RELATIVELY POINTWISE RECURRENT GRAPH MAP

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Abstract. Let $f$ be a self-continuous map of a graph $G$. Let $P(f)$ and $R(f)$ denote the sets of periodic points and recurrent points respectively. We say that the map $f$ is relatively recurrent if $R(f) = G$. In this paper, it is shown that $f$ is relatively recurrent if and only if one of the following statements holds:

(a) $G$ is a circle and $f$ is a homeomorphism topologically conjugate to an irrational rotation of the unit circle $S^1$;
(b) $P(f) = G$.

Part (b) extends a result of Blokh.

1. Introduction

A topological dynamical system is a pair $(X, f)$, where $X$ is a compact metric space and $f$ is a continuous map from $X$ to itself. Let $\mathbb{N}$ be the set of positive integers. Let $f^0$ be the identity map of $X$. Define, inductively, $f^n = f \circ f^{n-1}$ for any non-zero positive integer $n$. For $x \in X$, $\{f^n(x) : n \in \mathbb{N}\}$ is called the orbit of $x$ and is denoted by $O(x, f)$. Here $x$ is periodic if $f^n(x) = x$ for some non-zero positive integer $n$. Also, $x$ is called a recurrent point of $f$ if for any neighborhood $U$ of $x$ and any $m \in \mathbb{N}$ there exists $n > m$ such that $f^n(x) \in U$. It is easy to see that if $x$ is recurrent, then every iterate of $x$ is also recurrent. The converse is false. Here $x$ is called an almost periodic point of $f$ if for any neighborhood $U$ of $x$ there exists $N \in \mathbb{N}$ such that $\{f^n(x) : i = 0, 1, ..., N\} \cap U \neq \emptyset$ for all $n \in \mathbb{N}$. Also, $x$ is a non-wandering point of $f$ provided that for any open set $U$ containing $x$ there exist $y \in U$ and $n \in \mathbb{N}$ such that $f^n(y) \in U$. Let $P(f)$, $AP(f)$, $R(f)$ and $\Omega(f)$ denote the set of periodic points, almost periodic points, recurrent points and non-wandering points respectively. Notice that $\Omega(f)$ is closed and for the general topological system $(X, f)$ there are no further relations except for $P(f) \subset AP(f) \subset R(f) \subset \Omega(f)$. But for one-dimensional systems one can say more. For a dendrite map Illanes [8] proved that $\overline{P(f)} = \overline{R(f)}$ if and only if the dendrite does not contain any copy of the Gehman dendrite. For a graph map Mai and Shao [9] showed that $\overline{R(f)} = P(f) \cup R(f)$. In Lemma 3.1 for a graph map, we will show that $\overline{AP(f)} = \overline{R(f)}$. 

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It is interesting to study maps such that $P(f)$ or $R(f)$ satisfies some additional properties. Montgomery [12], Weaver [15], Epstein [4], and others studied homeomorphisms such that $P(f)$ is the whole space $X$ (X is a connected manifold or a continuum embedded in a 2-manifold). For an interval map, Nitecki [13] showed that if $P(f)$ is closed, then $P(f) = \Omega(f)$. For a graph map Mai [11] proved that $R(f)$ is the whole space $X$ if and only if one of the following statements holds:

1. $X$ is a circle, and $f$ is a homeomorphism topologically conjugate to an irrational rotation of the unit circle;
2. $f$ is a periodic homeomorphism.

In this paper we will study a relatively pointwise recurrent graph map. Our main result is the following theorem:

**Main Theorem.** Let $G$ be a graph and let $f : G \to G$ be a continuous map. Then $f$ is relatively pointwise recurrent if and only if one of the following two statements holds:

1. $G$ is a circle, and $f$ is a homeomorphism topologically conjugate to an irrational rotation of the unit circle $S^1$;
2. $P(f) = G$.

A set $W \subset X$ is called a minimal set of $f$ if it is nonempty, closed, invariant ($f(W) \subset W$) and no proper subset of $W$ satisfies these three properties, which is equivalent to the fact that the orbit of every element of $W$ is dense.

In a topological dynamical system there are two well-known theorems which exhibit the close relationship between almost periodic points and minimal sets; see Birkhoff [2]. In fact, if $X$ is a locally compact metric space, then the set of almost periodic points $AP(f)$ is the union of all minimal sets of $f$.

If $X$ has no isolated point, then $f$ is a transitive map if it has a dense orbit. If every orbit of $f$ is dense in $X$, the map $f$ is called minimal. For a transitive graph map Blokh [3] proved that $P(f)$ is dense. In this paper we show that for a graph map if $R(f)$ is dense, then $P(f)$ is also dense, which extends this result of Blokh [3] (see Corollary 3.4).

2. RELATIVELY POINTWISE RECURRENT MAP

A map $f$ of a compact metric space $(X,d)$ is recurrent if it admits iterates arbitrarily close to the identity, i.e. if there exists a sequence $n_k \to +\infty$ such that $d(f^{n_k}, Id) \to 0$ as $k \to +\infty$.

We say that $f$ is pointwise recurrent if $R(f) = X$.

We continue, motivated by a desire to understand the mechanics of recurrent maps, by a desire to extend some known result and by the following:

In [6] Gottschalk proved that if $X$ is a compact connected metric space, $f$ is a homeomorphism, and if $R(f) = X$, then every recurrent cut point of $X$ is periodic.

We start by the following definition.

**Definition.** $f$ is called a relatively pointwise recurrent map if $R(f) = X$.

For example a transitive map is a relatively pointwise recurrent map.

Let $X$ be a closed domain of finite volume of the $n$-Euclidian space $\mathbb{R}^n$ or the $n$-torus $\mathbb{T}^n$ and let $f$ be an invertible volume-preserving self-map of $X$. Then $f$ is a relatively pointwise recurrent map [7] Theorem 6.1.9].
We have the following implications:

recurrent $\Rightarrow^1$ pointwise recurrent $\Rightarrow^2$ relatively pointwise recurrent.

The following examples show that none of these implications can be reversed.

Examples 2.1. (1) An irrational rotation of $\mathbb{R}^2$ is a pointwise recurrent non-recurrent map.

(2) In $\mathbb{R}^2$ we consider the points $A_n(0, \frac{1}{n})$ and $B(n, \frac{1}{n})$ for every integer $n > 0$. Put $X = (\bigcup_{n>0} [A_n, B_n]) \cup [0, +\infty] \times \{0\}$. Define the map $f$ on $X$ by:

- $f(x, \frac{1}{n}) = (\varphi(x), \frac{1}{n})$, where
  $$\varphi(x) = \begin{cases} 
  x + 1 & \text{if } x \leq n - 1, \\
  x + 1 - n & \text{if } n - 1 < x \leq n.
  \end{cases}$$

- $f(x, 0) = (x + 1, 0)$.

$f$ is a relatively pointwise recurrent non-pointwise recurrent map.

In this example one can choose $X$ to be connected.

Theorem 2.2. If a continuous map of a topological space $X$ to itself is either (1) recurrent, (2) pointwise recurrent, or (3) relatively pointwise recurrent, then so is $f^k$, for each integer $k$.

Proof. (1) and (2) follow from [14] and [6] respectively.

We have $R(f) \subset R(f^k)$, and so if $R(f) = X$, then $R(f^k) = X$, which implies (3).

Proposition 2.3. If $f$ is relatively pointwise recurrent, then $f$ is surjective.

Proof. If $y \in R(f)$, then there exists a point $x \in X$ such that $f(x) = y$.

Let $y$ be an element of $X - R(f)$. Then there exists a sequence $y_n$ of $R(f)$ which converges to $y$. For all $n$ there exists $x_n$ such that $f(x_n) = y_n$. Since $X$ is compact, the sequence $x_n$ has a limit point $x$ and so $f(x) = y$. Therefore $f$ is surjective.

Proposition 2.4. If $f$ is relatively pointwise recurrent, then $\Omega(f) = X$.

Proof. Let $x$ be an element of $X$. If $x$ is a recurrent point, it is also non-wandering. If $x$ is a non-recurrent point, then every neighborhood of $x$ contains a recurrent point, and so it is in $\Omega(f)$.

The referee noticed that the converse of Proposition 2.4 also holds; see, for example, [5, Theorem 1.27].

Proposition 2.5. If $f$ is a relatively pointwise recurrent non-recurrent map, then $f$ is not equicontinuous.

Proof. Since $f$ is a relatively pointwise recurrent non-recurrent map, then $R(f)$ is not closed and so is not equal to $\Omega(f)$. From [10] Proposition 2.1 it follows that $f$ is not equicontinuous.

The following proposition can be derived from [10] Proposition 2.1.

Proposition 2.6. If $f$ is an equicontinuous relatively pointwise recurrent map, then $f$ is a pointwise recurrent map.
3. Proof of main theorem

Before going into the proof of the main theorem we recall the definition of a graph. A (finite) graph $G$ is a compact connected Hausdorff space which contains a finite non-empty set $V$ (the set of vertices), such that every connected component of $G \setminus V$ is homeomorphic to an open interval of the real line. These connected components are called edges. Since any graph can be embedded in $\mathbb{R}^3$, in what follows we will consider each graph endowed with the topology induced by the topology of $\mathbb{R}^3$. A graph map is a continuous map from a graph $G$ to itself.

To prove the main theorem we need the following lemmas:

**Lemma 3.1.** Let $f$ be a graph map. Then $R(f) = \overline{AP(f)}$.

**Proof.** We recall that $\overline{AP(f)}$ is the closure of the union of all minimal sets of $f$ and $\overline{AP(f)} \subseteq R(f)$.

In [1, Theorem 1] the authors showed that a minimal set of a graph map is a finite set or a Cantor set or a union of (finitely many) pairwise disjoint circles. One can deduce that each component of $G - (\overline{AP(f)} \cup V)$ is an open arc of $G$.

Let $[a, b]$ be a component of $G - (\overline{AP(f)} \cup V)$ such that $a \in \overline{AP(f)}$. We suppose that $[a, b] \cap R(f) \neq \emptyset$. Let $x$ be an element of $[a, b] \cap R(f)$. Since $x$ is recurrent, there exists an increasing sequence $(a_n)$ such that $(f^{n}(x))$ converges to $x$. There exist three integers $i < j < k$ such that one of the following two statement holds:

1. $a < x < f^{n_i}(x) < f^{n_j}(x) < f^{n_k}(x)$.
2. $a < f^{n_i}(x) < f^{n_j}(x) < f^{n_k}(x) < x$.

By applying [9, Lemma 2.2] we obtain:

1. $f^{n_i}(x) \in [a, b]$.
2. $f^{n_j-n_i}(a) \in [a, b]$.

In both cases the interval $[a, b]$ will intersect $\overline{AP(f)}$, which is impossible. \hfill $\square$

**Lemma 3.2.** Let $f$ be a relatively pointwise recurrent map of a graph $G$. If $W$ is a proper minimal set of $f$, then $W$ is not a union of (finitely many) pairwise disjoint circles.

**Proof.** In [1, Theorem 1] the authors showed that a minimal set of a graph map is a finite set or a Cantor set or a union of (finitely many) pairwise disjoint circles.

Suppose that $W$ is a union of (finitely many) pairwise disjoint circles. Then $G$ is not a circle (because $W \neq G$). From the fact that $G$ is connected it follows that $W$ contains a branching point $w$. Let $A$ be an arc of $G$ such that $A \cap W = \{w\}$. Since $R(f) = G$, there exists in $A$ a sequence $(w_n)$ of recurrent points which converges to $w$. From the fact that $O(w, f)$ is dense in $W$ it follows that there exists an integer $p$ such that $f^{p}(w)$ is not a branching point and so by continuity of $f^p$, there exists an integer $N$ such that $f^{p}(w_n) \in W$ for all $n > N$. The recurrence of $w_n$ implies that there exists an integer $q > p$ such that $f^{q}(w_n) \in A - \{w\}$, which implies that $f^{q-p}(f^{p}(w_n)) \in A - \{w\}$, which contradicts the fact that $W$ is invariant. \hfill $\square$

**Lemma 3.3.** Let $f$ be a relatively pointwise recurrent map of a graph $G$. If $f$ is not a minimal map, then $\overline{P(f)} = \overline{AP(f)}$.

**Proof.** We always have the inclusion $\overline{P(f)} \subseteq \overline{AP(f)}$.

By applying [1, Theorem 1] and Lemma [3.2] it follows that every minimal set of $f$ is a periodic orbit or a Cantor set. Let $W$ be a minimal set which is a
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Cantor set. Let \( w \) be an element of \( W \) and let \( w' \) be an element of \( G - V(G) \) such that the open arc \((w, w')\) does not meet \( W \). If \((w, w') \cap P(f) = \emptyset\), then from the fact that \((w, w') \cap R(f) \neq \emptyset\) and by applying [9, Lemma 2.2] it follows that \((w, w') \cap O(w, f) \neq \emptyset\), which is impossible. Thus \( w \in P(f) \). Since \( P(f) \) is invariant, \( O(w, f) \subset P(f) \) and so \( W \subset P(f) \). Therefore \( AP(f) = P(f) \). □

Proof of the main theorem. The two statements imply that \( f \) is a relatively pointwise recurrent map.

Conversely, (1) if \( f \) is a minimal map, then first by [11, Theorem 3.2] it follows that \( G \) is a circle, and second by [11, Lemma 3.1] it follows that \( f \) is a homeomorphism. From Proposition 2.4 it follows that \( f \) is a pointwise non-wandering circle map without periodic points. Thus it is topologically conjugate to an irrational rotation.

(2) If \( f \) is not a minimal map, then from Lemma 3.1 and Lemma 3.3 it follows that \( P(f) = G \). □

The fact that the set of periodic points of a transitive graph map is dense was proved by Blokh in [3]. The following corollary extends this result.

Corollary 3.4. Let \( f \) be a relatively pointwise recurrent map of a graph \( G \). If \( f \) is not a minimal map, then \( P(f) = G \).

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