INTEGRAL REPRESENTATION OF SKOROKHOD REFLECTION

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Abstract. We show that a certain integral representation of the one-sided Skorokhod reflection of a continuous bounded variation function characterizes the reflection in that it possesses a unique maximal solution which solves the Skorokhod reflection problem.

1. Introduction

The Skorokhod reflection problem has a long history. Skorokhod [10] introduced it as a method for representing a diffusion process with a reflecting boundary at zero. Given a continuous function \( X : [0, \infty) \to \mathbb{R} \), the standard Skorokhod reflection problem seeks to find \((Q(t), t \geq 0)\) and a continuous, nondecreasing function \( Y : [0, \infty) \to \mathbb{R}^+ \) with \( Y(0) = 0 \), such that \( Q(t) := X(t) + Y(t) \geq 0 \) for all \( t \), and \( \int_0^\infty Q(s) dY(s) = 0 \). Intuitively, the latter expresses the idea that \( Y \) can increase only at points \( t \) such that \( X(t) + Y(t) = 0 \). Skorokhod [10] showed that there is only one such \( Y \), namely, \( Y(t) = -\inf_{0 \leq s \leq t} (X(s) \wedge 0) \), and thus \( Q(t) = X(t) \vee \sup_{0 \leq s \leq t} (X(t) - X(s)) \).

We use the standard notation \( a \vee b := \max(a, b) \), \( a \wedge b := \min(a, b) \). The mapping \( X \mapsto Q \) is referred to as the (one-sided) Skorokhod reflection mapping and has now become a standard tool in probability theory and other areas. As an example, we recall that if \( X \) is the path of a Brownian motion, then \( Q \) is a reflecting Brownian motion and \( Q(t) \) has the same distribution as \( |X(t)| \) for all \( t \geq 0 \) [3, 9]. Several extensions of the Skorokhod reflection mapping exist generalizing the range of \( X \) (see, e.g., [11]) or its domain (see, e.g., [1]).

The question resolved in this paper was motivated by an application of the Skorokhod reflection in stochastic fluid queues [7, 6]. Suppose that \( A, C \) are two jointly stationary and ergodic random measures defined on a common probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with intensities \( a, c \), respectively, such that \( a < c \). Then there exists a unique stationary and ergodic stochastic process \((Q(t), t \in \mathbb{R})\) defined on...
Consider a locally finite signed measure $X$ on the Borel sets of $\mathbb{R}$. Assume that $X$ has no atoms, i.e. that $X(\{t\}) = 0$ for all $t \in \mathbb{R}$. Define
\begin{equation}
Q^*(t) := \sup_{0 \leq s \leq t} X(s, t], \quad t \geq 0,
\end{equation}
where $X(s, t] = X((s, t])$ is the value of $X$ at the interval $(s, t]$. In particular,
\[Q^*(0) = 0.\]

Let $X(t) := X(0, t]$ and write (2.1) as
\[Q^*(t) = X(t) - \inf_{0 \leq s \leq t} X(s).\]

The standard terminology \cite{3, 12} is that $Q^*$ solves the Skorokhod reflection problem for the function $t \mapsto X(t)$.

Decompose $X$ as the difference of two locally finite nonnegative measures $A, C$, without atoms; i.e. write
\begin{equation}
X = A - C.
\end{equation}

We stress that $A, C$ are not necessarily the positive and negative parts of $X$. In other words, the decomposition is not unique. For instance, we can add an arbitrary locally finite nonnegative measure without atoms to both $A$ and $C$. \footnote{Since $X, A, C$ are assumed to have no atoms, we may as well write $X[s, t]$ or $X(s, t]$ instead of $X(s, t]$, and likewise for $A$ and $C$, but we have chosen the notation to be consistent with possible generalizations.}
In [6] it was proved that (2.1) also satisfies the fixed point equation referred to as the “integral representation” of the reflected process:

\[(2.3) \quad Q(t) = \int_0^t 1(Q(s) > C(s, t)) \, dA(s), \quad t \geq 0.\]

A simpler version of this appeared earlier in [7]; this version was concerned with the case where \(C\) is a multiple of the Lebesgue measure. In an open problems session of the workshop on “New Topics at the Interface between Probability and Communications” [8], the second author asked whether and in what sense (2.3) implies (2.1); the question was actually asked for the special case where \(C\) is a multiple of the Lebesgue measure.

In this note we answer this question by proving the following:

**Theorem 2.1.** Let \(A, C\) be locally finite Borel measures on \(\mathbb{R}_+ = [0, \infty)\) without atoms and consider the integral equation (2.3). This integral equation admits a unique maximal solution, i.e. a solution which pointwise dominates any other solution. Further, this maximal solution is precisely the function \(Q^*\) defined by (2.1).

We proceed as follows. First, we present some auxiliary results and also give a proof of (2.1) \(\Rightarrow\) (2.3) which is different from the one found in [6]. Then we prove Theorem 2.1 by a successive approximation scheme and by proving a number of lemmas.

### 3. Proof of the Integral Representation and Auxiliary Results

We first exhibit some properties of \(Q^*\), defined by (2.1), and also show that \(Q^*\) satisfies the integral equation (2.3). The proof of the latter in the special case where \(C\) is a multiple of the Lebesgue measure can be found in [7, Lemma 1] and in [2, §3.5.3]. A more general case is dealt with in [6, Theorem 1]. We give a different proof in Proposition 3.4 below. The lemmas below are straightforward and well-known, but we give proofs for completeness. As before, \(X\) is a locally finite Borel measure without atoms and \(X = A - C\) is a decomposition as the difference of two nonnegative locally finite Borel measures without atoms. We set

\[A(t) := A(0, t], \quad C(t) := C(0, t].\]

**Lemma 3.1.** If \(0 \leq s \leq s' \leq t\) and if \(Q^*(s) > C(s, t]\), then \(Q^*(s') > C(s', t].\)

**Proof.** Assume that \(C(s, t] < Q^*(s) = \sup_{0 \leq u \leq s} X(u, s]\). This is equivalent to

\[C(t) - C(s) < \sup_{0 \leq u \leq s} \{A(s) - A(u) - (C(s) - C(u))\} = A(s) + \sup_{0 \leq u \leq s} \{-A(u) + C(u)\} - C(s),\]

that is, \(C(t) < A(s) + \sup_{0 \leq u \leq s} \{-A(u) + C(u)\}\).

The right-hand side of the latter is increasing in \(s\) and so replacing \(s\) by a larger \(s'\) we obtain

\[C(t) < A(s') + \sup_{0 \leq u \leq s'} \{-A(u) + C(u)\},\]

which is equivalent to \(Q^*(s') > C(s', t].\) \(\square\)
Lemma 3.2. \( Q^* \) satisfies

\[
Q^*(t) = \sup_{s \leq u \leq t} X(u, t] \vee (Q^*(s) + X(s, t]), \quad 0 \leq s \leq t.
\]

Proof. We show that the right-hand side of (3.1) equals the left-hand side:

\[
\sup_{s \leq u \leq t} X(u, t] \vee (Q^*(s) + X(s, t]) = \sup_{s \leq u \leq t} X(u, t] \vee \{ \sup_{0 \leq u \leq s} X(u, s]) + X(s, t]\}
\]

\[
= \sup_{s \leq u \leq t} X(u, t] \vee \sup_{0 \leq u \leq s} \{ X(u, s] + X(s, t]\}
\]

\[
= \sup_{s \leq u \leq t} X(u, t] \vee \sup_{0 \leq u \leq s} X(u, t]
\]

\[
= \sup_{0 \leq u \leq t} X(u, t] = Q^*(t).
\]

\[\square\]

Lemma 3.3. If \( 0 \leq s \leq t \) and if \( Q^*(s) \geq C(s, t] \), then \( Q^*(t) = Q^*(s) + X(s, t] \).

Proof. We use equation (3.1), rewritten as follows:

\[
Q^*(t) = \sup_{s \leq u \leq t} \{ X(u, t] \vee (Q^*(s) + X(s, t]) \}.
\]

Suppose \( 0 \leq s \leq u \leq t \) and that \( Q^*(s) \geq C(s, t] \). Then \( Q^*(s) \geq C(s, u] \) and so

\[
Q^*(s) + X(s, t] \geq C(s, u] + X(s, t]
\]

\[
= C(s, u] + A(s, t] - C(s, t]
\]

\[
= A(s, t] - C(u, t]
\]

\[
\geq A(u, t] - C(u, t] = X(u, t],
\]

and this inequality implies that the term \( X(u, t] \) inside the bracket of the right-hand side of (3.2) is not needed. Hence \( Q^*(t) = Q^*(s) + X(s, t] \), which is what we wanted to prove. \[\square\]

Define next

\[
\sigma^*(t) := \sup\{0 \leq s \leq t : Q^*(s) \leq C(s, t]\}.
\]

By Lemma 3.2,

(3.3)

\[
\sigma^*(t) := \sup\{0 \leq s \leq t : Q^*(s) \leq C(s, t]\}.
\]

By Lemma 3.2

(3.4a)

\[
Q^*(s) \leq C(s, t], \quad \text{if } 0 \leq s \leq \sigma^*(t),
\]

(3.4b)

\[
Q^*(s) \geq C(s, t], \quad \text{if } \sigma^*(t) < s \leq t,
\]

provided that the last inequality is nonvacuous. Since the function \( Q^* \) is nonnegative and continuous, we also have

\[
Q^*(\sigma^*(t)) = C(\sigma^*(t), t].
\]

Theorem 3.4. If \( X \) is a locally finite signed Borel measure on \([0, \infty)\) without atoms and if \( X = A - C \) is any decomposition of \( X \) as the difference of two nonnegative locally finite Borel measures without atoms, then the function \( Q^* \) defined by (2.1) satisfies (2.3).
Proof. By Lemma 3.3 and the last display,

\[ Q^*(t) = Q^*(\sigma^*(t)) + A(\sigma^*(t), t] - C(\sigma^*(t), t] \]
\[ = A(\sigma^*(t), t] \]
\[ = \int_{\sigma^*(t)}^t dA(s) \]
\[ = \int_0^t 1(Q^*(s) > C(s, t]) dA(s), \]

which is the integral representation formula (2.3). Note that, to obtain the last equality in the last display, we used (3.4a)-(3.4b). □

4. Proof of Theorem 2.1

A priori, it is not clear that (2.3) admits a maximal solution and, even if it does, whether it satisfies (2.1). We shall show the validity of these claims in the sequel.

We fix two locally finite measures \( A \) and \( C \) and define the map \( \Theta \) on the set of nonnegative measurable functions by

\[ \Theta(Q)(t) := \int_0^t 1(Q(s) > C(s, t]) dA(s), \quad t \geq 0. \]

The integral equation (2.3) then reads

\[ Q = \Theta(Q). \]

We observe that \( \Theta \) is increasing:

(4.2) If \( Q \leq \tilde{Q} \), then \( \Theta(Q) \leq \Theta(\tilde{Q}) \).

Here, and in the sequel, given two functions \( f, g : [0, \infty) \to \mathbb{R} \), we write \( f \leq g \) to mean that \( f(t) \leq g(t) \) for all \( t \geq 0 \). To see that (4.2) holds, simply observe that \( Q \leq \tilde{Q} \) implies \( 1(Q(s) > C(s, t]) \leq 1(\tilde{Q}(s) > C(s, t]) \) for all \( 0 \leq s \leq t \).

Define next a sequence of functions \((Q_k, k = 0, 1, 2, \ldots)\) by first letting \( Q_0 := \infty \), and then, recursively,

\[ Q_{k+1} := \Theta(Q_k), \quad k \geq 0. \]

Clearly, \( Q_1(t) = \int_0^t dA(s) = A(t) \). So \( Q_0 \geq Q_1 \). Since \( \Theta \) is an increasing map, we see that

\[ Q_k \geq Q_{k+1} \geq 0, \quad k \geq 0. \]

We can then define

\[ Q_\infty(t) := \lim_{k \to \infty} Q_k(t). \]

Lemma 4.1. If \( Q = \Theta(Q) \), then \( Q \leq Q_\infty \). Furthermore,

\[ Q^* \leq Q_\infty. \]

Proof. Suppose that \( Q \) satisfies \( Q = \Theta(Q) \). Since the integrand in the right-hand side of (4.1) is \( \leq 1 \), we have \( Q(t) \leq A(t) \) for all \( t \geq 0 \). Letting \( \Theta^{(k)} \) be the \( k \)-fold composition of \( \Theta \) with itself, we have

\[ Q = \Theta^{(k)}(Q) \leq \Theta^{(k)}(A) = Q_k, \]
and so \( Q \leq Q_\infty \). In particular, Proposition 3.4 states that \( Q^* = \Theta(Q^*) \). Hence \( Q^* \leq Q_\infty \).

However, it is not yet clear at this point that \( Q_\infty \) is a fixed point of \( \Theta \). We can only show that

\[
Q_\infty \geq \Theta(Q_\infty).
\]

Indeed, \( Q_\infty \leq Q_k \) for all \( k \), and so \( 1(Q_\infty(s) > C(s,t)) \leq 1(Q_k(s) > C(s,t)) \), for all \( 0 \leq s \leq t \), implying that \( \Theta(Q_\infty) \leq \Theta(Q_k) = Q_{k+1} \), and, by taking limits, that \( \Theta(Q_\infty) \leq Q_\infty \).

**Definition 4.2** (Regulating functions). Consider functions \( B : [0, \infty) \to [0, \infty) \) which are continuous, nondecreasing, with \( B(0) = 0 \), such that \( X(0,t) + B(t) \geq 0 \) for all \( t \geq 0 \). Call these functions regulating functions of \( X \). The set of regulating functions is denoted by \( \mathcal{R} \).

We define a mapping

\[
\Phi : \mathcal{R} \to \mathcal{R}
\]

in two steps as follows.

**Step 1.** Given \( B \in \mathcal{R} \), first define

\[
\sigma_B(t) := \sup \{0 \leq s \leq t : A(s) + B(s) - C(t) \leq 0\}, \quad t \geq 0.
\]

To motivate this definition, note that if \( B \) is chosen according to the formula \( B(t) = -\inf_{0 \leq s \leq t} \{A(s) - C(s)\} \), then \( \sigma_B(t) = \sigma^*(t) \) for all \( t \), where \( \sigma^* \) was defined in (3.3).

**Step 2.** Then let

\[
\Phi(B)(t) := B(\sigma_B(t)), \quad t \geq 0.
\]

We actually need to show that what is claimed in (4.3) holds, namely:

**Lemma 4.3.** If \( B \in \mathcal{R} \), then \( \Phi(B) \in \mathcal{R} \).

**Proof.** Clearly, \( \sigma_B(\cdot) \) is nondecreasing. Since \( B \) is nondecreasing, it follows that \( \Phi(B) = B \sigma_B \) is nondecreasing. Also, \( \Phi(B)(0) = B(\sigma_B(0)) = B(0) = 0 \). From the continuity of \( A \), \( B \) and the definition of \( \sigma_B \), we have

\[
A(\sigma_B(t)) + B(\sigma_B(t)) = C(t), \quad t \geq 0.
\]

We also have

\[
A(t) + \Phi(B)(t) - C(t) = A(t) + B(\sigma_B(t)) - C(t)
= [A(t) - A(\sigma_B(t))] + [A(\sigma_B(t)) + B(\sigma_B(t)) - C(t)]
= A(t) - A(\sigma_B(t)) \geq 0,
\]

where we used (4.4) in the third step. It remains to show that \( \Phi(B)(\cdot) \) is continuous. Note that \( \sigma_B(\cdot) \) need not be continuous. However, \( C(\cdot) \) is a continuous function and so, by (4.3), \( t \mapsto A(\sigma_B(t)) + B(\sigma_B(t)) \) is continuous. Hence

\[
[A(\sigma_B(t^+)) - A(\sigma_B(t^-))] + [B(\sigma_B(t^+)) - B(\sigma_B(t^-))] = 0, \quad \text{for all } t.
\]

Since \( A(\sigma_B(\cdot)) \) and \( B(\sigma_B(\cdot)) \) are both nondecreasing, it follows that \( A(\sigma_B(t^+)) - A(\sigma_B(t^-)) \geq 0 \) and \( B(\sigma_B(t^+)) - B(\sigma_B(t^-)) \geq 0 \) and, since their sum is zero, they are both zero, implying that \( A(\sigma_B(\cdot)) \) and \( B(\sigma_B(\cdot)) \) are continuous. \( \square \)
An immediate property of $\Phi$ is that
\begin{equation}
\Phi(B) \leq B \quad \text{for all } B \in \mathcal{R}.
\end{equation}
Indeed, for all $t \geq 0$, $\sigma_B(t) \leq t$ and so $B(\sigma_B(t)) \leq B(t)$.

Starting with the function
\begin{equation}
B_1(t) := C(t), \quad t \geq 0,
\end{equation}
we recursively define
\begin{equation}
B_{k+1} := \Phi(B_k), \quad k \geq 1.
\end{equation}
Therefore
\begin{equation}
B_1 \geq B_2 \geq \cdots \geq B_k \downarrow B_\infty, \quad \text{as } k \to \infty,
\end{equation}
where the inequalities and the limit are pointwise.

**Lemma 4.4.** The function $B_\infty$, defined via \(4.6\), \(4.7\) and \(4.8\), is a member of the class $\mathcal{R}$.

**Proof.** $B_\infty$ is nondecreasing since all the $B_k$ are nondecreasing. Also, $B_\infty(0) = 0$.
Since for all $k$, $A + B_k - C \geq 0$, we have $A + B_\infty - C \geq 0$. We proceed to show that $B_\infty$ is a continuous function. We observe that, for $0 \leq t \leq t'$,
\begin{equation*}
|\Phi(B)(t') - \Phi(B)(t)| = |B(\sigma_B(t')) - B(\sigma_B(t))| \\
= B(\sigma_B(t')) - B(\sigma_B(t)) \\
\leq A(\sigma_B(t')) - A(\sigma_B(t)) + B(\sigma_B(t')) - B(\sigma_B(t)) \\
= [A(\sigma_B(t')) + B(\sigma_B(t'))] - [A(\sigma_B(t)) + B(\sigma_B(t))] \\
= C(t') - C(t),
\end{equation*}
where we again used \(4.4\). It follows that the family of functions $\{\Phi(B), B \in \mathcal{R}\}$ is uniformly bounded and equicontinuous on each compact interval of the real line. By the Arzelà-Ascoli theorem, the family is compact and therefore $B_\infty$ is continuous. We have established that $B_\infty \in \mathcal{R}$. \(\square\)

We now claim that $B_\infty$ is a fixed point of $\Phi$.

**Lemma 4.5.** $\Phi(B_\infty) = B_\infty$.

**Proof.** By definition,
\begin{equation*}
\Phi(B_\infty)(t) = B_\infty(\sigma_{B_\infty}(t)),
\end{equation*}
where
\begin{equation*}
\sigma_{B_\infty}(t) = \sup\{0 \leq s \leq t : A(s) + B_\infty(s) \leq C(t)\}.
\end{equation*}
Now, since $B_k \geq B_{k+1}$ for all $k \geq 1$, it follows that $\sigma_{B_k} \leq \sigma_{B_{k+1}}$ for all $k \geq 1$, and so
\begin{equation*}
\sigma_L(t) := \lim_{k \to \infty} \sigma_{B_k}(t)
\end{equation*}
is well-defined. Since $B_k \geq B_\infty$ for all $k \geq 1$, we have $\sigma_{B_k} \leq \sigma_{B_\infty}$. Taking limits, we find
\begin{equation*}
\sigma_L \leq \sigma_{B_\infty}.
\end{equation*}
Using the last two displays and the fact that $B_k$ and $B_\infty$ are nondecreasing, we have
\[
\Phi(B_\infty)(t) = B_\infty(\sigma_{B_\infty}(t)) \geq B_\infty(\sigma_L(t))
\]
\[
= \lim_{k \to \infty} B_k(\sigma_L(t))
\]
\[
\geq \lim_{k \to \infty} B_k(\sigma_B(t))
\]
\[
= \lim_{k \to \infty} B_{k+1}(t) = B_\infty(t).
\]
By inequality (4.5), $\Phi(B) \leq B$ for all $B \in \mathcal{R}$ and since, by Lemma 4.4, $B_\infty \in \mathcal{R}$, it follows that we also have $B_\infty \leq \Phi(B_\infty)$. Therefore $B_\infty = \Phi(B_\infty)$, as claimed. □

**Lemma 4.6.** Consider the function $Q^*$ defined by (2.1) and define a function $B^*$ by
\[
B^*(t) := Q^*(t) - X(0,t), \quad t \geq 0.
\]
Then
(i) $B^* \in \mathcal{R}$;
(ii) $B^* = \Phi(B^*)$.

**Proof.** (i) We have $X(0,t] + B^*(t) = Q^*(t) \geq 0$ for all $t$. Using (2.1) and (2.2) we see that
\[
B^*(t) = \sup_{0 \leq s \leq t} \{-A(s) + C(s)\}.
\]
Therefore, $B^*(0) = 0$, and $B^*$ is continuous and nondecreasing. We conclude that $B^* \in \mathcal{R}$. To prove (ii), recall that $\Phi(B^*) = B^* \circ \sigma_{B^*}$, where
\[
\sigma_{B^*}(t) = \sup \{0 \leq s \leq t : A(s) + B^*(s) \leq C(t)\}.
\]
Splitting the supremum in (4.9) into two parts, we obtain
\[
B^*(t) = \sup_{0 \leq s \leq \sigma_{B^*}(t)} \{-A(s) + C(s)\} \vee \sup_{\sigma_{B^*}(t) \leq s \leq t} \{-A(s) + C(s)\}.
\]
For $s \geq \sigma_{B^*}(t)$, we have $A(s) + B^*(s) \geq C(t)$, i.e. $-A(s) + C(s) \leq B^*(s) - C(s,t]$. Therefore
\[
B^*(t) \leq B^*(\sigma_{B^*}(t)) \vee \sup_{\sigma_{B^*}(t) \leq s \leq t} \{B^*(s) - C(s,t]\}
\]
\[
= B^*(\sigma_{B^*}(t)) = \Phi(B^*)(t).
\]
Thus, $B^* \leq \Phi(B^*)$. On the other hand, since $B^* \in \mathcal{R}$, we have $\Phi(B^*) \leq B^*$, by (4.5).

**Lemma 4.7.** Let $B \in \mathcal{R}$ be any fixed point of $\Phi$. Then $B \leq B^*$.

**Proof.** Since $B = \Phi(B) = B \circ \sigma_B$ we have
\[
B = B \circ \sigma_B^{(k)},
\]
where $\sigma_B^{(k)} := \sigma_B \circ \cdots \circ \sigma_B$, $k$ times. Since
\[
t \geq \sigma_B(t) \geq \sigma_B \circ \sigma_B(t) \geq \cdots \geq \sigma_B^{(k)}(t),
\]
we may define
\[ \sigma_B^{(\infty)}(t) := \lim_{k \to \infty} \sigma_B^{(k)}(t). \]

By the continuity of \( B \),
\[ B = B \circ \sigma_B^{(\infty)}. \tag{4.10} \]

On the other hand, (4.4) gives
\[ A \circ \sigma_B^{(k+1)} + B \circ \sigma_B^{(k+1)} = C \circ \sigma_B^{(k)}, \quad k \geq 1. \]

Taking the limit as \( k \to \infty \), and using the continuity of \( A, B \), and \( C \), we have
\[ A \circ \sigma_B^{(\infty)} + B \circ \sigma_B^{(\infty)} = C \circ \sigma_B^{(\infty)}. \]

Since \( A(t) + B^*(t) \geq C(t) \) for all \( t \), we have
\[ A \circ \sigma_B^{(\infty)} + B^* \circ \sigma_B^{(\infty)} \geq C \circ \sigma_B^{(\infty)}, \]

and from the last two displays we conclude that
\[ B^* \circ \sigma_B^{(\infty)} \geq B \circ \sigma_B^{(\infty)}. \]

Since \( B^* \) is nondecreasing and since (4.10) holds, we have
\[ B^* \geq B^* \circ \sigma_B^{(\infty)} \geq B \circ \sigma_B^{(\infty)} = B, \]
as claimed. \( \square \)

We are now ready to prove Theorem 2.1. We already know from Lemma 4.1 that \( Q^* \leq Q_{\infty} \). So we only have to prove the opposite inequality. Recall that \( Q_1 = A \) and \( B_1 = C \). Trivially then
\[ Q_1(t) + C(t) = A(t) + B_1(t), \quad t \geq 0. \]

Thus, for \( 0 \leq s \leq t \) we have
\[ Q_1(s) > C(s, t) \iff Q_1(s) + C(s) > C(t) \iff A(s) + B_1(s) > C(t) \iff s > \sigma_{B_1}(t). \]

From this we get
\[ Q_2(t) = \int_0^t \mathbf{1}(Q_1(s) > C(s, t)) \, dA(s) \]
\[ = \int_0^t \mathbf{1}(s > \sigma_{B_1}(t)) \, dA(s) \]
\[ = A(t) - A(\sigma_{B_1}(t)). \]

But (4.4) gives
\[ A(\sigma_{B_1}(t)) + B_1(\sigma_{B_1}(t)) = C(t), \]
and so
\[ Q_2(t) + C(t) = A(t) + B_1(\sigma_{B_1}(t)) = A(t) + B_2(t), \quad t \geq 0. \]
We now claim that
\[ Q_k(t) + C(t) = A(t) + B_k(t), \quad t \geq 0, \quad k \geq 1. \]
This can be proved by induction along the same lines as above. Taking limits as \( k \to \infty \), we conclude
\[ Q_\infty(t) + C(t) = A(t) + B_\infty(t), \quad t \geq 0. \]
Lemma 4.5 tells us that \( B_\infty \) is a fixed point of \( \Phi \), and so, by Lemma 4.7,
\[ B_\infty \leq B^*. \]
Hence
\[ Q_\infty(t) + C(t) = A(t) + B_\infty(t) \leq A(t) + B^*(t) = Q^*(t) + C(t), \quad t \geq 0, \]
and this gives
\[ Q_\infty \leq Q^*, \]
as needed.

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