HAMILTON’S GRADIENT ESTIMATES
AND LIOUVILLE THEOREMS
FOR FAST DIFFUSION EQUATIONS
ON NONCOMPACT RIEMANNIAN MANIFOLDS

XIAOBAO ZHU

(Communicated by Chuu-Lian Terng)

Abstract. Let $M$ be a complete noncompact Riemannian manifold of dimension $n$. In this paper, we derive a local gradient estimate for positive solutions of fast diffusion equations

$$\partial_t u = \Delta u^\alpha, \quad 1 - \frac{2}{n} < \alpha < 1$$

on $M \times (-\infty, 0]$. We also obtain a theorem of Liouville type for positive solutions of the fast diffusion equation.

1. Introduction

In this paper we study the fast diffusion equation (FDE for short)

$$\partial_t u = \Delta u^\alpha,$$

where $\alpha \in (0, 1)$. FDE arises in the study of fast diffusions, in particular in diffusion in plasma ([3]), in thin liquid film dynamics driven by Van der Waals forces ([7], [8]), and in models of gas-kinetics ([4]). It also arises in geometry: the case $\alpha = \frac{n-2}{n}$ in dimensions $n > 3$ describes the evolution of a conformal metric by the Yamabe flow ([13]); the case $\alpha = 0$, $n = 2$ describes the Ricci flow on surfaces ([10], [6], [16]). Precisely, we can find the relationship from $\partial_t u = \Delta (\frac{1}{\alpha} u^\alpha) = \text{div}(u^\alpha \nabla u)$ and $\text{div}(\nabla u) = \Delta \log u = \Delta \log u$. We refer the reader to the book by Daskalopoulos-Kenig ([5]) and the references therein for more about FDE.

As a nonlinear problem, the mathematical theory of FDE is based on a priori estimates. In 1979, Aronson and Bénilan obtained a celebrated second-order differential inequality of the form ([11])

$$\sum_i \frac{\partial}{\partial x_i} (\alpha u^{\alpha-2} \frac{\partial u}{\partial x_i}) \geq -\kappa \frac{1}{t}, \quad \kappa := \frac{n}{n(\alpha-1)+2},$$

which applies to all positive solutions of (1.1) defined on the whole Euclidean space on the condition that $\alpha > 1 - \frac{2}{n}$.
There are few results about FDE on manifolds. In 2008, Lu, Ni, Vázquez and Villani studied the FDE on manifolds ([13]) and got a local Aronson-Bénilan estimate. We do not state their result here. What we will do in this paper is to get Hamilton’s gradient estimates. First, let us recall what Hamilton’s result is:

**Theorem (Hamilton [9]).** Let $M$ be a closed Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(M) \geq -k$ for some $k \geq 0$. Suppose that $u$ is any positive solution to the heat equation with $u \leq M$ for all $(x,t) \in M \times (0, \infty)$. Then

$$\frac{\| \nabla u \|^2}{u^2} \leq \left( \frac{1}{t} + 2k \right) \log \frac{M}{u},$$

Hamilton’s estimate tells us that when the temperature is bounded we can compare the temperature of two different points at the same time.

In 2006, P. Souplet and Qi S. Zhang ([14]) generalized Hamilton’s estimate to complete noncompact Riemannian manifolds. In 2007, B. Kotschwar ([11]) used Hamilton’s estimate and obtained a global gradient estimate for heat kernels on complete noncompact manifolds. In 2010, M. Bailesteanu, X. Cao and A. Pulemotov ([2]) generalized Souplet and Zhang’s result to Ricci flow. Also in 2010, on complete noncompact Riemannian manifolds, J. Wu ([15]) obtained a localized Hamilton-type gradient estimate for the positive smooth bounded solutions to the nonlinear diffusion equation

$$u_t = \Delta u - \nabla \phi \cdot \nabla u - au \log u - bu,$$

where $\phi$ is a $C^2$ function and $a \neq 0$ and $b$ are two real constants. We would like to remark here that this equation was also studied by Y. Yang in ([17]), where they derived a parabolic gradient estimate.

In this paper, we consider the positive solution for FDE (1.1). Like what they did for the heat equation, we derive a similar Hamilton’s estimate for FDE. Inspired by the inequality of Aronson and Bénilan, we let $\alpha > 1 - \frac{2}{n}$ throughout this paper. Note that the pressure $\tilde{v} := \frac{\alpha - 1}{\alpha - 1} u^{\alpha - 1} < 0$,

$$\partial_t \tilde{v} = (\alpha - 1) \tilde{v} \Delta \tilde{v} + |\nabla \tilde{v}|^2.$$

Conveniently, we let $v = -\tilde{v}$. Then $v > 0$ and satisfies

$$v_t = (1 - \alpha) v \Delta v - |\nabla v|^2.

(1.2)$$

Our main result is the following:

**Theorem 1 (Gradient estimates).** Let $M$ be a Riemannian manifold of dimension $n \geq 2$ with $\text{Ricci}(M) \geq -k$ for some $k \geq 0$. Suppose that $v$ is any positive solution to the equation (1.2) in $Q_{R,T} = B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$. Suppose also that $v \leq M$ in $Q_{R,T}$. Then there exists a constant $C = C(\alpha, M)$ such that

$$\frac{\| \nabla v \|}{v^{1/2}} \leq CM^{1/2} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right)$$

in $Q_{\frac{R}{2}, \frac{T}{2}}$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Corollary 1.1 (Liouville type theorem). Let $M$ be a complete, noncompact manifold of dimension $n$ with nonnegative Ricci curvature. Let $u$ be a positive ancient solution to the equation \((1.1)\) with $1 - \frac{2}{n} < \alpha < 1$ such that $\frac{1}{u(x,t)} = o(\sqrt{t})^{\frac{n}{n-2}}$. Then $u$ is a constant.

2. Proof of Theorem [1]

Let $w \equiv \frac{\nabla v^2}{v^\beta}$, $\beta > 0$ to be determined.

We will derive an equation for $w$. First notice that

\[
 w_t = \frac{2v_tv_u}{v^\beta} - \beta \frac{v^2_v}{v^{\beta+1}} = \frac{2v_t(1-\alpha)v\Delta v - |\nabla v|^2}{v^\beta} - \beta \frac{v^2((1-\alpha)v\Delta v - |\nabla v|^2)}{v^{\beta+1}}
\]

\[
 = 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} - 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} - 4\frac{v^2_v}{v^{\beta+1}}
\]

\[
 = -\beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}} + \beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}} + \beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}}
\]

(2.1)

\[
 w_j = \frac{2v_tv_j}{v^\beta} - \beta \frac{v^2_v}{v^{\beta+1}},
\]

(2.2)

\[
 w_{jj} = \frac{2v^2_v}{v^{\beta+1}} + 2\frac{v^2_v}{v^{\beta+1}} - 4\beta \frac{v^2_v}{v^{\beta+1}} - \beta \frac{v^2_v}{v^{\beta+1}} + \beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}}.
\]

By (2.1) and (2.2),

\[
 (1-\alpha)v\Delta w - w_t
\]

\[
 = 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} + 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} - 4\beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}}
\]

\[
 - \beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}} + \beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}}
\]

\[
 - 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} + 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} + 4\beta \frac{v^2_v}{v^{\beta+1}}
\]

\[
 + \beta(1-\alpha)\frac{v^2_v}{v^{\beta+1}} - \beta \frac{v^2_v}{v^{\beta+1}}
\]

\[
 = 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} - 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}} + 2(1-\alpha)\frac{v^2_v}{v^{\beta+1}}
\]

\[
 + 4(1-\beta(1-\alpha))\frac{v^2_v}{v^{\beta+1}} + (\beta(1-\alpha) - \beta)\frac{v^2_v}{v^{\beta+1}}.
\]

Here, in the second equality, we use the Ricci formula: $v_{ij} - \frac{R_{ij}}{v^\beta} = v_{ij}$.

Notice that

\[
 \nabla w \cdot \nabla v = w_j v_j = \frac{2v_tv_j}{v^\beta} - \beta \frac{v^2_v}{v^{\beta+1}}.
\]

(2.3)
By straightforward calculation, one has

\[ (1 - \alpha)v \Delta w - w_t \]

\[ = 2(1 - \alpha)\frac{v_{ij}^2}{v^{\beta+1}} - 2(1 - \alpha)\frac{v_i^2v_j}{v^\beta} + 2(1 - \alpha)\frac{R_{ij}v_i v_j}{v^{\beta+1}} \]

\[ + 4(1 - \alpha)(\beta + 1)(1 - \alpha))\frac{\alpha}{v^{\beta+1}} + (\beta + 1)(1 - \alpha)\frac{v_i^2v_j}{v^{\beta+1}} \]

\[ \geq - \frac{n(1 - \alpha)}{2} |\nabla v|^2 v^{\beta+1} - 2(1 - \alpha)k v^{\beta+1} + 2(1 - \alpha)\nabla w \cdot \nabla v \]

\[ + 2\beta(1 - \beta(1 - \alpha))\frac{v_i^2v_j}{v^{\beta+1}} + (\beta + 1)(1 - \alpha) - \beta)\frac{v_i^2v_j}{v^{\beta+1}} \]

\[ = 2(1 - \beta(1 - \alpha))\nabla w \cdot \nabla v - 2(1 - \alpha)k v^{\beta+1} + 2(1 - \beta(1 - \alpha)) \frac{|\nabla v|^2}{v^{\beta+1}}. \]

For the purpose of obtaining the gradient estimates as in [14], we need to have the coefficient of \( w^2 \) be positive. Fortunately, we can do this by choosing a suitable \( \beta \). In fact,

\[ \beta(1 - \alpha) - \beta - \frac{n(1 - \alpha)}{2} + 2\beta(1 - \beta(1 - \alpha)) \]

\[ = - (1 - \alpha)(\beta^2 - \frac{2 - \alpha}{1 - \alpha} \beta + \frac{n}{2}). \]

It is easily found that the discriminant is \((-\frac{2 - \alpha}{1 - \alpha})^2 - 2n\), which is positive when \( \alpha \in (1 - \frac{2}{n}, 1) \). So we can choose a suitable \( \beta \) to make sure the term will be positive.

Rearranging, we have

\[(1 - \alpha)v \Delta w - w_t \]

\[ = 2(1 - \beta(1 - \alpha))\nabla w \cdot \nabla v - 2(1 - \alpha)k v w - (1 - \alpha)(\beta^2 - \frac{2 - \alpha}{1 - \alpha} \beta + \frac{n}{2}) v^{\beta-1} w^2. \]

From here, we will use the well-known cutoff function of Li and Yau to derive the desired bounds. We caution the reader that the calculation is not the same as that in [12] due to the difference of the first order.

Let \( \psi = \psi(x, t) \) be a smooth cutoff function supported in \( Q_{R, T} \), satisfying the following properties:

1. \( \psi = \psi(d(x, x_0), t) \equiv \psi(r, t) \equiv 1 \) in \( Q_{R/2, T/2}, 0 \leq \psi \leq 1 \).
2. \( \psi \) is decreasing as a radial function in the spatial variables.
3. \( |\partial_r \psi|/\psi^a \leq C_a/R, |\partial^2_r \psi|/\psi^a \leq C_a/R^2 \) when \( 0 < a < 1 \).
4. \( |\partial_r \psi|/\psi^{1/2} \leq C/T \).

By straightforward calculation, one has

\[ (1 - \alpha)v \Delta(\psi w) + b \cdot \nabla (\psi w) - 2(1 - \alpha)v \nabla\psi \cdot \nabla(\psi w) - (\psi w)_t \]

\[ \geq - (1 - \alpha)(\beta^2 - \frac{2 - \alpha}{1 - \alpha} \beta + \frac{n}{2}) v^{\beta-1} \psi w^2 + (b \cdot \nabla \psi) w \]

\[ - 2(1 - \alpha)v \frac{|\nabla \psi|^2}{\psi} w + (1 - \alpha)v (\Delta \psi) w - \psi_t w - 2(1 - \alpha)kv \psi w, \]
where \( b = -2(1 - \beta(1 - \alpha))\nabla v \).

Suppose that the maximum of \( \psi w \) is reached at \((x_1, t_1)\). By [12], we can assume, without loss of generality, that \( x_1 \) is not on the cut-locus of \( M \). Then at \((x_1, t_1)\), one has \( \Delta (\psi w) \leq 0 \), \( (\psi w)_t \geq 0 \) and \( \nabla (\psi w) = 0 \). Therefore,

\[
\begin{align*}
- (1 - \alpha)(\beta^2 - 2 - \beta + \frac{n}{2})v^{\beta - 1}\psi w^2(x_1, t_1) \\
\leq - [(b \cdot \nabla \psi)w - 2(1 - \alpha)v \frac{\nabla \psi}{\psi} w] \\
(2.4) \\
+ (1 - \alpha)v(\Delta \psi)w - \psi w - 2(1 - \alpha)kv \psi w](x_1, t_1).
\end{align*}
\]

Denote \(- (1 - \alpha)(\beta^2 - 2 - \beta + \frac{n}{2}) = \frac{2}{\gamma}\). Then \( \gamma > 0 \) only depends on \( \alpha, \beta \).

Rearranging, we have

\[
\begin{align*}
2\psi w^2(x_1, t_1) \\
\leq [-\gamma(b \cdot \nabla \psi)v^{1-\beta} w + 2(1 - \alpha)\gamma v^{2-\beta}\frac{\nabla \psi}{\psi} w] \\
(2.5) \\
- (1 - \alpha)\gamma v^{2-\beta}(\Delta \psi)w + \gamma v^{1-\beta}\psi w + 2(1 - \alpha)\gamma k v^{2-\beta}\psi w](x_1, t_1).
\end{align*}
\]

We need to find an upper bound for each term on the right-hand side of (2.5). For the first term,

\[
\begin{align*}
- \gamma(b \cdot \nabla \psi)v^{1-\beta} w &\leq \gamma(b \cdot \nabla \psi)v^{1-\beta} w \\
&= C|\nabla v||\nabla \psi|v^{1-\beta} w \\
&\leq CM^{1-\beta/2}w^{3/2}|\nabla \psi| \\
&\leq \frac{1}{4}\psi w^2 + C(M^{1-\beta/2}|\nabla \psi|)^4 \\
(2.6) \\
&\leq \frac{1}{4}\psi w^2 + CM^{4-2\beta} \frac{1}{R^4}.
\end{align*}
\]

Here we used the fact that \( 0 < v \leq M \).

For the second term on the right-hand side of (2.5), we proceed as follows:

\[
\begin{align*}
2(1 - \alpha)\gamma v^{2-\beta}\frac{\nabla \psi}{\psi} w &\leq CM^{2-\beta}\frac{\nabla \psi}{\psi} \psi^2 w \\
&\leq \frac{1}{4}\psi w^2 + CM^{4-2\beta}\left(\frac{|\nabla \psi|}{\psi^2}\right)^2 \\
(2.7) \\
&\leq \frac{1}{4}\psi w^2 + CM^{4-2\beta} \frac{1}{R^4}.
\end{align*}
\]
Furthermore, by the properties of $\psi$ and the assumption on the Ricci curvature, one has
\[
-(1-\alpha)\gamma v^{2-\beta}(\Delta \psi)w
=-(1-\alpha)\gamma(\partial_r^2 \psi + (n-1) \frac{\partial_r \psi}{r} + \partial_r \psi \partial_r \log \sqrt{g})v^{2-\beta}w
\leq CM^{2-\beta}(|\partial_r^2 \psi| + (n-1) \frac{|\partial_r \psi|}{r} + \sqrt{k}|\partial_r \psi|)w
\leq CM^{2-\beta}\left(\frac{\partial_r^2 \psi}{\psi^2} + 2(n-1) \frac{|\partial_r \psi|}{R\psi^2} + \sqrt{k}|\partial_r \psi| \psi^{1/2} \right)w
\leq \frac{1}{4} \psi w^2 + CM^{4-2\beta}\left(\frac{\partial_r^2 \psi}{\psi^2} \right)^2 + \left(\frac{|\partial_r \psi|}{R\psi^2} \right)^2 + \left(\sqrt{k}|\partial_r \psi| \psi^{1/2} \right)^2
\leq \frac{1}{4} \psi w^2 + CM^{4-2\beta}\left(\frac{1}{R^4} + k \frac{1}{T^2}\right).
\tag{2.8}
\]

Now we estimate $\gamma v^{1-\beta} \psi_t w$ as
\[
\gamma v^{1-\beta} \psi_t w \leq \gamma v^{1-\beta} |\psi_t| w
\leq \gamma \frac{|\psi_t|}{\psi^{1/2}} \psi^{1/2} w M^{1-\beta}
\leq \frac{1}{4} \psi w^2 + CM^{2-2\beta} \frac{1}{T^2}.
\tag{2.9}
\]

Here we suppose $\beta \leq 1$.

Finally, for the last term, we have
\[
2(1-\alpha)\gamma kv^{2-\beta} \psi w \leq C \psi^{1/2} w k M^{2-\beta}
\leq \frac{1}{4} \psi w^2 + CM^{4-2\beta} k^2.
\tag{2.10}
\]

Substituting (2.8), (2.10) into the right-hand side of (2.5), we deduce that
\[
2\psi w^2(x_1, t_1) \leq \frac{5}{4} \psi w^2(x_1, t_1) + CM^{4-2\beta}\left(\frac{1}{R^4} + \frac{1}{T^2} + k^2\right).
\]

Therefore,
\[
\psi w^2(x_1, t_1) \leq CM^{4-2\beta}\left(\frac{1}{R^4} + \frac{1}{T^2} + k^2\right).
\]

What we get shows, for all $(x, t) \in Q_{R, T}$, that
\[
\psi^2(x, t) w^2(x, t) \leq \psi^2(x_1, t_1) w^2(x_1, t_1)
\leq \psi^2(x_1, t_1) w^2(x_1, t_1)
\leq CM^{4-2\beta}\left(\frac{1}{R^4} + \frac{1}{T^2} + k^2\right).
\]
Notice that $\psi(x,t) = 1$ in $Q_{R/2,T/2}$: $w = \frac{1}{\sqrt{v}}$. We have
\[ \frac{|\nabla v(x,t)|}{v^{\beta/2}(x,t)} \leq CM^{1-\beta/2}(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}), \]
where $C = C(\alpha, \beta, M)$.

Then we choose $\beta = 1$. This ends the proof of Theorem 1.

3. Simple proof of Corollary 1.1

From Theorem 1 we know that, when $v$ is a positive ancient solution to the equation (1.2) such that $v(x,t) = o(d(x,x_0) + \sqrt{\lvert t \rvert})$, then $v$ is a constant.

Notice that $v = \frac{1}{u^\alpha} u^{\alpha-1} = \frac{1}{u^\alpha} (\frac{1}{u})^{1-\alpha}$, so when $u$ is a positive ancient solution to the equation (1.1) such that $\frac{1}{u(x,t)} = o(d(x,x_0) + \sqrt{\lvert t \rvert})$, then $u$ is a constant. This ends the proof of Corollary 1.1.

ACKNOWLEDGEMENT

The author would like to thank Professor Jiayu Li for introducing him to the present problem and for his encouragement.

REFERENCES


Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China

E-mail address: zhuxiaobao@amss.ac.cn