) \sim (X^*)^\gamma \hat{\hat{\otimes}} Y^\theta \sim X^* \hat{\hat{\otimes}} Y \hat{\hat{\otimes}} \mathbb{R}^\gamma \hat{\hat{\otimes}} \mathbb{R}^\theta \sim X^* \hat{\hat{\otimes}} Y \hat{\hat{\otimes}} \mathbb{R}^{\theta\gamma} \sim X^* \hat{\hat{\otimes}} Y^\theta. \tag{2}$$

On the other hand, if $\omega \le \gamma$, since $l_1([1,\gamma], X^*)$ has the approximation property (see for instance [2, page 285]), it follows from [3, Proposition 5.3] that

$$\mathcal{K}(X^\gamma, Y^\theta) \sim l_1([1,\gamma], X^*) \hat{\hat{\otimes}} Y^\theta. \tag{3}$$

Now suppose that $\beta < \alpha^\omega$. We have by Theorem 1.1 that $\mathbb{R}^\alpha \sim \mathbb{R}^\beta$ and thus $Y^\alpha \sim Y^\beta$. Hence according to (2) and (3) and our hypothesis on the cardinalities of ξ and η, we conclude that $\mathcal{K}(X^\xi, Y^\alpha) \sim \mathcal{K}(X^\eta, Y^\beta)$.

Conversely, assume that $\omega \le \alpha \le \beta < \omega_1$ and $\mathcal{K}(X^\xi, Y^\alpha) \sim \mathcal{K}(X^\eta, Y^\beta)$. Then bearing in mind (2) and (3) we conclude that

$$\mathbb{R}^\beta \hookrightarrow Y^\beta \hookrightarrow l_1([1,\eta], X^*) \hat{\hat{\otimes}} Y^\beta \hookrightarrow l_1([1,\xi], X^*) \hat{\hat{\otimes}} Y^\alpha \sim (l_1([1,\xi], X^*) \hat{\hat{\otimes}} Y)^\alpha. \tag{4}$$

Next we distinguish two cases: $\alpha < \omega^\omega$ and $\omega^\omega \le \alpha$.

Case 1. $\alpha < \omega^\omega$. We thus obtain from Theorem 1.1 that $\mathbb{R}^\alpha \sim \mathbb{R}^\omega$. So, by (4) we infer that

$$\mathbb{R}^\beta \hookrightarrow (l_1([1,\xi], X^*) \hat{\hat{\otimes}} Y)^\alpha \sim (l_1([1,\xi], X^*) \hat{\hat{\otimes}} Y)^\omega \sim l_1([1,\xi], X^*) \hat{\hat{\otimes}} Y^\omega.$$

Then, it follows from Proposition 2.2 that $\beta < \omega^\omega$. Again, an appeal to Theorem 1.1 tells us that $\mathbb{R}^\beta \sim \mathbb{R}^\omega$ and $\beta < \alpha^\omega$.

Case 2. $\omega^\omega \leq \alpha$. In this case, by Proposition 2.1, $l_1([1,\xi], X^*)\hat{\hat{\otimes}}Y$ contains no subspace isomorphic to $\mathbb{R}^{\omega^\omega}$. Consequently, by (3) and Proposition 2.2, $\mathbb{R}^\beta \hookrightarrow \mathbb{R}^\alpha$. Once again by Theorem 1.1, we get that $\beta < \alpha^\omega$. Thus the theorem is established.

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