NEW INTEGRAL IDENTITIES
FOR ORTHOGONAL POLYNOMIALS ON THE REAL LINE

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Abstract. Let $\mu$ be a positive measure on the real line, with associated orthogonal polynomials $\{p_n\}$ and leading coefficients $\{\gamma_n\}$. Let $h \in L_1(\mathbb{R})$. We prove that for $n \geq 1$ and all polynomials $P$ of degree $\leq 2n - 2$,
\[
\int_{-\infty}^{\infty} P(t) h\left(\frac{p_{n-1}(t)}{p_n(t)}\right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left(\int_{-\infty}^{\infty} h(t) dt\right) \left(\int P(t) \, d\mu(t)\right).
\]
As a consequence, we establish weak convergence of the measures on the left-hand side.

1. Introduction

Let $\mu$ be a positive measure on the real line with infinitely many points in its support, and let $\int x^j \, d\mu(x)$ be finite for $j = 0, 1, 2, \ldots$. Then we may define orthonormal polynomials
\[
p_n(x) = \gamma_n x^n + \cdots, \quad \gamma_n > 0,
\]
satisfying
\[
\int_{-\infty}^{\infty} p_n p_m \, d\mu = \delta_{mn}.
\]
Let
\[
L_n(x, y) = \frac{\gamma_{n-1}}{\gamma_n} \left(p_n(x) p_{n-1}(y) - p_{n-1}(x) p_n(y)\right)
\]
and for non-real $a$,
\[
E_{n,a}(z) = \sqrt{\frac{2\pi}{|L_n(a, \bar{a})|}} L_n(a, z).
\]

In a recent paper [6], we used the theory of de Branges spaces [1] to show that for $\text{Im} \, a > 0$, and all polynomials $P$ of degree $\leq 2n - 2$, we have
\[
\int_{-\infty}^{\infty} \frac{P(t)}{|E_{n,a}(t)|} dt = \int P(t) \, d\mu(t).
\]

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This may be regarded as an analogue of Geronimus’ formula for the unit circle, where instead of $E_{n,2}$, we have a multiple of the orthonormal polynomial on the unit circle in the denominator [3, Thm. V.2.2, p. 198], [8, pp. 95, 955]. There is an earlier real line analogue, due to Barry Simon [9] Theorem 2.1, p. 5], namely

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{P(t)}{(\gamma_{n-1}/\gamma_n)^2 p_n^2(t) + p_{n-1}^2(t)} \, dt = \int P(t) \, d\mu(t).
\]

Simon calls this a real line orthogonal polynomial analogue of Carmona’s formula and refers also to earlier work of Krutikov and Remling [5] and Carmona [2]. The latter is the special case of (1.3) with $(p_{n-1}/p_n) (\bar{a}) = \pm i \gamma_{n-1}/\gamma_n$. In a subsequent paper, we gave a self-contained proof of (1.3), and deduced results on weak convergence, discrepancy, and Gauss quadrature.

In this paper, we first establish the following alternative form of (1.3):

**Proposition 1.1.** Let $\mu$ be a positive measure on the real line with infinitely many points in its support, and with $\int x^2 \, d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then for all polynomials $P$ of degree $\leq 2n - 2$,

\[
\frac{1}{\pi} |\text{Im} \, z| \int_{-\infty}^{\infty} \frac{P(t)}{|zp_n(t) - p_{n-1}(t)|^2} \, dt = \frac{\gamma_{n-1}}{\gamma_n} \int P(t) \, d\mu(t)
\]

and

\[
\frac{1}{\pi} |\text{Im} \, z| \int_{-\infty}^{\infty} \frac{P(t)}{|p_n(t) - zp_{n-1}(t)|^2} \, dt = \frac{\gamma_{n-1}}{\gamma_n} \int P(t) \, d\mu(t).
\]

The factor involving $z$ inside the integral above is essentially the Poisson kernel for the upper-half plane. By using limiting properties of Poisson integrals, we deduce our main result, a new integral identity for orthogonal polynomials:

**Theorem 1.2.** Let $\mu$ be a positive measure on the real line with infinitely many points in its support, and with $\int x^2 \, d\mu(x)$ finite for $j = 0, 1, 2, \ldots$. Let $\{p_n\}$ and $\{\gamma_n\}$ denote, respectively, the orthogonal polynomials, and leading coefficients corresponding to $\mu$. Let $h \in L_1(\mathbb{R})$. Then for all polynomials $P$ of degree $\leq 2n - 2$,

\[
\int_{-\infty}^{\infty} \frac{P(t)}{p_n(t)^2} h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) \, dt = \frac{\gamma_{n-1}}{\gamma_n} \left( \int_{-\infty}^{\infty} h(t) \, dt \right) \left( \int P(t) \, d\mu(t) \right)
\]

and

\[
\int_{-\infty}^{\infty} \frac{P(t)}{p_{n-1}(t)^2} h \left( \frac{p_n(t)}{p_{n-1}(t)} \right) \, dt = \frac{\gamma_{n-1}}{\gamma_n} \left( \int_{-\infty}^{\infty} h(t) \, dt \right) \left( \int P(t) \, d\mu(t) \right).
\]

Note that if we choose $P = p_{n-1}^2$ in (1.7), we obtain, if the denominator integral is not 0,

\[
\frac{\gamma_{n-1}}{\gamma_n} = \frac{\int_{-\infty}^{\infty} h \left( \frac{p_n(t)}{p_{n-1}(t)} \right) \, dt}{\int_{-\infty}^{\infty} h(t) \, dt}.
\]

It might be possible to derive this special case in an alternative way, i.e., from the partial fraction expansion of $\frac{p_{n-1}}{p_n} (x)$ and known formulae for the distribution function, meas $\left\{ x : \frac{p_{n-1}}{p_n} (x) > t \right\}$. We may replace $h(t) \, dt$ in (1.6) and (1.7) by a signed measure $dv(t)$ of finite total mass, provided one appropriately defines
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dν \left( \frac{p_n(t)}{p_{n-1}(t)} \right) \quad \text{over each interval in which} \quad \frac{p_n(t)}{p_{n-1}(t)} \quad \text{is monotone. If we choose} \quad h(x) = \frac{\log x^2}{x^2 - 1}, \quad \text{in Theorem 1.2, we obtain an entropy-type integral:}

**Corollary 1.3.** With the notation of Theorem 1.2,

\begin{equation}
\frac{2}{\pi^2} \int_{-\infty}^{\infty} P(t) \frac{\ln |p_{n-1}(t)| - \ln |p_n(t)|}{p_{n-1}(t)^2 - p_n(t)^2} \, dt = \frac{\gamma_{n-1}}{\gamma_n} \int P(t) \, d\mu(t).
\end{equation}

We also obtain a weak convergence-type result: recall that \( \mu \) is said to be **determinate** if the moment problem

\begin{equation}
\int x^j \, d\nu(x) = \int x^j \, d\mu(x), \quad j = 0, 1, 2, \ldots,
\end{equation}

has the unique solution \( \nu = \mu \) from the class of positive measures. We also say that a function \( f \) has **polynomial growth at \( \infty \)** if for some \( L > 0 \) and for large enough \( |x| \),

\[ |f(x)| \leq |x|^L. \]

**Theorem 1.4.** Assume the hypotheses of Theorem 1.2, and in addition assume that \( \mu \) is determinate. Then for all functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) having polynomial growth at \( \infty \), and such that they are Riemann-Stieltjes integrable with respect to \( \mu \), we have

\begin{equation}
\lim_{n \to \infty} \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{f(t)}{p_{n-1}(t)^2} h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) \, dt = \left( \int_{-\infty}^{\infty} h(t) \, dt \right) \left( \int f(t) \, d\mu(t) \right)
\end{equation}

and

\begin{equation}
\lim_{n \to \infty} \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{f(t)}{p_{n-1}(t)^2} h \left( \frac{p_n(t)}{p_{n-1}(t)} \right) \, dt = \left( \int_{-\infty}^{\infty} h(t) \, dt \right) \left( \int f(t) \, d\mu(t) \right).
\end{equation}

Of course, if \( f \) is continuous on the real line, it will be locally Riemann-Stieltjes integrable with respect to \( \mu \). Simon \[9\] proved weak convergence involving his Carmona-type formula.

2. **Proof of the results**

**Proof of Proposition 1.1.** Fix \( z \in \mathbb{C} \setminus \mathbb{R} \). Choose \( a \in \mathbb{C} \) such that

\[ p_{n-1}(\bar{a}) = z p_n(\bar{a}). \]

There are \( n \) choices for \( a \), counting multiplicity. Then from (1.1), we see that

\[ L_n(\bar{a},t) = -\frac{\gamma_{n-1}}{\gamma_n} p_n(\bar{a})(zp_n(t) - p_{n-1}(t)) \]

and

\[ L_n(a,\bar{a}) = 2i\frac{\gamma_{n-1}}{\gamma_n} \Im(z) |p_n(a)|^2. \]
Hence
\[ |E_{n,a}(t)|^2 = \frac{2\pi}{|L_n(a,\bar{a})|} |L_n(a,t)|^2 \]
\[ = \frac{\pi}{\text{Im} z} \frac{\gamma_{n-1}}{\gamma_n} |p_n(t) - p_{n-1}(t)|^2. \]
Substituting into (1.3) gives (1.4), while replacing \( z \) by \( \frac{1}{2} \) in (1.4) gives (1.5). \( \square \)

**Proof of (1.6) of Theorem 1.2.**

**Step 1: A Poisson integral identity.** Let \( z = x + iy \), where \( y > 0 \). We can recast (1.4) as
\[ \int_{-\infty}^{\infty} P(t) \frac{1}{\pi} \frac{y}{(p_n(t)x - p_{n-1}(t))^2 + y^2p_n^2(t)} dt = \frac{\gamma_{n-1}}{\gamma_n} \int P(t) \, d\mu(t). \] (2.1)
Let \( h \in L_1(\mathbb{R}) \). We multiply (2.1) by \( h(x) \), integrate over the real line, and interchange integrals, obtaining
\[ \int_{-\infty}^{\infty} P(t) \left[ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y h(x)}{(p_n(t)x - p_{n-1}(t))^2 + y^2p_n^2(t)} \, dx \right] dt = \frac{\gamma_{n-1}}{\gamma_n} \left( \int_{-\infty}^{\infty} h(t) \, dt \right) \left( \int P(t) \, d\mu(t) \right). \] (2.2)
This is justified if the integral on the left converges absolutely, namely,
\[ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{|P(t)| |h(x)|}{(p_n(t)x - p_{n-1}(t))^2 + y^2p_n^2(t)} \, dx \right] dt < \infty. \] (2.3)
To prove this, choose \( A \) such that all zeros of \( p_n \) lie in \((-A, A)\). Let
\[ c = \inf_{t,x \in \mathbb{R}} \left[ (p_n(t)x - p_{n-1}(t))^2 + y^2p_n^2(t) \right]. \]
This is positive, as \( p_{n-1} \) and \( p_n \) do not have common zeros. Then we can bound the left-hand side in (2.3) above by
\[ \int_{|t| \geq A} \frac{|P(t)|}{y^2p_n^2(t)} \left( \int_{-\infty}^{\infty} |h(x)| \, dx \right) dt + \int_{|t| \leq A} |P(t)| \left( \int_{-\infty}^{\infty} |h(x)| \, dx \right) dt / c < \infty. \]
Thus (2.3) is valid. Recall that if \( h \in L_1(\mathbb{R}) \), its Poisson integral for the upper-half plane is
\[ \mathcal{P}[h](\alpha + i\beta) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\beta}{(x - \alpha)^2 + \beta^2} h(x) \, dx. \]
We can recast (2.2) as
\[ \int_{-\infty}^{\infty} P(t) \mathcal{P}[h] \left( \frac{p_{n-1}(t)}{p_n(t)} + iy \right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left( \int_{-\infty}^{\infty} h(t) \, dt \right) \left( \int P(t) \, d\mu(t) \right). \] (2.4)

**Step 2: The case where \( h \) is bounded and has compact support.** Firstly, as \( h \) is bounded, we have the elementary bound
\[ \left| \mathcal{P}[h] \left( \frac{p_{n-1}(t)}{p_n(t)} + iy \right) \right| \leq \| h \|_{L_\infty(\mathbb{R})}. \]
valid for all \( y \) and \( t \). Next, if \( \frac{p_{n-1}(t)}{p_n(t)} \) is a Lebesgue point of \( h \), we have the classic result
\[
(2.5) \quad \lim_{y \to 0^+} \mathcal{P} [h] \left( \frac{p_{n-1}(t)}{p_n(t)} + iy \right) = h \left( \frac{p_{n-1}(t)}{p_n(t)} \right).
\]
Now, if \( u \) is not a Lebesgue point of \( h \) (and such points have measure 0), the equation \( \frac{p_{n-1}(t)}{p_n(t)} = u \) has at most \( n \) solutions for \( t \), and locally these vary differentiably with \( u \). It follows that (2.5) holds for a.e. \( t \).

Let \( \varepsilon > 0 \) and \( \mathcal{E}_\varepsilon \) denote the union of \( n \) closed intervals of radius \( \varepsilon \), centered on the zeros of \( p_n \). Since \( P(t)/p_n^2(t) = O(t^{-2}) \) at \( \infty \), we may use Lebesgue’s Dominated Convergence Theorem to deduce that
\[
(2.6) \quad \lim_{y \to 0^+} \int_{\mathbb{R} \setminus \mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} \mathcal{P} [h] \left( \frac{p_{n-1}(t)}{p_n(t)} + iy \right) dt = \int_{\mathbb{R} \setminus \mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) dt.
\]
It remains to estimate
\[
I_{\varepsilon,y} = \int_{\mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} \mathcal{P} [h] \left( \frac{p_{n-1}(t)}{p_n(t)} + iy \right) dt
\]
and
\[
I_{\varepsilon,0} = \int_{\mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) dt.
\]
As \( p_{n-1} \) and \( p_n \) have no common zeros, if \( \varepsilon > 0 \) is small enough,
\[
\inf_{\mathcal{E}_\varepsilon} |p_{n-1}| > 0.
\]
Moreover, as \( h \) has compact support, we may choose \( \varepsilon > 0 \) so small that for \( x \) in the support of \( h \) and \( t \in \mathcal{E}_\varepsilon \), we have
\[
|p_n(t) x - p_{n-1}(t)| \geq \frac{1}{2} |p_{n-1}(t)|.
\]
Then for \( 0 < y \leq 1 \)
\[
|I_{\varepsilon,y}| = \left| \frac{y}{\pi} \int_{\mathcal{E}_\varepsilon} \left[ \int_{-\infty}^{\infty} \frac{P(t) h(x)}{(p_n(t) x - p_{n-1}(t))^2 + y^2 p_n^2(t)} dx \right] dt \right|
\]
\[
\leq \frac{1}{\pi} \int_{\mathcal{E}_\varepsilon} \left[ \int_{-\infty}^{\infty} \frac{|P(t)| |h(x)|}{\left( \frac{1}{2} |p_{n-1}(t)| \right)^2} dx \right] dt
\]
\[
\leq \frac{4}{\pi} \sup_{t \in \mathcal{E}_\varepsilon} \frac{P(t)}{p_n^2(t)} \left( \int_{-\infty}^{\infty} |h(x)| dx \right) \int_{\mathcal{E}_\varepsilon} 1 dt.
\]
This is a bound independent of \( y \) and decreases to 0 as \( \varepsilon \) decreases to 0. Finally, if \( \varepsilon > 0 \) is small enough, \( h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) = 0 \) for \( t \in \mathcal{E}_\varepsilon \) (recall that \( h \) has compact support), so for such an \( \varepsilon \),
\[
I_{\varepsilon,0} = 0.
\]
Combining the above, we obtain
\[
(2.7) \quad \lim_{y \to 0^+} \int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} \mathcal{P} [h] \left( \frac{p_{n-1}(t)}{p_n(t)} + iy \right) dt = \int_{-\infty}^{\infty} \frac{P(t)}{p_n^2(t)} h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) dt,
\]
and hence, from (2.4),
\[
\int_{-\infty}^{\infty} \frac{P(t)}{p_n(t)^2} h \left( \frac{p_n(t)}{p_n(t)} \right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left( \int_{-\infty}^{\infty} h(t) dt \right) \left( \int P(t) \, d\mu(t) \right).
\]
Thus we have (1.6) for the case where \( h \) is bounded and has compact support.

**Step 3: The case where \( h \) is bounded but has non-compact support.**

Let
\[
h_m = h\chi_{[-m,m]}, \quad m \geq 1.
\]
We have (1.6) for \( h_m \); that is,
\[
\int_{-\infty}^{\infty} \frac{P(t)}{p_n(t)^2} h_m \left( \frac{p_n(t)}{p_n(t)} \right) dt = \frac{\gamma_{n-1}}{\gamma_n} \left( \int_{-\infty}^{\infty} h_m dt \right) \int P d\mu.
\]
Now for each \( t \) with \( p_n(t) \neq 0 \) and all large enough \( m \),
\[
h_m \left( \frac{p_n(t)}{p_n(t)} \right) = h \left( \frac{p_n(t)}{p_n(t)} \right).
\]
Next,
\[
\left| \frac{P(t)}{p_n(t)^2} h_m \left( \frac{p_n(t)}{p_n(t)} \right) \right| \leq \left| \frac{P(t)}{p_n(t)^2} h \left( \frac{p_n(t)}{p_n(t)} \right) \right|.
\]
This upper bound is independent of \( m \) and moreover is integrable over \((-\infty, \infty)\), since it is \( O(t^{-2}) \) at \( \infty \) and has an integrable singularity at each zero of \( p_n \). To see the latter, we proceed as follows. Let \( x_{jn} \) be a zero of \( p_n \). We can write, in \((x_{jn}, x_{jn} + \varepsilon)\), with small enough \( \varepsilon > 0 \),
\[
\frac{p_n(t)}{p_n(t)} = \frac{g(t)}{t-x_{jn}},
\]
where \( g \) is non-vanishing and continuously differentiable. If \( \varepsilon > 0 \) is small enough, we have for some appropriate constant \( C \) and \( t \in (x_{jn}, x_{jn} + \varepsilon) \):
\[
\left| \frac{P(t)}{p_n(t)^2} h \left( \frac{p_n(t)}{p_n(t)} \right) \right| \leq C \frac{1}{(t-x_{jn})^2} \left| h \left( \frac{g(t)}{t-x_{jn}} \right) \right|\]
\[
\leq C \left| \frac{g'(t) (t-x_{jn}) - g(t)}{(t-x_{jn})^2} \right| \left| h \left( \frac{g(t)}{t-x_{jn}} \right) \right|\]
\[
= C \left| \frac{d}{dt} \left( \frac{g(t)}{t-x_{jn}} \right) \right| \left| h \left( \frac{g(t)}{t-x_{jn}} \right) \right|.
\]
In the second to last line, we use the fact that if \( \varepsilon \) is small enough, \( |g(t)| \gg |g'(t)(t-x_{jn})| \), while \( |g| \) is bounded below. Then, if \( g(x_{jn}) > 0 \), the substitution \( s = \frac{g(t)}{t-x_{jn}} \) gives
\[
\int_{x_{jn}}^{x_{jn} + \varepsilon} \left| \frac{P(t)}{p_n(t)^2} h \left( \frac{p_n(t)}{p_n(t)} \right) \right| dt \leq C \int_{x_{jn}}^{x_{jn} + \varepsilon} \left| h \left( \frac{g(t)}{t-x_{jn}} \right) \right| \left| \frac{d}{dt} \left( \frac{g(t)}{t-x_{jn}} \right) \right| dt\]
\[
= C \int_{g(x_{jn} + \varepsilon)}^{g(x_{jn})} |h(s)| ds \leq C \int_{-\infty}^{\infty} |h(s)| ds.
\]
If \( g(x_n) < 0 \), we proceed similarly. Thus, indeed, the function \( \left| \frac{p(t)}{p_n(t)} \right| h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) \) provides an integrable bound independent of \( m \). Then Lebesgue’s Dominated Convergence Theorem allows us to let \( m \to \infty \) in (2.9) to obtain (1.6) for the case where \( h \) is bounded but has non-compact support.

**Step 4: The case where \( h \) is unbounded.** Let us define

\[
H_m(t) = \begin{cases} 
  h(t), & \text{if } |h(t)| \leq m, \\
  0, & \text{otherwise.}
\end{cases}
\]

We have that (1.6) holds for \( h = H_m \). Next, for each \( t \) with \( p_n(t) \neq 0 \), \( h \left( \frac{p_{n-1}(t)}{p_n(t)} \right) \) finite, and all large enough \( m \),

\[
H_m \left( \frac{p_{n-1}(t)}{p_n(t)} \right) = h \left( \frac{p_{n-1}(t)}{p_n(t)} \right).
\]

Moreover, \( \left| \frac{p(t)}{p_n(t)} \right| H_m \left( \frac{p_{n-1}(t)}{p_n(t)} \right) \) admits the same integrable bound as in Step 3. Then Lebesgue’s Dominated Convergence Theorem gives the result. \( \square \)

**Proof of (1.7) of Theorem 1.2.** For the given \( h \), define a new function \( \tilde{h} \) by

\[
\tilde{h}(x) = x^{-2} h(x^{-1}).
\]

A substitution shows that also \( \tilde{h} \in L_1(\mathbb{R}) \) and

\[
\frac{1}{p_n^2(t)} \tilde{h} \left( \frac{p_{n-1}(t)}{p_n(t)} \right) = \frac{1}{p_{n-1}^2(t)} h \left( \frac{p_n(t)}{p_{n-1}(t)} \right).
\]

So applying (1.6) to \( \tilde{h} \) gives (1.7) for \( h \). \( \square \)

**Proof of Corollary 1.3.** Choose in (1.6) of Theorem 1.2

\[
h(x) = \log \frac{x^{-2}}{1-x^2},
\]

which has \( h \in L_1(\mathbb{R}) \). Moreover, the fact that \( h \) is even and a substitution show that \([8, p. 533, 4.231.13]\)

\[
\int_{-\infty}^{\infty} h = 8 \int_0^1 \log \frac{x^{-1}}{1-x^2} dx = \pi^2.
\]

**Proof of Theorem 1.4.** We may prove the result for non-negative \( h \), because every \( h \) satisfying the hypotheses of Theorem 1.2 is the difference of two non-negative functions satisfying the same hypotheses. Let \( f \) be Riemann-Stieltjes integrable with respect to \( \mu \) and of polynomial growth at \( \infty \), and let \( \varepsilon > 0 \). Since \( \mu \) is determinate, there exist upper and lower polynomials \( P_u \) and \( P_l \) such that

\[
P_l \leq f \leq P_u \quad \text{in } (-\infty, \infty)
\]

and

\[
\int (P_u - P_l) \, d\mu < \varepsilon.
\]
See, for example, [3, Theorem 3.3, p. 73]. Then for \( n \) so large that \( 2^n - 2 \) exceeds the degree of \( P_u \) and \( P_\ell \), (1.3) gives
\[
\left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{f}{p_{n-1}^2} h \left( \frac{p_n}{p_{n-1}} \right) - \int f \, d\mu
\]
\[
= \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{f - P_\ell}{p_{n-1}^2} h \left( \frac{p_n}{p_{n-1}} \right) - \int (f - P_\ell) \, d\mu
\]
\[
\leq \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} P_u - P_\ell \frac{p_n}{p_{n-1}} h \left( \frac{p_n}{p_{n-1}} \right) - 0
\]
\[
= \int (P_u - P_\ell) \, d\mu < \varepsilon.
\]

Similarly, for large enough \( n \),
\[
\left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{f}{p_{n-1}^2} h \left( \frac{p_n}{p_{n-1}} \right) - \int f \, d\mu
\]
\[
= \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} \frac{f - P_u}{p_{n-1}^2} h \left( \frac{p_n}{p_{n-1}} \right) - \int (f - P_u) \, d\mu
\]
\[
\geq \left( \frac{\gamma_{n-1}}{\gamma_n} \right)^{-1} \int_{-\infty}^{\infty} P_\ell - P_u \frac{p_n}{p_{n-1}} h \left( \frac{p_n}{p_{n-1}} \right) - 0
\]
\[
= \int (P_\ell - P_u) \, d\mu > -\varepsilon. \quad \Box
\]

References


