

THE DIHEDRAL GROUP \mathcal{D}_5 AS A GROUP OF SYMPLECTIC AUTOMORPHISMS ON K3 SURFACES

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ABSTRACT. We prove that if a K3 surface X admits $\mathbb{Z}/5\mathbb{Z}$ as a group of symplectic automorphisms, then it actually admits \mathcal{D}_5 as a group of symplectic automorphisms. The orthogonal complement to the \mathcal{D}_5 -invariants in the second cohomology group of X is a rank 16 lattice, L . It is known that L does not depend on X : we prove that it is isometric to a lattice recently described by R. L. Griess Jr. and C. H. Lam. We also give an elementary construction of L .

1. INTRODUCTION

A finite group of symplectic automorphisms on a K3 surface X has the property that the desingularization of the quotient of X by this group is again a K3 surface. In [Nik1] the finite abelian groups of symplectic automorphisms on a K3 surface are classified. The main result of Nikulin in [Nik1] is that the isometries induced by finite abelian groups of symplectic automorphisms on the second cohomology group of a K3 surface are essentially unique. The uniqueness of the isometries induced by G on $H^2(X, \mathbb{Z})$ implies that the lattice $\Omega_G := (H^2(X, \mathbb{Z})^G)^\perp$ does not depend on X . Thanks to this result it is possible to associate the lattice Ω_G to each finite abelian group G of symplectic automorphisms on a K3 surface. From this one obtains information on the coarse moduli space of K3 surfaces admitting G as a group of symplectic automorphisms (cf. [Nik1], [GS1], [GS2]). In [GS1] and [GS2] the lattices Ω_G are computed for each finite abelian group G of symplectic automorphisms on a K3 surface.

In [Mu] and [X] the finite (not necessary abelian) groups of symplectic automorphisms on a K3 surface are classified. Under some conditions (cf. Section 2) Nikulin's result on the uniqueness of the isometries induced by finite groups of symplectic automorphisms on the second cohomology group of the K3 surfaces can be extended to finite (not necessary abelian) groups (cf. [W]). As a consequence one can attach the lattice $\Omega_G := (H^2(X, \mathbb{Z})^G)^\perp$, which depends only on G , also to some finite nonabelian groups G of symplectic automorphisms on a K3 surface X .

Let us consider a pair of finite groups (G, H) such that G acts symplectically on a K3 surface and H is a subgroup of G . It is evident that the K3 surfaces admitting

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G as a group of symplectic automorphisms also admits H as a group of symplectic automorphisms. It is more surprising that for certain pairs of groups (G, H) the converse also holds. Indeed, for certain pairs (G, H) the condition “a K3 surface X admits G as a group of symplectic automorphisms” is equivalent to the condition “ X admits H as a group of symplectic automorphisms”. For these pairs the lattices Ω_G and Ω_H coincide. The aim of this paper is to describe this situation and to give explicitly one pair (G, H) . We observe that in order to find (G, H) with the described property, one has to consider nonabelian groups acting symplectically on K3 surfaces. Indeed, the lattices Ω_K associated to abelian groups K are completely described in [GS1], [GS2], and one can check that Ω_G and Ω_H never coincide if both G and H are abelian and $G \neq H$.

In Section 2 we describe some known results on symplectic automorphisms over kählerian K3 surfaces and prove our main results (Proposition 2.14 and Corollary 2.15). In Proposition 2.14 we give sufficient conditions on G and H to prove that a K3 surface admits G as a group of symplectic automorphisms if and only if it admits H as a group of symplectic automorphisms. Applying this proposition we prove (Corollary 2.15) that a K3 surface admits $\mathbb{Z}/5\mathbb{Z}$ as a group of symplectic automorphisms if and only if it admits \mathcal{D}_5 (the dihedral group of order 10) as a group of symplectic automorphisms. In particular we prove that $\Omega_{\mathbb{Z}/5\mathbb{Z}} \simeq \Omega_{\mathcal{D}_5}$.

In [GS1] the isometry induced on $\Omega_{\mathbb{Z}/5\mathbb{Z}}$ by a symplectic automorphism of order 5 is described. Since we prove that $\Omega_{\mathbb{Z}/5\mathbb{Z}} \simeq \Omega_{\mathcal{D}_5}$, there is also an involution acting on this lattice. In order to describe both the isometry of order 5 and the involution generating \mathcal{D}_5 on $\Omega_{\mathcal{D}_5}$, in Section 3 we give a different description of this lattice: it is an overlattice of $A_4(-2)^{\oplus 4}$ (the isometry of order 5 is induced by the natural one on A_4). In the proof of Corollary 3.5 also the action of the involution is described. Moreover, we show that the lattice $\Omega_{\mathbb{Z}/5\mathbb{Z}}$ (computed in [GS1]) is isometric to a lattice describe by Griess and Lam in [GL].

In Section 4 we consider algebraic K3 surfaces and in particular 3-dimensional families of K3 surfaces admitting a symplectic automorphism of order 5, σ_5 , and a polarization, invariant under σ_5 . It follows from the results of Section 2 that the K3 surfaces in these families also admit an involution ι , generating together with σ_5 the dihedral group \mathcal{D}_5 . For each of these families we exhibit the automorphism σ_5 and find the automorphism ι .

2. SYMPLECTIC AUTOMORPHISMS ON K3 SURFACES

Definition 2.1. Let X be a smooth compact complex surface. The surface X is a K3 surface if the canonical bundle of X is trivial and the irregularity of X , $q(X) := h^{1,0}(X)$, is 0.

The second cohomology group of a K3 surface, equipped with the cup product, is isometric to a lattice, which is the unique, up to isometries, even unimodular lattice with signature $(3, 19)$. This lattice will be denoted by Λ_{K3} and is isometric to $U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$, where U is the unimodular lattice with bilinear form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $E_8(-1)$ is the lattice obtained by multiplying by -1 the lattice associated to the Dynkin diagram E_8 .

The Néron-Severi group of a K3 surface X , $NS(X)$, coincides with its Picard group. The transcendental lattice of X , T_X , is the orthogonal to $NS(X)$ in $H^2(X, \mathbb{Z})$.

Definition 2.2. An isometry α of $H^2(X, \mathbb{Z})$ is an effective isometry if it preserves the Kähler cone of X . An isometry α of $H^2(X, \mathbb{Z})$ is a Hodge isometry if its \mathbb{C} -linear extension to $H^2(X, \mathbb{C})$ preserves the Hodge decomposition of $H^2(X, \mathbb{C})$.

Theorem 2.3 ([BR]). *Let X be a K3 surface and g be an automorphism of X . Then g^* is an effective Hodge isometry of $H^2(X, \mathbb{Z})$. Conversely, let f be an effective Hodge isometry of $H^2(X, \mathbb{Z})$. Then f is induced by an automorphism of X .*

Definition 2.4. An automorphism σ on a K3 surface X is symplectic if σ^* acts as the identity on $H^{2,0}(X)$; that is, $\sigma^*(\omega_X) = \omega_X$, where ω_X is a nowhere vanishing holomorphic two form on X .

Equivalently, σ is symplectic if the isometry induced by σ^* on the transcendental lattice is the identity.

Remark 2.5. Let σ be an automorphism of finite order on a K3 surface. The desingularization of X/σ is a K3 surface if and only if σ is symplectic.

In [Nik1] the finite abelian groups acting symplectically on a kählerian K3 surface are analyzed. Now it is known that every K3 surface is a Kähler variety, [S], so there are no restrictions on the K3 surfaces analyzed in [Nik1].

From now on we always assume that G is a finite group of symplectic automorphisms on the K3 surface X .

Definition 2.6. Let the K3 surface Y be the minimal desingularization of the quotient X/G . Let M_j be the curves arising from the desingularization of the singularities of X/G .

The singularities of the quotient X/G are computed by Nikulin ([Nik1, Section 6]) if G is an abelian group and by Xiao ([X, Table 2]) for all the other finite groups. If either $G = Q_8$ (binary dihedral group of order 8) or $G = T_{24}$ (binary tetrahedral group of order 24), then there are two possible configurations for the singularities of X/G , and hence for the exceptional curve M_j on Y . For all the other groups the number and the type of the singularities of X/G are determined by G .

Definition 2.7. Let us assume that $G \neq Q_8$, $G \neq T_{24}$. The minimal primitive sublattice of $NS(Y)$ containing the curves M_j does not depend on X (cf. [Nik1], [X]). It will be denoted by M_G .

The lattice M_G is computed by Nikulin ([Nik1, Section 7]) for each abelian group G and by Xiao ([X, Table 2]) for the all the other groups G .

Remark 2.8. For $G = \mathbb{Z}/2\mathbb{Z}$, the lattice $M_{\mathbb{Z}/2\mathbb{Z}}$ (called a Nikulin lattice) is an even overlattice of index 2 of $A_1(-1)^{\oplus 8}$. Its discriminant group is $(\mathbb{Z}/2\mathbb{Z})^6$ (cf. [Nik1]), and its discriminant form is the same as $U(2)^{\oplus 3}$ (cf. [Mo]).

Definition 2.9 ([Nik1, Definition 4.6]). We say that G has a unique action on Λ_{K3} if, given two embeddings $i : G \hookrightarrow \text{Aut}(X)$, $i' : G \hookrightarrow \text{Aut}(X')$ such that G is a group of symplectic automorphisms on the K3 surfaces X and X' , there exists an isometry $\phi : H^2(X, \mathbb{Z}) \rightarrow H^2(X', \mathbb{Z})$ such that $i'(g)^* = \phi \circ i(g) \circ \phi^{-1}$ for all $g \in G$.

Theorem 2.10 ([Nik1, Theorem 4.7], [W, Corollary 3.0.1]). *Let G be a finite group acting symplectically on a K3 surface, $G \neq Q_8$, $G \neq T_{24}$. If M_G admits a unique primitive embedding in Λ_{K3} , then G has a unique action on Λ_{K3} .*

Definition 2.11. Under the assumptions of Theorem 2.10 the lattice $(\Lambda_{K3}^G)^\perp$ is uniquely determined by G , up to isometry. It will be called Ω_G .

Remark 2.12. By [Nik1, (8,12)], it follows that $\text{rank}(\Omega_G) = \text{rank}(M_G)$.

Theorem 2.13 ([Nik1, Theorem 4.15]). *Let G be a finite group acting symplectically on a K3 surface, such that G has a unique action on Λ_{K3} . A K3 surface X admits G as a group of symplectic automorphisms if and only if the lattice Ω_G is primitively embedded in $NS(X)$.*

Nikulin proved that for each abelian group acting symplectically on a K3 surface, the hypothesis of Theorem 2.10 (and hence the ones of Theorem 2.13) are satisfied.

Proposition 2.14. *Let G be a finite group acting symplectically on a K3 surface and let H be a subgroup of G . Let us assume that both G and H are neither Q_8 nor T_{24} . We assume that both M_H and M_G admit a unique primitive embedding in Λ_{K3} and $\text{rank}(M_G) = \text{rank}(M_H)$. Then $\Omega_H \simeq \Omega_G$, and so a K3 surface X admits G as a group of symplectic automorphisms if and only if X admits H as a group of symplectic automorphisms.*

Proof. Since H is a subgroup of G , Ω_H is a sublattice of Ω_G . Moreover, $\text{rank}(\Omega_G) = \text{rank}(\Omega_H)$ by Remark 2.12 and the condition on the rank of the lattices M_G and M_H . This implies that $\Omega_H \hookrightarrow \Omega_G$ with a finite index. Let X be a K3 admitting G as a group of symplectic automorphisms. Then both Ω_G and $\Omega_H(\hookrightarrow \Omega_G)$ are primitively embedded in $NS(X)$; hence the index of the inclusion $\Omega_H \hookrightarrow \Omega_G$ is 1, i.e. $\Omega_G \simeq \Omega_H$.

The K3 surface X admits G as a group of symplectic automorphisms if and only if Ω_G is primitively embedded in $NS(X)$. By the isometry $\Omega_H \simeq \Omega_G$ this condition is equivalent to requiring that Ω_H be primitively embedded in $NS(X)$, which holds if and only if X admits H as a group of symplectic automorphisms. \square

Corollary 2.15. *A K3 surface admits $\mathbb{Z}/5\mathbb{Z}$ as a group of symplectic automorphisms if and only if it admits \mathcal{D}_5 as a group of symplectic automorphisms.*

Proof. The lattice $M_{\mathbb{Z}/5\mathbb{Z}}$ is computed in [Nik1], where it is proved that it admits a unique primitive embedding in Λ_{K3} and that its rank is 16. The lattice $M_{\mathcal{D}_5}$ is described in [X] as an overlattice of index 2 of the lattice $A_4(-1)^{\oplus 2} \oplus A_1(-1)^{\oplus 8}$. In particular, $\text{rank}(M_{\mathcal{D}_5}) = 16$ and $M_{\mathcal{D}_5} \simeq A_4(-1)^{\oplus 2} \oplus M_{\mathbb{Z}/2\mathbb{Z}}$, where $M_{\mathbb{Z}/2\mathbb{Z}}$ is the Nikulin lattice (see Remark 2.8). Thus the discriminant group of $M_{\mathcal{D}_5}$ is $(\mathbb{Z}/5\mathbb{Z})^2 \oplus (\mathbb{Z}/2\mathbb{Z})^6$. By [Nik2, Theorem 1.14.4], $M_{\mathcal{D}_5}$ admits a unique primitive embedding in Λ_{K3} . The corollary immediately follows from Proposition 2.14. \square

Remark 2.16. We proved that if a K3 surface admits a symplectic automorphism of order five, σ_5 ; hence it also admits a symplectic involution generating \mathcal{D}_5 together with σ_5 . This result cannot be improved; i.e., it is **not** true that if a K3 surface X admits $\mathcal{D}_5 = \langle \sigma_5, \iota \rangle$ as a group of symplectic automorphisms, then it also admits a symplectic automorphism α such that $J := \langle \alpha, \sigma_5, \iota \rangle \supsetneq \mathcal{D}_5$ is a finite group. By contradiction assuming there exists such an α , then $\mathcal{D}_5 \subsetneq J$ and $\Omega_{\mathcal{D}_5} \simeq \Omega_J$. In particular, $\text{rank}\Omega_J = \text{rank}\Omega_{\mathcal{D}_5} = 16$, but there are no finite groups J of symplectic automorphisms on a K3 surface such that $\mathcal{D}_5 \subsetneq J$ and $\text{rank}M_J (= \text{rank}\Omega_J) = 16$ (cf. [X, Table 2]).

3. CONSTRUCTION OF $\Omega_{\mathbb{Z}/5\mathbb{Z}} \simeq \Omega_{\mathcal{D}_5}$

The aim of this section is to construct the lattice $\Omega_{\mathbb{Z}/5\mathbb{Z}}$ as an overlattice of $A_4(-2)^{\oplus 4}$ and to describe the action of \mathcal{D}_5 on this lattice. The automorphism of order five on $\Omega_{\mathbb{Z}/5\mathbb{Z}}$ will be induced by the automorphism of order five on each copy of $A_4(-2)$.

We will add some rational linear combinations of the elements of $A_4(-2)^{\oplus 4}$ to obtain an even overlattice of $A_4(-2)^{\oplus 4}$. The main point is that we would like to extend the automorphism of $A_4(-2)^{\oplus 4}$ to the lattice $\Omega_{\mathbb{Z}/5\mathbb{Z}}$, so if we add an element to $A_4(-2)^{\oplus 4}$ we have to add all elements in its orbit.

We recall that the standard basis of A_4 is expressed in terms of the standard basis $\{\varepsilon_i\}$ of \mathbb{R}^5 in the following way: $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, hence $\alpha_5 = -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$. The cyclic permutation of the basis vectors of \mathbb{R}^5 induces the automorphism γ on A_4 ($\gamma(\alpha_i) = \alpha_{i+1}$).

Proposition 3.1. *Let us consider the lattice $A_4(-2)^{\oplus 4}$ and the automorphism $g := (\gamma, \gamma, \gamma, \gamma)$ (acting as γ on each copy of $A_4(-2)$). Let $a_{j,i}$, $j, i = 1, 2, 3, 4$, be the element α_i in the j -copy of $A_4(-2)$. Let*

$$\mu := \frac{1}{2}(a_{1,1} + a_{2,1} + a_{3,1} + a_{4,1}), \quad \nu := \frac{1}{2}(a_{2,1} + a_{3,3} + a_{3,4} + a_{4,1} + a_{4,3} + a_{4,4}).$$

Then:

- The lattice

$$L := A_4(-2)^{\oplus 4} + \langle g^i(\mu), g^i(\nu) \rangle_{i=0,1,2,3},$$

generated by $A_4(-2)^{\oplus 4}$ and the eight vectors $g^i(\mu), g^i(\nu)$, for $i = 0, 1, 2, 3$, is an even overlattice of $A_4(-2)^{\oplus 4}$ of rank 16.

- The index of $A_4(-2)^{\oplus 4}$ in L is 2^8 .
- There are no vectors of length -2 in L .

Proof. Since μ has an integer intersection with the basis $\{a_{i,j}\}$ of $A_4(-2)^{\oplus 4}$ and has self intersection -4 , we can add the element μ to the lattice $A_4(-2)^{\oplus 4}$ obtaining an even overlattice. All the elements $g^i(\mu)$ in the orbit of μ have an integer intersection with this basis of $A_4(-2)^{\oplus 4}$ and $g^i(\mu)g^j(\mu) \in \mathbb{Z}$ for all $i, j = 0, 1, 2, 3$. Thus we can add the four vectors $\mu, g(\mu), g^2(\mu), g^3(\mu)$ to the lattice $A_4(-2)^{\oplus 4}$.

It is easy to show that the vectors $g^i(\nu)$, $i = 0, 1, 2, 3$, have an integer intersection pairing with all the vectors in $A_4(-2)^{\oplus 4}$ and that $g^i(\nu)g^j(\nu) \in \mathbb{Z}$, $(g^i(\nu))^2 \in 2\mathbb{Z}$ (indeed these are properties of all the vectors of type $v_{i,j,k,h,l,m} = \frac{1}{2}(0, \varepsilon_i - \varepsilon_{i+1}, \varepsilon_j - \varepsilon_{j+2}, \varepsilon_k - \varepsilon_h + \varepsilon_l - \varepsilon_m)$, $k < h < l < m$ and $\nu = v_{1,3,1,2,3,5}$).

Moreover $g^i(\nu)g^j(\mu) \in \mathbb{Z}$, $i, j \in \{0, 1, 2, 3\}$ (indeed this is a property of the vectors of type $v_{i,j,k,h,l,m}$ such that $\{i, i + 1\} \cap \{j, j + 2\} = \emptyset$ and $\{k, h, l, m\} = \{i, i + 1, j, j + 2\}$).

Thus adding the vectors in the orbit of ν and of μ to $A_4(-2)^{\oplus 4}$ we construct an even overlattice L of $A_4(-2)^{\oplus 4}$. By the computation of the discriminant of this lattice, it follows that $A_4(-2)^{\oplus 4}$ has index 2^8 in L (indeed, to construct L we add to $A_4(-2)^{\oplus 4}$ exactly eight vectors of type $\frac{1}{2}v$, $v \in A_4(-2)^{\oplus 4}$, and they are independent over \mathbb{Z}).

Now we prove that there are no vectors with length -2 in L . Let $y = \sum_{i=0}^3 b_i g^i(\mu) + \sum_{i=0}^3 c_i g^i(\nu)$, $b_i, c_i \in \mathbb{Z}$. In $A_4(-2)^{\oplus 4} \otimes \mathbb{Q}$ we have

$$y := \frac{1}{2} \left(\sum_{i=0}^3 b_i \alpha_{i+1}, \quad \sum_{i=0}^3 b_i \alpha_{i+1} + \sum_{i=0}^3 c_i \alpha_{i+1}, \right. \\ \left. \sum_{i=0}^3 b_i \alpha_{i+1} + (-c_1 + c_3) \alpha_1 + (-c_1 - c_2 + c_3) \alpha_2 + (c_0 - c_1 - c_2) \alpha_3 + (c_0 - c_2) \alpha_4, \right. \\ \left. \sum_{i=0}^3 b_i \alpha_{i+1} + (c_0 - c_1 + c_3) \alpha_1 + (-c_2 + c_3) \alpha_2 + (c_0 - c_1) \alpha_3 + (c_0 - c_2 + c_3) \alpha_4 \right).$$

If we require that at least two components of y are equal to zero, we obtain that $b_i = c_i = 0$ for all i . So if $y \neq 0$, then y has at most one component equal to zero.

Each vector w in L is of the form $y + z$ with y as above, $b_i, c_i \in \{0, 1\}$, and $z \in A_4(-2)^{\oplus 4}$; moreover, such y and z are uniquely determined by w .

If $y = (0, 0, 0, 0)$, $w \in A_4(-2)^{\oplus 4}$, and hence $w^2 \leq -4$.

If $y \neq (0, 0, 0, 0)$, then $w = \frac{1}{2}(w_1, w_2, w_3, w_4)$ with $w_i \in A_4(-2)$ and at most one of $w_i = 0$. Since $w_i^2 \leq -4$ and $w_i \cdot w_j = 0$ if $i \neq j$, we get $w^2 \leq \frac{3}{4}(-4)$. Hence there are no vectors of length -2 in L . □

Proposition 3.2. *The lattice L is isometric to the lattice $\Omega_{\mathbb{Z}/5\mathbb{Z}} = \Omega_{\mathcal{D}_5}$.*

Proof. By uniqueness of $\Omega_{\mathbb{Z}/5\mathbb{Z}}$, to prove the proposition it suffices to show that there exists a K3 surface S such that $G = \mathbb{Z}/5\mathbb{Z}$ is a group of symplectic automorphisms on S and $(H^2(S, \mathbb{Z})^G)^\perp \simeq L$.

By construction L admits an automorphism of order 5, g , acting trivially on the discriminant group. Moreover L is negative definite and its discriminant group is $(\mathbb{Z}/5\mathbb{Z})^4$. Hence it admits a primitive embedding in Λ_{K3} ([Nik2, Theorem 1.14.4]). Since g acts trivially on the discriminant group, $G := \langle g \rangle$ extends to a group of isometries on Λ_{K3} which acts as the identity on $L^{\perp \Lambda_{K3}}$.

Let S be a K3 surface such that $L \subset NS(S)$ (such a K3 surface exists by the surjectivity of the period map). By Proposition 3.1, L does not contain elements of length -2 . This is enough to prove that the isometries of G defined above (if necessary, composed with a reflection in the Weil group) are effective isometries for S (the proof of this fact is essentially given in [Nik1, Theorem 4.3]; see also [GS1, Step 4, proof of Proposition 5.2]). By construction, these are Hodge isometries (cf. [Nik1, Theorem 4.3]), so they are induced by automorphisms on S (by the Torelli theorem; cf. [BR]). Since these automorphisms act as the identity on $T_S \subset L^{\perp \Lambda_{K3}}$, they are symplectic. By construction of the isometries of G , $L \simeq (H^2(S, \mathbb{Z})^G)^\perp$, and so $L \simeq \Omega_{\mathbb{Z}/5\mathbb{Z}}$. □

Since $\Omega_{\mathcal{D}_5} \simeq L$, the dihedral group and in particular an involution acts on L . This implies that the lattice $\Omega_{\mathbb{Z}/2\mathbb{Z}} \simeq E_8(-2)$ (cf. [Mo]) is primitively embedded in L and there exists an involution on L acting as -1 on this lattice and as the identity on its orthogonal. In the following remark we give an embedding of $E_8(-2)$ in L , and in the proof of the Corollary 3.5 we describe the involution associated to this embedding.

Remark 3.3. The vectors

$$\begin{aligned} e_1 &:= \mu & e_2 &:= g^2(\mu) + g^3(\mu) & e_3 &:= \nu & e_4 &:= \mu + g^2(\mu) + g^3(\mu) - g^2(\nu) - g^3(\nu) \\ e_5 &:= a_{1,1} & e_6 &:= a_{1,3} + a_{1,4} & e_7 &:= a_{2,1} & e_8 &:= a_{2,3} + a_{2,4} \end{aligned}$$

generate a copy of $E_8(-2)$ embedded in L . Indeed, the lattice generated by e_i is such that multiplying its bilinear form by $\frac{1}{2}$ one obtains a negative definite even unimodular lattice of rank 8, i.e. a copy of $E_8(-1)$.

Remark 3.4. Let $f_{i+8} = g(e_i)$, $i = 1, \dots, 8$, where $\{e_i\}_{i=1, \dots, 8}$ is the basis of $E_8(-2)$ defined in Remark 3.3. Since g is an isometry of the lattice it is clear that f_i , $i = 9, \dots, 16$, generate a copy of $E_8(-2)$ embedded in L . A direct computation shows that the classes e_i , f_{i+8} , $i = 1, \dots, 8$, generate a lattice of rank 16 and discriminant 5^4 embedded in L and so they are a \mathbb{Z} -basis for L .

The paper [GL] classifies positive definite lattices which have dihedral groups \mathcal{D}_n (for $n = 2, 3, 4, 5, 6$) in the group of the isometries and which have the properties:

- the lattices are rootless (i.e. there are no elements of length 2),
- they are the sum of two copies of $E_8(2)$,
- there are two involutions in \mathcal{D}_n acting as minus the identity on each copy of $E_8(2)$.

In [GL, Section 7] it is proved that there is a unique lattice with all these properties and admitting \mathcal{D}_5 in the group of isometries, called $DIH_{10}(16)$.

Corollary 3.5. *The lattice $DIH_{10}(16)$ described in [GL] is isometric to the lattice $L(-1) \simeq \Omega_{\mathcal{D}_5}(-1)$.*

Proof. The even lattice L has no vectors of length -2 (Proposition 3.1).

On L there is an isometry of order 5, g (Proposition 3.1).

Let us define a map h on the lattice L which acts as -1 on the copy of $E_8(-2)$ generated by e_i , $i = 1, \dots, 8$ (cf. Remark 3.3) and as the identity on the orthogonal complement. Since h acts trivially on the discriminant group of $E_8(-2) \simeq \langle e_i \rangle_{i=1, \dots, 8}$, h is an isometry of L and in particular an involution. One can directly check that its action on the basis of $A_4(-2)^{\oplus 4}$ is $h(a_{i,1}) = -a_{i,1}$, $h(a_{i,2}) = -a_{i,5}$, $h(a_{i,3}) = -a_{i,4}$, $h(a_{i,4}) = -a_{i,3}$, $i = 1, 2, 3, 4$. This action extends to a \mathbb{Z} -basis of L .

The group $\langle g, h \rangle$ is \mathcal{D}_5 . The involutions h and $g^2 \circ h$ are two involutions generating the group \mathcal{D}_5 . By construction h and $g^2 \circ h$ act as minus the identity respectively on the lattice $E_8(-2) \simeq \langle e_i \rangle_{i=1, \dots, 8}$ and on the lattice $E_8(-2) \simeq \langle f_i \rangle_{i=9, \dots, 16}$. These two copies of $E_8(-2)$ generate L (by Remark 3.4). So $L(-1)$ satisfies the conditions which define $DIH_{10}(16)$ and hence $DIH_{10}(16) \simeq L(-1) \simeq \Omega_{\mathcal{D}_5}(-1)$.

4. EXAMPLES: ALGEBRAIC K3 SURFACES WITH A POLARIZATION OF A LOW DEGREE

Here we give some very explicit examples of families of K3 surfaces admitting $\mathbb{Z}/5\mathbb{Z}$, and hence \mathcal{D}_5 , as a group of symplectic automorphisms. In particular, in this section we consider algebraic K3 surfaces.

We recall that a polarization L , with $L^2 = 2d$, on a K3 surface X defines a map $\phi_L : X \rightarrow \mathbb{P}^{d+1}$. In this section we consider K3 surfaces with a polarization L such that $\phi_L(X)$ is a complete intersection in a certain projective space and K3 surfaces with a polarization of degree 2, which exhibits the K3 surfaces as double covers of the plane.

Let X be a general member of a family of K3 surfaces admitting an automorphism of order 5, σ_5 , and a polarization, L , invariant under σ_5 . In [GS1, Proposition 5.1] the possible Néron–Severi groups of X are computed. In particular, if $L^2 = 2d < 10$,

one obtains that $NS(X) \simeq \mathbb{Z}L \oplus \Omega_{\mathbb{Z}/5\mathbb{Z}} =: \mathcal{L}_{2d}$. This lattice admits a unique primitive embedding in Λ_{K3} . The family of K3 surfaces with a polarization L of degree $L^2 < 10$, invariant under a symplectic automorphism of order 5, is then the family of the \mathcal{L}_{2d} -polarized K3 surfaces. In particular, for each $d < 5$ we find a 3-dimensional family of K3 with such a polarization L , and hence we have the following possibilities:

- $\phi_L : X \xrightarrow{2:1} \mathbb{P}^2$, so X is a double cover of \mathbb{P}^2 branched along a plane sextic curve: in this case $NS(X) \simeq \mathcal{L}_2$;
- $\phi_L(X)$ is a quartic in \mathbb{P}^3 : in this case $NS(X) \simeq \mathcal{L}_4$;
- $\phi_L(X)$ is the complete intersection of a quadric and a cubic in \mathbb{P}^4 : in this case $NS(X) \simeq \mathcal{L}_6$;
- $\phi_L(X)$ is the complete intersection of three quadrics in \mathbb{P}^5 : in this case $NS(X) \simeq \mathcal{L}_8$.

Now we construct a general member of each of these families and show that it also admits a symplectic involution ι generating, together with σ_5 , the group \mathcal{D}_5 . Since the automorphisms ι and σ_5 leave invariant the polarization, both these automorphisms can be extended to automorphisms of the ambient projective space.

We will denote by ω a primitive 5-th root of unity.

4.1. $L^2 = 4$. This polarization gives a map to \mathbb{P}^3 , where the K3 surfaces are realized as quartic surfaces. Let us consider the automorphism

$$\sigma_{\mathbb{P}^3} : (x_0 : x_1 : x_2 : x_3) \rightarrow (x_0 : \omega^3 x_1 : \omega x_2 : \omega^2 x_3).$$

The quartic surfaces in \mathbb{P}^3 , defined as

$$(4.1) \quad V(ax_0^3 x_2 + bx_0^2 x_1^2 + cx_0 x_3^3 + dx_0 x_1 x_2 x_3 + ex_1^3 x_3 + fx_1 x_2^3 + gx_2^2 x_3^2),$$

are invariant for $\sigma_{\mathbb{P}^3}$. Hence the restriction of $\sigma_{\mathbb{P}^3}$ to K3 surfaces with equation (4.1) is an automorphism σ_5 of the surfaces. To show that this automorphism is symplectic it suffices to apply σ_5 to the following holomorphic two-form in local coordinates $x = x_1/x_0, y = x_2/x_0, z = x_3/x_0$:

$$\left(\frac{\partial f}{\partial z}\right)^{-1} dx \wedge dy,$$

where f denotes the equation of the quartic in the local coordinate x, y, z .

Equation (4.1) depends on 7 parameters. The automorphisms of \mathbb{P}^3 commuting with $\sigma_{\mathbb{P}^3}$ are $diag(\alpha, \beta, \gamma, \delta)$ (which is a four-dimensional group); hence this family of $\sigma_{\mathbb{P}^3}$ -invariant quartics has

$$(7 - 1) - (4 - 1) = 3$$

moduli. So the family of K3 surfaces given by equation (4.1) is the family of K3 surfaces admitting an automorphism of order 5, leaving invariant a polarization of degree 4.

Up to a projectivity, commuting with $\sigma_{\mathbb{P}^3}$, equation (4.1) becomes

$$(4.2) \quad a'x_0^3 x_2 + b'x_0^2 x_1^2 + c'x_0 x_3^3 + d'x_0 x_1 x_2 x_3 + a'x_1^3 x_3 + c'x_1 x_2^3 + g'x_2^2 x_3^2 = 0.$$

Let us define an involution of \mathbb{P}^3 :

$$\iota_{\mathbb{P}^3} : (x_0 : x_1 : x_2 : x_3) \rightarrow (x_1 : x_0 : x_3 : x_2).$$

Equation (4.2) is invariant under $\iota_{\mathbb{P}^3}$, hence $\iota_{\mathbb{P}^3}$ induces an automorphism of the quartic surfaces with equation (4.2). We call this automorphism ι . The fixed point set of $\iota_{\mathbb{P}^3}$ in \mathbb{P}^3 is the union of the lines

$$l_1 = \begin{cases} x_0 = x_1, \\ x_2 = x_3, \end{cases} \quad l_2 = \begin{cases} x_0 = -x_1, \\ x_2 = -x_3. \end{cases}$$

Hence ι fixes eight points on the quartics (4.2): the intersection of the quartics with l_1 and l_2 . This is enough to show that ι is a symplectic involution. Indeed the involutions which are not symplectic are either fixed point free or fix some curves [Z, Theorem 1]. Hence the quartics given in (4.2) (and hence, up to a projectivity, in (4.1)) admit both the automorphisms σ_5 and ι . It is easy to check that $\langle \sigma_{\mathbb{P}^3}, \iota_{\mathbb{P}^3} \rangle = \mathcal{D}_5$ and hence $\langle \sigma_5, \iota \rangle = \mathcal{D}_5$. So the family of smooth quartic surfaces in \mathbb{P}^3 admitting the symplectic automorphism σ_5 also admits a symplectic involution ι and in fact the group $\mathcal{D}_5 = \langle \sigma_5, \iota \rangle$.

4.2. $L^2 = 6$. This polarization gives a map to \mathbb{P}^4 , where the K3 surfaces are realized as complete intersections of a cubic and a quadric.

Let us consider the automorphism

$$\sigma_{\mathbb{P}^4} : (x_0 : x_1 : x_2 : x_3 : x_4) \rightarrow (x_0 : \omega x_1 : \omega^2 x_2 : \omega^3 x_3 : \omega^4 x_4).$$

Let

$$(4.3) \quad \begin{aligned} Q &:= V(ax_0^2 + bx_1x_4 + cx_2x_3); \\ C &:= V(dx_0^3 + ex_0x_1x_4 + fx_0x_2x_3 + gx_1^2x_3 + hx_2x_4^2 + lx_1x_2^2 + mx_3^2x_4). \end{aligned}$$

Then Q and C are $\sigma_{\mathbb{P}^4}$ -invariant hypersurfaces in \mathbb{P}^4 . We observe that the complete intersection of these two hypersurfaces is generically smooth and thus it is a K3 surface.

The complete intersection of Q and C is also the complete intersection of Q and $C + \lambda x_0 Q$. Hence there is a 1-dimensional family of invariant cubics giving the same complete intersection: the cubics giving different complete intersections depend on $7 - 1 = 6$ parameters. The automorphisms of \mathbb{P}^4 which commute with $\sigma_{\mathbb{P}^4}$ are of the form $diag(\alpha, \beta, \gamma, \delta, \varepsilon)$. So the family of complete intersections of a cubic and a quadric invariant under the automorphism $\sigma_{\mathbb{P}^4}$ has $(3 - 1) + (6 - 1) - (5 - 1) = 3$ moduli.

Let X be the complete intersection of Q and C . The automorphism $\sigma_{\mathbb{P}^4}$ induces a symplectic automorphism on X (this can be shown as in case $L^2 = 4$ considering the two holomorphic form, in local coordinates x, y, z, t , $(dx \wedge dy)/(Q_z C_t - C_t Q_z)$, where F_x is the partial derivative of F w.r.t. x).

Up to the action of the projectivities commuting with $\sigma_{\mathbb{P}^4}$, we can assume that $g = h$ and $l = m$ in the equation of C in (4.3). Hence the involution

$$\iota_{\mathbb{P}^4} : (x_0 : x_1 : x_2 : x_3 : x_4) \rightarrow (x_0 : x_4 : x_3 : x_2 : x_1)$$

fixes Q and C . So its restriction to X is an involution, ι , of X . Moreover, ι has eight fixed points (six on the plane $x_1 = x_4, x_2 = x_3$ and two on the lines $x_0 = 0, x_1 + x_4 = 0, x_2 + x_3 = 0$). Thus σ_5 and ι are symplectic automorphisms of the K3 surface X and they generate the group \mathcal{D}_5 .

4.3. $L^2 = 8$. This polarization gives a map to \mathbb{P}^5 , where the K3 surfaces are realized as complete intersection three quadrics.

Let us consider the map

$$\sigma_{\mathbb{P}^5} : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \rightarrow (x_0 : x_1 : \omega x_2 : \omega^2 x_3 : \omega^3 x_4 : \omega^4 x_5)$$

and the complete intersection of the quadrics

$$\begin{aligned} Q'_1 &:= V(ax_0^2 + bx_0x_1 + cx_1^2 + dx_2x_5 + ex_3x_4), \\ Q'_2 &:= V(fx_0x_2 + gx_1x_2 + hx_3x_5 + lx_4^2), \\ Q'_3 &:= V(mx_0x_5 + nx_1x_5 + ox_2x_4 + px_3^2). \end{aligned}$$

The group of automorphisms of \mathbb{P}^5 commuting with $\sigma_{\mathbb{P}^5}$ is $(GL(2) \times GL(1)^4)/GL(1)$, which has dimension $7 = 8 - 1$. So these complete intersections in \mathbb{P}^5 have $(5 - 1) + (4 - 1) + (4 - 1) - (8 - 1) = 3$ moduli. Up to automorphisms of \mathbb{P}^5 commuting with $\sigma_{\mathbb{P}^5}$ we can assume that the quadrics have the following equation:

$$\begin{aligned} Q_1 &:= V(x_0^2 + bx_0x_1 + x_1^2 + dx_2x_5 + ex_3x_4), \\ Q_2 &:= V(x_0x_2 + x_1x_2 + x_3x_5 + x_4^2), \\ Q_3 &:= V(x_0x_5 + x_1x_5 + x_2x_4 + x_3^2), \end{aligned}$$

and in fact they depend on 3 parameters.

The complete intersection X of the quadrics Q_1, Q_2, Q_3 is smooth for a generic choice of the parameters b, d, e (one can check it directly by putting $e = b = 0, d = 1$). Moreover X is invariant under the automorphism $\sigma_{\mathbb{P}^5}$, so $\sigma_{\mathbb{P}^5}$ induces an automorphism σ_5 on X and σ_5 is symplectic (this can be shown as in the case $L^2 = 6$).

The involution

$$\iota_{\mathbb{P}^5} : (x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \rightarrow (x_0 : x_1 : x_5 : x_4 : x_3 : x_2)$$

fixes the quadric Q_1 and switches the quadrics Q_2 and Q_3 . So its restriction to the K3 surface X is an involution, ι , of the surface X . Moreover ι has eight fixed points on the space $x_2 = x_5, x_3 = x_4$. Thus σ_5 and ι are symplectic automorphisms of the K3 surface X and they generate the group \mathcal{D}_5 .

4.4. $L^2 = 2$. This polarization gives a $2 : 1$ map to \mathbb{P}^2 , and the K3 surfaces are realized as double cover of \mathbb{P}^2 branched along a sextic plane curve.

The map

$$\sigma_{\mathbb{P}^2} : (x_0 : x_1 : x_2) \rightarrow (x_0 : \omega x_1 : \omega^4 x_2)$$

is an automorphism of \mathbb{P}^2 . Up to projectivity of \mathbb{P}^2 commuting with $\sigma_{\mathbb{P}^2}$, the invariant sextics for $\sigma_{\mathbb{P}^2}$ are

$$\mathcal{C}_6 := V(x_0^6 + x_0x_1^5 + x_0x_2^5 + ax_0^4x_1x_2 + bx_0^2x_1^2x_2^2 + cx_1^3x_2^3).$$

Let X be the double cover of \mathbb{P}^2 branched along \mathcal{C}_6 , i.e. X is $V(u^2 - (x_0^6 + x_0x_1^5 + x_0x_2^5 + ax_0^4x_1x_2 + bx_0^2x_1^2x_2^2 + cx_1^3x_2^3))$ in the weighted projective space $\mathbb{P}\mathbb{W}(3, 1, 1, 1)$. The automorphism $\sigma_{\mathbb{P}^2}$ lifts to a symplectic automorphism $\sigma_5 : (u : x_0 : x_1 : x_2) \rightarrow (u : x_0 : \omega x_1 : \omega^4 x_2)$ of X . So we constructed the 3-dimensional family of K3 surfaces which are double covers of \mathbb{P}^2 and have a symplectic automorphism of order 5 which leaves invariant the polarization.

The involution

$$\alpha_{\mathbb{P}^2} : (x_0 : x_1 : x_2) \rightarrow (x_0 : x_2 : x_1)$$

leaves the curve \mathcal{C}_6 invariant, so it lifts to an involution $\alpha_X : (u : x_0 : x_1 : x_2) \rightarrow (u : x_0 : x_2 : x_1)$ of the surface X . The involution α_X fixes a curve (the pullback

of the line $x_1 = x_2$ in \mathbb{P}^2), so it is not symplectic. Let $i : (u : x_0 : x_1 : x_2) \rightarrow (-u : x_0 : x_1 : x_2)$ be the covering involution on X . It is a nonsymplectic involution (indeed the quotient X/ι is rational) and it commutes both with α_X and σ_5 . The involution $\iota = \alpha_X \circ i$ is a symplectic involution on X (because it is the composition of two commuting nonsymplectic involutions). Moreover one has $\iota \circ \sigma_5 = \sigma_5^{-1} \circ \iota$, and hence $\mathcal{D}_5 = \langle \sigma_5, \iota \rangle$ acts symplectically on X .

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